Journal of Mathematical Extension
Vol. 8, No. 2, (2014), 25-31

Best Proximity Point and Geometric Contraction Maps

M. R. Haddadi
Ayatollah Boroujerdi University

Abstract. In this paper we introduce geometric contraction map and give a new condition for the existence and uniqueness of best proximity point of geometric contractions. We also consider the convergence of iterates to proximity points in metric spaces.

AMS Subject Classification: 54E15; 46A03; 54E50; 41A17
Keywords and Phrases: Best proximity pair, best proximity point, fixed point, geometric contraction.

1. Introduction

Fixed point theory is an important tool for solving the equation $T(x) = x$. However, if T does not have any fixed point, then one often tries to find an element x which is in some sense closest to $T(x)$. A classical result in this direction is a best approximation theorem due to Ky Fan [1].
A best proximity pair evolves as a generalization of the best approximation considered in [1-5], of exploring the sufficient conditions for the existence of the best proximity point.
In this paper we consider sufficient conditions that ensure the existence of an element $x \in X$ for two subsets A, B in metric space (X, d) such that $d(x, Tx) = dist(A, B)$ for $T : A \cup B \to A \cup B$, where

$$dist(A, B) = \inf_{(a,b)\in A\times B} d(a, b).$$

Received: July 2013; Accepted: May 2014

In this case, we say that x is a best proximity point of T with respect to A and B. It is clear that if x is a best proximity point of T, then $dist(A,B) = 0$ if and only if x is a fixed point of T.

2. Main Results

In this section at first we give a new definition and use it to present new results.

Definition 2.1. *Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is a geometric contraction map if*

i) $T(A) \subset B$ *and* $T(B) \subset A$.
ii) *For some $\alpha \in (0,1)$ and all $x \in A$ and $y \in B$ we have*

$$d(Tx, Ty) \leqslant d(x,y)^{\alpha} dist(A,B)^{1-\alpha}.$$

For example, if

$$A = \{(x,0) : x \geqslant 1\}, \ B = \{(0,y) : y \geqslant 1\},$$

and $T(x,y) = (\sqrt{y}, \sqrt{x})$. Then $dist(A,B) = \sqrt{2}$ and $\alpha = \frac{1}{2}$,

$$\begin{aligned} \|T(x,0) - T(0,y)\| &= \|(0,\sqrt{x}) - (\sqrt{y},0)\| \\ &= \|(\sqrt{y},\sqrt{x})\| \\ &= \sqrt{x+y} \\ &\leqslant \sqrt{\sqrt{2}\sqrt{x^2+y^2}} \\ &= \sqrt{dist(A,B)\|(x,0)-(0,y)\|}. \end{aligned}$$

Hence T is a geometric contraction map with respect to $\alpha = \frac{1}{2}$.

Proposition 2.2. *Let A and B be nonempty subsets of a metric space X, $T : A \cup B \to A \cup B$ be a geometric contraction map and $x_n = T^n x_0$ for $x_0 \in A \cup B$. Then $d(x_n, Tx_n) \to dist(A,B)$.*

Proof. By definition of geometric contraction map we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant d(x_{n-1}, x_n)^{\alpha} dist(A,B)^{1-\alpha} \\ &\leqslant d(x_{n-2}, x_{n-1})^{\alpha^2} dist(A,B)^{1-\alpha^2} \\ &\ \ \vdots \\ &\leqslant d(x_0, x_1)^{\alpha^n} dist(A,B)^{1-\alpha^n}. \end{aligned}$$

Thus $d(x_n, x_{n+1}) \to dist(A,B)$. $\square$

Theorem 2.3. *Let A and B be nonempty subsets of a metric space X and $T : A \cup B \to A \cup B$ be a geometric contraction map. Let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence to $x \in A$. Then $d(x, Tx) = dist(A,B)$.*

Proof. Suppose $\{x_{2n_k}\}$ is a subsequence of $\{x_{2n}\}$ converges to some $x \in A$. Now

$$dist(A,B) \leqslant d(x, x_{2n_k-1}) \leqslant d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

Thus, by Proposition 2.2, we have $d(x, x_{2n_k-1})$ converges to $d(A,B)$. Since

$$dist(A,B) \leqslant d(x_{2n_k}, Tx) \leqslant d(x_{2n_k-1}, x)^{\alpha} dist(A,B)^{(1-\alpha)} \leqslant d(x_{2n_k-1}, x),$$

so $d(x, Tx) = dist(A,B)$. $\square$

Proposition 2.4. *Let A and B be nonempty subsets of a metric space X, $T : A \cup B \to A \cup B$ a geometric contraction map, $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose $x_0 \in A$ then since by Proposition 2.2, $d(x_{2n}, x_{2n+1})$ converges to $dist(A,B)$, it is enough to prove that $\{x_{2n+1}\}$ is bounded. Suppose $\{x_{2n+1}\}$ is not bounded, then for $M > dist(A,B)$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_2, x_{2n_0+1}) > M, \ d(x_0, x_{2n_0-1}) < M.$$

Hence by the geometric contraction property of T,

$$M < d(x_2, x_{2n_0+1}) \leqslant d(x_0, x_{2n_0-1})^{\alpha^2} dist(A,B)^{1-\alpha^2},$$

and so

$$M^{\frac{1}{\alpha^2}} dist(A,B)^{1-\frac{1}{\alpha^2}} < d(x_0, x_{2n_0-1}) < M.$$

Therefore $M < dist(A,B)$ which is a contradiction. The proof is similar when $x_0 \in B$. $\square$

The Proposition 2.4 leads us to an existence result when one of the sets A or B is boundedly compact. We remember that a subset A of a metric space is boundedly compact if every bounded sequence in A has a convergent subsequence.

Corollary 2.5. *Let A and B be nonempty closed subsets of a metric space X and $T : A \cup B \to A \cup B$ a geometric contraction map. If either A or B is boundedly compact, then there exists $x \in A \cup B$ with $d(x,Tx) = dist(A,B)$.*

Proof. It follows directly from Theorem 2.3 and Proposition 2.4. $\square$
In continue, we will give some new results of the existence of the best proximity points.

Theorem 2.6. *Let A and B be nonempty closed subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B, T(B) \subset A$ and*

$$d(Tx,Ty) \leqslant d(x,y)^{\alpha}[d(x,Tx)d(y,Ty)]^{\beta} dist(A,B)^{\gamma},$$

for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + \gamma = 1$. If A (or B) is boundedly compact, then there exists $x \in A \cup B$ with $d(x,Tx) = dist(A,B)$.

Proof. Suppose x_0 is an arbitrary point of $A \cup B$ and define $x_{n+1} = Tx_n$. Now

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leqslant d(x_n, x_{n+1})^{\alpha}[d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})]^{\beta} dist(A,B)^{\gamma}. \end{aligned}$$

So

$$d(x_{n+1}, x_{n+2}) \leqslant d(x_n, x_{n+1})^{\frac{\alpha+\beta}{1-\beta}} dist(A, B)^{\frac{\gamma}{1-\beta}},$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leqslant d(x_n, x_{n+1})^k dist(A, B)^{1-k},$$

where $k = \frac{\alpha+\beta}{1-\beta} < 1$. Hence inductively we have

$$d(x_{n+1}, x_n) \leqslant d(x_1, x_0)^{k^n} dist(A, B)^{1-k^n},$$

and so

$$d(x_{n+1}, x_n) \to dist(A, B).$$

Therefore, by repeat the technique of the proof of Proposition 2.4, both sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. Now since A (or B) is boundedly compact then $\{x_{2n}\}$ has a convergent subsequence and so by repeat the technique of the proof of Proposition 2.3, there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$. □

Theorem 2.7. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B, T(B) \subset A$ and*

$$d(Tx, T^2x) \leqslant d(x, Tx)^k dist(A, B)^{1-k},$$

for all $x \in A \cup B$, where $0 \leqslant k < 1$. If there are $u \in A \cup B$ and $n \in \mathbb{N}$ such that $T^n u = u$, then $d(u, Tu) = dist(A, B)$.

Proof. Let $u \in A \cup B$ and $n \in \mathbb{N}$ be such that $T^n u = u$. If $dist(A, B) < d(u, Tu)$, we have

$$\begin{aligned}
d(u, Tu) &= d(T(T^{n-1}u), T^2(T^{n-1}u)) \\
&\leqslant d(T^{n-1}u, T(T^{n-1}u))^k \ dist(A, B)^{1-k} \\
&\leqslant d(T^{n-2}u, T^{n-1}u)^{k^2} dist(A, B)^{1-k^2} \\
&\vdots \\
&\leqslant d(u, Tu)^{k^n} dist(A, B)^{1-k^n} \\
&< d(u, Tu),
\end{aligned}$$

which is a contradiction, so $d(u,Tu) = dist(A,B)$. □

In the following, we give some new conditions on the mapping T to find uniqueness of best proximity points. Remember that if X is a uniformly convex Banach space with modulus of convexity δ. Then $\delta(\epsilon) > 0$ for $\epsilon > 0$, and $\delta(.)$ is strictly increasing. Moreover, if $x, y, z \in X$, $d > 0$, and $r \in [0, 2d]$ such that $\|x - z\| \leqslant d$, $\|y - z\| \leqslant d$ and $\|x - y\| \geqslant r$, then

$$\|\frac{x+y}{2} - z\| \leqslant (1 - \delta(\frac{r}{d}))d. \qquad (*)$$

Theorem 2.8. *Let A and B be two nonempty closed and convex subsets of a Banach space X and $T : A \cup B \to A \cup B$ a geometric contraction map. Then there exists unique $x \in A$ with $\|x - Tx\| = dist(A,B)$.*

Proof. Suppose $x_0 \in A$ and $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$. By Proposition 2.4, the sequence $\{x_{2n}\}$ is bounded. Hence since X is uniformly convex, $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ that weakly converges to x and so $\|x - Tx\| \leqslant \liminf_{k\to\infty} \|x_{2n_k} - Tx_{2n_k}\|$. Therefore by Proposition 2.2,

$$\|x - Tx\| = dist(A,B).$$

If there exist $x, y \in A$ and $x \neq y$ such that $\|x - Tx\| = \|y - Ty\| = dist(A,B)$ where necessarily, $T^2x = x$ and $T^2y = y$. Therefore

$$\|Tx - y\| = \|Tx - T^2y\| \leqslant \|x - Ty\| \text{ and } \|Ty - x\| = \|Ty - T^2x\| \leqslant \|y - Tx\|,$$

then

$$\|Ty - x\| = \|y - Tx\|. \qquad (**)$$

On the other hands, $\|y - Tx\| > dist(A,B)$. If $\|y - Tx\| = dist(A,B)$, since X is uniformly convex by $(*)$ we have

$$\|\frac{x+y}{2} - Tx\| \leqslant (1 - \delta(\frac{r}{d}))dist(A,B) < dist(A,B),$$

since A is convex, $\frac{x+y}{2} \in A$ and it is a contradiction. Hence $\|y - Tx\| > dist(A,B)$, and so $\|y - Tx\| = \|T^2y - Tx\| < \|Ty - x\|$ which is a contradiction by $(**)$. Therefore $x = y$. □

Theorem 2.9. *Let A and B be two nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subset B,\ T(B) \subset A$ and*

$$d(Tx, Ty) \leqslant d(x, y)^{\alpha}[d(x, Tx)d(y, Ty)]^{\beta} dist(A, B)^{\gamma},$$

for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + \gamma = 1$. Then there exists a unique element $x \in A$ such that $\|x - Tx\| = dist(A, B)$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the above unique element.

Proof. One can prove this theorem by the method of the Proposition 2.8. □

References

[1] K. Fan, Extensions of two fixed point theorems of F. E. Browder, *Math. Zeit.*, 112 (1969), 234-240.

[2] M. R. Haddadi and S. M. Moshtaghioun, *Some results on the best proximity pair*, Abstract and Applied Analysis, (2011), ID 158430, 9 pages.

[3] S. Sadiq Basha, Best proximity points: global optimal approximate solutions, *J. Global Optimization*, 49 (1) (2011), 15-21.

[4] M. A. Shahzad Naseer, Best proximity pairs and equilibrium pairs for Kakutani multimaps, *Nonlinear Anal.*, 70 (3) (2009), 1209-1216.

[5] I. Singer, *Best Approximations Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, 1970.

Mohammad Reza Haddadi
Department of Mathematics
Assistant Professor of Mathematics
Ayatollah Boroujerdi University
Boroujerd, Iran
E-mail: haddadi@abru.ac.ir