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Hypothesis Testing in Weighted Distributions

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Abstract. There are many situations in which experiments are not available or data are recorded from the population proportion to a nonnegative function called weight function. In a such situations the classical methods for inferencing about unknown parameters are not useful. In this study the problem of statistical hypothesis testing is considered for weighted distributions to obtain (uniformly) most powerful tests.

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1. Introduction

Consider a random vector $\mathbf{X}$ distributed according to the density $f_\theta(\mathbf{x})$; it is desired to make testing about the unknown values of the parameter $\theta(\theta \in \Theta \subset R)$. In many situations the usual random sample from the poqulation is not available, due to the data having unequal properties of entering or recording the sample. Suppose that the probability that the observation $\mathbf{x}$ enter the sample gets multiplied by some nonnegative weight function $w(x,\gamma)$, where γ is a parameter which may or may not depend on θ. Then the observed sample is a random sample from the following weighted distribution:

$$f_\theta^w(x) = \frac{w(x,\gamma) f_\theta(x)}{E_\theta[w(X,\gamma)]}, \tag{1}$$

where $w(x,\gamma)$ is nonnegative and

$$E_\theta[w(X,\gamma)] = \int_{-\infty}^{\infty} w(x,\gamma) f_\theta(x) dx < \infty, \tag{2}$$

is just a normalizing constant.

The weighted distributions may be applied in many statistical fields. For example in meta-analysis weight functions can be used to model selection process and develop estimation procedures, Iyengar and Greenhouse [12], Silliman [19], Fleiss [11]. Such other examples can be found in Oil Discovery West [19, 20] and Line Transact Sampling Cook and Martin [8] and Drummer and McDonald [9]. More examples are also given in Patil [14].

The concept of weighted distributions was originally introduced by Fisher ([10]) to the study of effect of methods of ascertainment upon estimation of frequencies. However, it was Rao [16,17] who presented a unified theory of weighted distributions. Patil and Taillie [15] calculated the Fisher information for certain exponential family of weighted distributions see also Bayarri and DeGroot [4] and Bayarri et al, [5]. Some other properties of weighted distributions are studied by many authors see Alavi and Chinipardaz [1,2]. The weighted distributions have also been studied in bayesian analysis. For the finite population models West [20,21] considered known or parametric forms of $w(x, \gamma)$. The parametric forms of $w(x, \gamma)$ have also been investigated and employed for selection models in infinite population models by Bayarri and Berger (1998) and Bayarri and DeGroot [6,7].

This article is devoted to the problem of hypothesis testing in weighted distributions. Motivation of the study is that the information about the parameter, θ, provided with the weighted distributions may have more (less) than unweighted distributions.

The article includes five sections; in Section 2 the problem of hypothesis testing is discussed for simple versus simple hypothesis. The Neyman-Pearson Lemma is generalized for weighted distributions. In Section 3, focus is on selecting an appropriate weight function from two or a set of candidates weight functions based on Neyman-Pearson Lemma. Section 4 is devoted to Uniformly Most Powerful (UMP) tests. A necessarily and sufficient condition is obtained for a weighted distribution to have Monotone Likelihood Ratio (MLR) property. Finally, in Section 5 UMP unbiased tests for exponential family under the weighted samples are studied

2. General Concepts under Weighted Sampling

Suppose we want to test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$, where $\Theta_0 \cup \Theta_1 = \Theta$ is parameter space and $\Theta_0 \cap \Theta_1 = \emptyset$. Assume that sampling is under weight function $w(x, \gamma)$ or $w(x, \gamma, \theta)$. Therefore, X^w=$(X_1^w, X_2^w, \ldots, X_n^w) \sim f_\theta^w(x)$ where $f_\theta^w(x)$ is given in (1) and $\mathbf{x}$=$(x_1, x_2, \ldots, x_n)$ is observed vector of X^w. In such a case the test function based on weighted function with α size is defined by

$$\phi(\mathbf{X}^w) = \begin{cases} 1 & \text{if } H_0 \text{ is rejected} \\ 0 & \text{if } H_0 \text{ is not rejected,} \end{cases}$$

with $\sup_{\theta \in \Theta_0} E_{\theta_0}[\phi(X^w)] = \alpha$ and the power function $\beta^* = E_\theta[\phi(X^w)]$. Consider simple versus simple hypothesis testing ($H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1$). The most powerful, MP, test with size α in classical theory is based on Fundamental Neyman-Pearsom lemma given by

$$\phi(\mathbf{X}) = \begin{cases} 1 & \frac{f_{\theta_1}(\mathbf{X})}{f_{\theta_0}(\mathbf{X})} > c \\ 0 & \text{otherwisw,} \end{cases}$$

where threshold value c is obtained with $E_{\theta_0}[\phi(\mathbf{X})] = \alpha$. Note that $f_{\theta_1}(x)$ and $f_{\theta_0}(x)$ can have different structures.

Now suppose that weight function is $w(x, \gamma, \theta) = w(x, \gamma)$. Replacing $f_{\theta_i}^w(x), i = 0, 1$ in (1)

$$\frac{f_{\theta_1}^w(\mathbf{x})}{f_{\theta_0}^w(\mathbf{x})} = \frac{\prod_{i=1}^n f_{\theta_1}(x_i)[E_{\theta_0} w(X, \gamma)]^n}{\prod_{i=1}^n f_{\theta_0}(x_i)[E_{\theta_1} w(X, \gamma)]^n} = g(\theta_0, \theta_1)\frac{f_{\theta_1}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})},$$

where $g(\theta_0, \theta_1)$ is positive and independent of test statistic.

$$\frac{f_{\theta_1}(\underline{\mathrm{x}})}{f_{\theta_0}(\underline{\mathrm{x}})} > c \Longleftrightarrow \frac{f_{\theta_1}^w(\underline{\mathrm{x}})}{f_{\theta_0}^w(\underline{\mathrm{x}})} > k,$$

where $k = c\frac{[E_{\theta_0} w(X,\gamma)]^n}{[E_{\theta_1} w(X,\gamma)]^n}$ is obtained from $E_{\theta_0}[\phi(X^w)] = \alpha$. It means that the structure of test is the same for weighted and unweighted sampling with possibly a different critical region.

Example 1. As an example consider $X_1, X_2, \dots, X_n \sim f_\theta(x) = \theta \exp(-\theta x)$. A MP α size test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1 (\theta_1 > \theta_0)$ is

$$\phi(\mathbf{X}) = \begin{cases} 1 & \bar{X} < c_1 \\ 0 & \bar{X} \geqslant c_1, \end{cases}$$

where $c_1 = \frac{\chi^2_{(2n)}(\alpha)}{2n\theta_0}$ and $\chi^2_m(q)$ is qth quantile value of chi-square random variable with m degrees of freedom and power function

$$\beta^* = E_{\theta_1}[\phi(\mathbf{X})] = P_{\theta_1}\left(2n\theta_1 \bar{X} < \frac{2n\theta_1}{2n\theta_0}\chi^2_{(2n)}(\alpha)\right) = F_{\chi^2_{(2n)}}\left(\frac{\theta_1}{\theta_0}\chi^2_{(2n)}(\alpha)\right),$$

Table 1. The power of test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ for n and γ

	$\alpha = 0.05$					$\alpha = 0.01$				
	n					n				
γ	1	5	10	20	50	1	5	10	20	50
0	0.0975	0.3595	0.6431	0.9185	0.999	0.0199	0.1167	0.3162	0.7059	0.9949
1	0.1595	0.6431	0.9185	0.997	1	0.0363	0.3162	0.7059	0.9767	0.9999
2	0.2258	0.8224	0.9885	0.999	1	0.0583	0.5297	0.9077	0.999	1
5	0.4236	0.9855	0.9999	1	1	0.1518	0.9077	0.999	1	1
10	0.6872	0.999	1	1	1	0.3599	0.9978	0.999	1	1

where $F_{\chi^2_{(2n)}}$ stands for the distribution function with $2n$ degree of freedom.

Now under size biased weight function $w(x,\gamma)=x^{\gamma}$, $g(\theta_0,\theta_1)=\left(\frac{\theta_1}{\theta_0}\right)^n$. Then MP test α level is again

$$\phi(\mathbf{X})=\begin{cases} 1 & \bar{X}^w < c_2(\gamma) \\ 0 & \bar{X}^w \geqslant c_2(\gamma), \end{cases}$$

where $c_2(\gamma)$ is obtained from $P(\bar{X}^w < c_2(\gamma))=\alpha$ with $c_2(\gamma)=\frac{\chi^2{}_{(2n(1+\gamma))}(\alpha)}{2n\theta_0}$, because $X^w \sim gamma(1+\gamma,\theta)$, see Alavi and Chnipardaz [2].

Table 1. shows the power of test $H_0:\theta=1$ versus $H_1:\theta=2$ for various n and γ. As can be seen from the table the power increases as γ increases. For the case of a weight function which depends on parameter, $w(x,\gamma,\theta)$

$$\frac{f^w_{\theta_1}(\mathbf{x})}{f^w_{\theta_0}(\mathbf{x})}=\left\{\frac{f_{\theta_1}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})}\right\}\cdot\left\{\frac{[E_{\theta_0}w(X,\beta,\theta_0)]^n}{[E_{\theta_1}w(X,\beta,\theta_1)]^n}\frac{\prod_{i=1}^n w(x_i,\beta,\theta_1)}{\prod_{i=1}^n w(x_i,\beta,\theta_0)}\right\}.$$

Let unweighted test statistic be $T(\mathbf{X})$. If two expressions are both decreasing (increasing) of $T(\mathbf{x})$ the test function in weighted and random sampling have the same structure and otherwise nothing can be said and a direct computation has to be used.

Example 2. Suppose that $X_1,X_2,\ldots,X_n$ is a weighted sample from $X\sim N(\theta,\sigma^2)$, σ^2 known, under the weight function $w(x,\gamma,\theta,\sigma^2)=\exp\left[\gamma(\frac{x-\theta}{\sigma})^2\right]$ with $\gamma<\frac{1}{2}$. Now, $X_1^w,X_2^w,\ldots,X_n^w$ is a random sample from $X^w\sim N(\theta,\frac{\sigma^2}{1-2\gamma})$ (Alavi and Chinipardaz, 2009). Thus $\bar{X}^w\sim$

$N(\theta, \frac{\sigma^2}{n(1-2\gamma)})$. For testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$,

$$\begin{aligned} \frac{[E_{\theta_0} w(X,\gamma,\theta_0)]^n \prod_{i=1}^n w(x_i,\gamma,\theta_1)}{[E_{\theta_1} w(X,\gamma,\theta_1)]^n \prod_{i=1}^n w(x_i,\gamma,\theta_0)} &= 1 \times \frac{\exp[\frac{\gamma}{\sigma^2}\sum_{i=1}^n (x_i-\theta_1)^2]}{\exp[\frac{\gamma}{\sigma^2}\sum_{i=1}^n (x_i-\theta_0)^2]} \\ &= \exp\left[\frac{n\gamma}{\sigma^2}(\theta_1-\theta_0)(\theta_1+\theta_0-2\bar{x})\right], \end{aligned}$$

is an increasing function of test statistic, $\bar{\mathbf{X}}^w$, when $\gamma < 0$, because

$$\begin{aligned} E_\theta[w(X,\gamma,\theta)]^n &= \int_{-\infty}^{\infty} \exp\,[\gamma(\frac{x-\theta}{\sigma})^2]\frac{1}{\sqrt{2\pi}\sigma}\exp\,[-\frac{1}{2}(\frac{x-\theta}{\sigma})^2]dx \\ &= \int_{-\infty}^{\infty} \exp\,[\gamma z^2]\frac{1}{\sqrt{2\pi}}\exp\,[-\frac{1}{2}z^2]dz, \end{aligned}$$

does not depend on θ. Therefore for $\gamma < 0$, the test

$$\phi(\bar{X}^w) = \begin{cases} 1 & \bar{X}^w > \theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n(1-2\gamma)}} \\ 0 & \text{otherwise}, \end{cases}$$

is MP test with size α and power function

$$P\left(\bar{X}^w > \theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n(1-2\gamma)}}\right) = 1 - \Phi\left(z_{1-\alpha} - \frac{(\theta_1-\theta_0)\sqrt{n(1-2\gamma)}}{\sigma}\right),$$

which is a decreasing function of γ with the maximum value equal to $1 - \Phi\left(z_{1-\alpha} - \frac{\sqrt{n}(\theta_1-\theta_0)}{\sigma}\right)$.

Note that for $0 < \gamma < \frac{1}{2}$, in which two expressions have different behavior with respect to $\bar{X}$, nothing can be said.

3. Hypothesis Testing for Weight Function

In this section we consider the testing problem for two different weight functions. Suppose that the researcher is asked to know which weight function is satisfied for the obtained data. i.e. $H_0 : w(x) = w_1(x)$ versus $H_1 : w(x) = w_2(x)$.

Clearly, for random sampling $w_1(x) = constant$. In this case the researcher want to know if the sampling is random versus weighted sampling with weight function $w(x) = w_2(x)$. According to Neyman Pearson Lemma, a MP test is given by

$$\begin{aligned}\phi(\mathbf{X}^w) &= \begin{cases} 1 & \dfrac{f^{w_2}(\mathbf{x})}{f^{w_1}(\mathbf{x})} > c \\ 0 & \dfrac{f^{w_2}(\mathbf{x})}{f^{w_1}(\mathbf{x})} \leqslant c \end{cases} \\ &= \begin{cases} 1 & \dfrac{[Ew_1(X)]^n \prod_{i=1}^n w_2(x_i)}{[Ew_2(X)]^n \prod_{i=1}^n w_1(x_i)} > c \\ 0 & \dfrac{[Ew_2(X)]^n \prod_{i=1}^n w_2(x_i)}{[Ew_1(X)]^n \prod_{i=1}^n w_1(x_i)} \leqslant c \end{cases} \\ &= \begin{cases} 1 & T > k \\ 0 & T \leqslant k, \end{cases}\end{aligned}$$

where the test statistic is $T = \prod_{i=1}^n \frac{w_2(x_i)}{w_1(x_i)}$ and c is obtained to give α-size test, i.e.,

$$P_{w_1}(T > c) = \alpha.$$

When the data are considered to be random, $w_1(x) = 1$, the test is rejected if

$$T^* = \prod_{i=1}^n w_2(x_i) > c^*,$$

where c^* is given by $P_{Random}(T^* > c^*) = \alpha$.

Example 3. Consider example 1 with known θ. Suppose that we want to test $H_0 : w(x, \gamma) = 1$ versus $H_1 : w(x, \gamma) = x^\gamma$ where γ is specified. A MP test is

$$\phi(\mathbf{x}^w) = \begin{cases} 1 & \prod_{i=1}^n x_i^\gamma > k(\gamma) \\ 0 & otherwise. \end{cases}$$

Table 2. The values of two power of test $H_0 : w(x) = 1$ versus $H_1 : w(x, \gamma) = x^\gamma$

	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
α	c	β^*	c	β^*	c	β^*
0.01	4.61	0.056	21.25	0.1616	97.97	0.325
0.025	3.69	0.117	13.62	0.2870	50.24	0.496
0.05	2.99	0.201	8.94	0.4247	26.73	0.648
0.1	2.3	0.331	5.29	0.595	12.17	0.799
0.15	1.89	0.435	3.57	0.704	6.51	0.875
0.2	1.61	0.522	2.59	0.781	4.17	0.920

where $k(\gamma)$ is obtained from $P\left(\prod_{i=1}^n X_i^\gamma > k(\gamma)\right) = \alpha$. For $n = 1$, $k(\gamma) = \left(-\frac{\ln \alpha}{\theta}\right)^\gamma$ the power of the test is

$$\beta_\gamma^* = P\left\{(X^w)^\gamma > \left(-\frac{\ln \alpha}{\theta}\right)^\gamma\right\} = \int_{-\frac{\ln \alpha}{\theta}}^\infty \frac{\theta^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma e^{-\theta x} dx,$$

because $X^w \sim Gamma(\gamma + 1, \theta)$ Alavi and Chinipardaz [2].

Table 2. shows the values of the power of the test $H_0 : w(x) = 1$ versus $H_1 : w(x, \gamma) = x^\gamma, \gamma = 1, 2, 3$ and $\theta = 1$ for various values of α. As can be seen the power of the test increases as γ increases for fixed α. For $n > 1$, distribution of T^* is very complicated. However k and β_γ^* can be easily calculated using Monte Carlo simulation.

4. Uniformly Most Powerful(UMP) Tests

Suppose that the weight function is independent of unknown parameter and suppose that θ_1 and θ_2 $(\theta_1 < \theta_2)$ are two distinct real values of Θ. Then

$$\frac{f^w_{\theta_2}(\mathbf{x})}{f^w_{\theta_1}(\mathbf{x})} = \frac{f_{\theta_2}(\mathbf{x})}{f_{\theta_1}(\mathbf{x})} \frac{[E_{\theta_2} w(\mathbf{x},\gamma)]^n}{[E_{\theta_1} w(\mathbf{x},\gamma)]^n} \,.$$

If $f_\theta(x)$ has MLR in $T(\mathbf{x})$, $T(\mathbf{x})$ is also MLR for $f^w_\theta(x)$ because $\frac{[E_{\theta_2} w(\mathbf{x},\gamma)]^n}{[E_{\theta_1} w(\mathbf{x},\gamma)]^n}$ is positive for any $\theta \in \Theta$. This allows us to give the UMP test for one-sided hypotheses $H_0 : \theta \leqslant \theta_0$ $(\theta \geqslant \theta_0)$ versus $H_1 : \theta > \theta_0$ $(\theta < \theta_0)$ for the large family of distributions with MLR property, including exponential family. Therefore,

$$\phi(\mathbf{X}^w) = \begin{cases} 1 & T(\mathbf{X}^w) > (<)\, c \\ 0 & otherwise., \end{cases}$$

where c is given by $E_\theta[\phi(\mathbf{X}^w)] = \alpha$ is the UMP test with increasing power function.

Example 4. Consider $X \sim Gamma(p,\theta)$, with known p and weight function $w(x,\gamma_1,\gamma_2) = x^{\gamma_1}\exp\{-\gamma_2 x\}$, $\gamma_1 > -p$, $\gamma_2 > -\theta$. Now for $H_0 :$ $\theta \leqslant \theta_0$ versus $H_1 : \theta > \theta_0$, the following test is UMP

$$\phi(\mathbf{X}^w) = \begin{cases} 1 & \bar{\mathbf{X}}^w > (<)\, [2n(\theta_0+\gamma_2)]^{-1}\chi^2_{(2n(p+\gamma_1))}(\alpha) \\ \\ 0 & otherwise. \end{cases}$$

Note that for $\theta_1 < \theta$

$$\begin{aligned}\frac{E_{\theta_2}[X^{\gamma_1}\exp\{-\gamma_2 X\}]}{E_{\theta_1}[X^{\gamma_1}\exp\{-\gamma_2 X\}]} &= \left(\frac{\theta_2^p}{(\theta_2+\gamma_2)^{p+\gamma_1}}\right)\Big/\left(\frac{\theta_1^p}{(\theta_1+\gamma_2)^{p+\gamma_1}}\right)\\ &= \left(\frac{\theta_2(\theta_1+\gamma_2)}{\theta_1(\theta_2+\gamma_2)}\right)^p\left(\frac{\theta_1+\gamma_2}{\theta_2+\gamma_2}\right)^{\gamma_1} \geqslant 0.\end{aligned}$$

In a more complex case suppose that $w(x,\gamma,\theta)$ involves unknown parameter, θ. Lehmann and Romano [13, page 98] showed that a necessary and sufficient condition for densities $f_\theta(x)$ to have MLR in x is that mixed second derivative for $\log f_\theta(x)$ exists and is nonnegative for all θ and all x;

$$\frac{\partial^2}{\partial\theta\partial x}\log f_\theta(x) \geqslant 0.$$

Replacing the weighted density function we have

$$\begin{aligned}\frac{\partial^2}{\partial\theta\partial x}\log f_\theta^w(x) &= \frac{\partial^2}{\partial\theta\partial x}\log\frac{w(x,\gamma,\theta)f_\theta(x)}{E_\theta w(x,\gamma,\theta)}\\ &= \frac{\partial^2}{\partial\theta\partial x}\log f_\theta(x) + \frac{\partial^2}{\partial\theta\partial x}\log\left[\frac{w(x,\gamma,\theta)}{E_\theta[w(X,\gamma,\theta)]}\right]\\ &= \frac{\partial^2}{\partial\theta\partial x}\log f_\theta(x) + \frac{\partial^2}{\partial\theta\partial x}\log w(x,\gamma,\theta). \qquad (3)\end{aligned}$$

Therefore, if $f_\theta(x)$ has MLR in x, $f_\theta^w(x)$ has also MLR in x only if the second term of (3) exists and is log concave. i. e.

$$\frac{\partial^2}{\partial\theta\partial x}\log w(x,\gamma,\theta) \geqslant 0. \qquad (4)$$

Condition (4) is easy to check for any weight function. For example if $X \sim \frac{1}{\theta}\exp\{-\frac{1}{\theta}(x-\frac{\theta}{2})\}\quad x > \frac{\theta}{2} > 0$ and $w(x,\gamma,\theta) = \gamma(x-\frac{\theta}{2})$, we have

$$\frac{\partial^2}{\partial\theta\partial x}\log\gamma\left(x-\frac{\theta}{2}\right) = \left(x-\frac{\theta}{2}\right)^{-2} > 0.$$

It means that $f^w_\theta(x) = \frac{x-\frac{\theta}{2}}{\theta^2}\exp\{-\frac{1}{\theta}(x-\frac{\theta}{2})\}, \quad x > \frac{\theta}{2}$ has MLR with respect to x.

In example 2 we have

$$\frac{\partial^2}{\partial\theta\partial x}\log w(x,\gamma,\theta,\sigma^2) = -2\frac{\gamma}{\sigma^2} \geqslant 0, \quad \gamma < 0,$$

and nothing can be said for $0 < \gamma < \frac{1}{2}$

5. UMP Unbiased Tests in Weighted Distributions

Unfortunately, UMP test does not exist for two sided tests, $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ for many statistical densities, including exponential family see Rohatgi and Saleh [18] for an example. Instead there is UMP unbiased tests in exponential family, see Lehman and Romano [13]. Now suppose that X is from exponential family given as

$$f(x,\theta) = \exp\{a(\theta)T(x) + c(\theta) + d(x)\} \tag{5}$$

and $w(x,\gamma) \geqslant 0$, γ is known and independent of θ, then pdf of X^w is given by

$$\begin{aligned} f^w(x,\theta,\beta) &= \frac{w(x,\gamma)\exp\{a(\theta)T(x) + c(\theta) + d(x)\}}{E_\theta[w(X,\beta)]} \\ &= \exp\{a(\theta)T(x) + c(\theta) + d(x) + \log w(x,\gamma) - \log E_\theta[w(X,\gamma)]\} \\ &= \exp\{a(\theta)T(x) + c^w(\theta) + d^w(x)\}, \end{aligned}$$

where $c^w(\theta) = c(\theta) - \log E_\theta[w(X,\gamma)]$ and $d^w(x) = d(x) + \log w(x,\gamma)$. Therefore, for $X_1,\ldots,X_n$ taken from (5), $T(\mathbf{X}^w)$ is still a minimal sufficient statistics and the UMP unbiased test is

$$\phi(\mathbf{X}^w) = \begin{cases} 1 & T(\mathbf{X}^w) < c_1 \text{ or } T(\mathbf{X}^w) > c_2 \\ \\ 0 & c_1 \leqslant T(\mathbf{X}^w) \leqslant c_2 \end{cases},$$

with $P_{\theta_0}(c_1 \leqslant T(\mathbf{X}^w) \leqslant c_2) = 1-\alpha$ and

$$E_{\theta_0}\left[\phi(\mathbf{X}^w)T(\mathbf{X}^w)\right] = \alpha E_{\theta_0}(T(\mathbf{X}^w)),$$

or

$$E_{\theta_0}\left[1-\phi(\mathbf{X}^w)\right] = (1-\alpha)\left\{\frac{\frac{\partial}{\partial\theta}[E_\theta(w(x,\gamma))] - E_\theta(w(x,\gamma))\frac{\partial}{\partial\theta}[c(\theta)]}{E_\theta(w(x,\gamma))\frac{\partial}{\partial\theta}[a(\theta)]}\right\}_{\theta=\theta_0}. \quad (6)$$

This is similar to UMP unbiased test in random sampling. The only difference is that equation (6) has to be changed to

$$E_{\theta_0}\left[1-\phi(\mathbf{X})\right] = (1-\alpha)\left\{-\frac{\frac{\partial}{\partial\theta}[c(\theta)]}{\frac{\partial}{\partial\theta}[a(\theta)]}\right\}_{\theta=\theta_0}.$$

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