Journal of Mathematical Extension
Vol. 14, No. 3, (2020), 117-132
ISSN: 1735-8299
URL: http://www.ijmex.com
Original Research Paper

On Generalization of Midpoint and Trapezoid Type Inequalities Involving Fractional Integrals

Mehmet Zeki Sarikaya*
Düzce University

Sümeyye Sönmezoğlu
Düzce University

Abstract. In this paper, we first give a lemma for twice differentiable function to obtain trapezoid and midpoint inequalities. By using this lemma, we establish some inequalities for mapping whose second derivatives in absolute value are convex via Riemann-Liouville fractional integrals. These results generalize the midpoint and trapezoid inequalities involving Riemann-Liouville fractional integrals given in earlier studies.

AMS Subject Classification: 26D15; 26B25; 26D10
Keywords and Phrases: Hermite-Hadamard inequality, midpoint inequality, fractional integral operators, convex function

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[6], [9], [20, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Received: June 2019; Accepted: May 2020
*Corresponding author

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[4], [7], [12], [14], [18], [19], [21], [23], [27], [28], [31], [32]) and the references cited therein.

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult ([8], [13], [15]).

Definition 1.1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al.[25] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leqslant a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leqslant \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$.

Sarıkaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Lioville fractional integrals in [22].

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \leqslant \frac{f(a)+f(b)}{2}. \tag{3}$$

For the more information fractional calculus and related inequalities please refer to ([5], [10], [11], [16], [17], [24], [26], [29], [30], [33]).

2. Generalized Midpoint and Trapezoid Type Inequalities

In this section, we will first present a lemma for twice differentiable functions to obtain trapezoid and midpoint inequalities. By using this lemma, we establish some inequalities which generalize the midpoint and trapezoid inequalities involving Riemann-Liouville fractional integrals obtained in previous works.

Lemma 2.1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$. If $f : I \to \mathbb{R}$ is a twice differentiable mapping such that f'' is integrable and $0 \leqslant \lambda \leqslant 1$, $\alpha \geqslant 1$, then we have*

$$\begin{aligned} & \left[\left(\lambda - \frac{\alpha+1}{2^{\alpha}}\right) f\left(\frac{a+b}{2}\right) - \lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \\ & \left. + \frac{\Gamma(\alpha+2)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right)\right] \\ = & \frac{(b-a)^2}{2}\int_0^1 k(t) f''(ta+(1-t)b)\,dt \end{aligned} \tag{4}$$

where

$$k(t) = \begin{cases} t(t^{\alpha}-\lambda) & 0 \leqslant t \leqslant \frac{1}{2} \\ (1-t)\left((1-t)^{\alpha}-\lambda\right) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 k(t) f''(ta+(1-t)b)\,dt \\ &= \int_0^{\frac{1}{2}} t(t^\alpha-\lambda) f''(ta+(1-t)b)\,dt \\ &\quad +\int_{\frac{1}{2}}^1 (1-t)\left((1-t)^\alpha-\lambda\right) f''(ta+(1-t)b)\,dt \\ &= I_1+I_2. \end{aligned} \tag{5}$$

Integrating by parts twice, we can state:

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} t(t^\alpha-\lambda) f''(ta+(1-t)b)\,dt \\ &= t(t^\alpha-\lambda)\frac{f'(ta+(1-t)b)}{a-b}\bigg|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{f'(ta+(1-t)b)}{a-b}\left((\alpha+1)t^\alpha-\lambda\right)dt \\ &= \frac{-1}{2(b-a)}\left(\frac{1}{2^\alpha}-\lambda\right) f'\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^2}\left(\frac{\alpha+1}{2^\alpha}-\lambda\right) f\left(\frac{a+b}{2}\right) \\ &\quad -\frac{\lambda}{(b-a)^2} f(b) + \frac{\alpha(\alpha+1)}{(b-a)^2}\int_0^{\frac{1}{2}} f(ta+(1-t)b)\,t^{\alpha-1}dt \\ &\quad +\frac{\alpha(\alpha+1)}{(b-a)^2}\frac{1}{(b-a)^\alpha}\Gamma(\alpha)\, J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \end{aligned} \tag{6}$$

and similarly, we get

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 (1-t)\left((1-t)^\alpha-\lambda\right) f''(ta+(1-t)b)\,dt \\ &= (1-t)\left((1-t)^\alpha-\lambda\right)\frac{f'(ta+(1-t)b)}{a-b}\bigg|_{\frac{1}{2}}^1 \\ &\quad +\int_{\frac{1}{2}}^1 \frac{f'(ta+(1-t)b)}{a-b}\left((\alpha+1)(1-t)^\alpha-\lambda\right) \\ &= \frac{1}{2(b-a)}\left(\frac{1}{2^\alpha}-\lambda\right) f'\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)}{(b-a)^2} \\ &\quad -\frac{1}{(b-a)^2}\left(\frac{\alpha+1}{2^\alpha}-\lambda\right) f\left(\frac{a+b}{2}\right) + \frac{\alpha(\alpha+1)}{(b-a)^{\alpha+2}}\Gamma(\alpha)\, J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a). \end{aligned} \tag{7}$$

Using (6) and (7) in (5), it follows that

$$\begin{aligned} I &= I_1 + I_2 = \frac{-2}{(b-a)^2}\left(\frac{\alpha+1}{2^\alpha} - \lambda\right) f\left(\frac{a+b}{2}\right) - \frac{2\lambda}{(b-a)^2}\left(\frac{f(a)+f(b)}{2}\right) \\ &\quad + \frac{\alpha(\alpha+1)}{(b-a)^{\alpha+2}}\Gamma(\alpha)\left(J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right) \\ &= \frac{2}{(b-a)^2}\left[\left(\lambda - \frac{\alpha+1}{2^\alpha}\right) f\left(\frac{a+b}{2}\right) - \lambda\left(\frac{f(a)+f(b)}{2}\right)\right] \\ &\quad + \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right). \end{aligned}$$

Then by multipling the above equality with $\frac{(b-a)^2}{2}$, this completes the proof. □

Theorem 2.2. *Let $I \subset \mathbb{R}$ be an open intervial, $a, b \in I$ with $a < b$ and $f : I \to \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable and $0 \leqslant \lambda \leqslant 1$, $\alpha \geqslant 1$. If $|f''|$ is a convex on $[a, b]$, then the following inequalities hold:*

$$\begin{aligned} &\left|\left(\lambda - \frac{\alpha+1}{2^\alpha}\right) f\left(\frac{a+b}{2}\right) - \lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \\ &\left. + \frac{\Gamma(\alpha+2)}{2(b-a)^\alpha}\left(J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right)\right| \end{aligned} \tag{8}$$

$$\leqslant \frac{(b-a)^2}{2}\begin{cases} \left(\dfrac{1}{2^{\alpha+2}(\alpha+2)} - \dfrac{\lambda}{8} + \dfrac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}\right)\left[|f''(a)| + |f''(b)|\right], & 0 \leqslant \lambda \leqslant \frac{1}{2} \\ \left(\dfrac{\lambda}{8} - \dfrac{1}{2^{\alpha+2}(\alpha+2)}\right)\left[|f''(a)| + |f''(b)|\right], & \frac{1}{2} \leqslant \lambda \leqslant 1. \end{cases}$$

$$\begin{aligned}
&\left|\left(\lambda-\frac{\alpha+1}{2^{\alpha}}\right)f\left(\frac{a+b}{2}\right)-\lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \qquad (9)\\
&\left.+\frac{\Gamma(\alpha+2)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right)\right|\\
\leqslant\ & \frac{(b-a)^2}{2}\int_0^1|k(t)|\,|f''(ta+(1-t)b)|\,dt\\
=\ & \frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}}|t(t^{\alpha}-\lambda)|\,|f''(ta+(1-t)b)|\,dt\right.\\
&\left.+\int_{\frac{1}{2}}^{1}|(1-t)((1-t)^{\alpha}-\lambda)|\,|f''(ta+(1-t)b)|\,dt\right\}\\
=\ & \frac{(b-a)^2}{2}\{J_1+J_2\}.
\end{aligned}$$

We assume that $0\leqslant\lambda\leqslant\frac{1}{2}$, then using the convexity of $|f''|$,we get

$$\begin{aligned}
J_1 &\leqslant \int_0^{\frac{1}{2}}|t(t^{\alpha}-\lambda)|\left[t|f''(a)|+(1-t)|f''(b)|\right]dt \qquad (10)\\
&= \int_0^{\lambda^{\frac{1}{\alpha}}}t(\lambda-t^{\alpha})\left[t|f''(a)|+(1-t)|f''(b)|\right]dt\\
&\quad+\int_{\lambda^{\frac{1}{\alpha}}}^{\frac{1}{2}}t(t^{\alpha}-\lambda)\left[t|f''(a)|+(1-t)|f''(b)|\right]dt\\
&= |f''(a)|\left[\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}-\frac{\lambda}{24}\right]\\
&\quad+|f''(b)|\left[\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}-\frac{\lambda}{12}\right]
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
J_2 &\leqslant \int_{\frac{1}{2}}^{1-\lambda^{\frac{1}{\alpha}}}(1-t)((1-t)^{\alpha}-\lambda)\left[t|f''(a)|+(1-t)|f''(b)|\right]dt \qquad (11)\\
&\quad+\int_{1-\lambda^{\frac{1}{\alpha}}}^{1}(1-t)(\lambda-(1-t)^{\alpha})\left[t|f''(a)|+(1-t)|f''(b)|\right]dt
\end{aligned}$$

$$= |f''(a)|\left[\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}-\frac{\lambda}{12}\right]$$

$$+|f''(b)|\left[\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}-\frac{\lambda}{24}\right].$$

Using (10) and (11) in (9), we see that the first inequality of (8) holds. On the other hand, let $\frac{1}{2}\leqslant\lambda\leqslant 1$, then, using the convexity of $|f''|$ and by simple computation we have

$$\begin{aligned} J_1' &\leqslant \int_0^{\frac{1}{2}} |t(t^\alpha-\lambda)|\left[t\left|f''(a)\right|+(1-t)\left|f''(b)\right|\right]dt \qquad (12)\\ &= \int_0^{\frac{1}{2}} t(\lambda-t^\alpha)\left[t\left|f''(a)\right|+(1-t)\left|f''(b)\right|\right]dt\\ &= \left(\frac{\lambda}{24}-\frac{1}{2^{\alpha+3}(\alpha+3)}\right)\left|f''(a)\right|+\left(\frac{\lambda}{12}-\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\right)\left|f''(b)\right| \end{aligned}$$

and similarly

$$\begin{aligned} J_2' &\leqslant \int_{\frac{1}{2}}^{1} |(1-t)((1-t)^\alpha-\lambda)|\,|f''(ta+(1-t)b)|\,dt \qquad (13)\\ &= \int_{\frac{1}{2}}^{1} (1-t)(\lambda-(1-t)^\alpha)\,[t\,|f''(a)|+(1-t)\,|f''(b)|]\,dt\\ &= \left(\frac{\lambda}{12}-\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\right)|f''(a)|+\left(\frac{\lambda}{24}-\frac{1}{2^{\alpha+3}(\alpha+3)}\right)|f''(b)|. \end{aligned}$$

Thus if we write (12) and (13) in (9), we obtain the second inequality of (8). This completes the proof. $\square$

Corollary 2.3. *Under the assumptions of Theorem 2.2 with* $\lambda = 0$, *then we get the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right)-f\left(\frac{a+b}{2}\right)\right|$$

$$\leqslant \frac{(b-a)^2}{(\alpha+1)(\alpha+2)}\left[\frac{|f''(a)|+|f''(b)|}{8}\right]$$

which is proved by Noor and Awan in [16, Theorem 2 (for s=1)].

Remark 2.4. *If we take $\alpha = 1$ in Corollary 2.3, then we get the following inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{(b-a)^2}{24} \left[\frac{|f''(a)| + |f''(b)|}{2} \right]$$

which is given by Sarikaya et al. in [27].

Corollary 2.5. *Under the assumptions of Theorem 2.2 with $\lambda = \frac{\alpha+1}{2^\alpha}$, then we get the following inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left(J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right) \right|$$
$$\leqslant \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)} \left(\alpha(\alpha+1)^{1+\frac{2}{\alpha}} + 1 - \frac{(\alpha+1)(\alpha+2)}{2} \right) \left[|f''(a)| + |f''(b)| \right]$$

for $\alpha \geqslant 3$ and

$$\left| \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left(J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right) \right|$$
$$\leqslant \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)} \left(\frac{(\alpha+1)(\alpha+2)}{2} - 1 \right) \left[|f''(a)| + |f''(b)| \right]$$

for $1 \leqslant \alpha \leqslant 3$.

Remark 2.6. *If we take $\alpha = 1$ in Corollary 2.5, then we get the following inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leqslant \frac{(b-a)^2}{12} \left[\frac{|f''(a)| + |f''(b)|}{2} \right]$$

which is given by Sarikaya and Aktan in [23].

Remark 2.7. *Under the assumptions of Theorem 2.2 with* $\lambda = \frac{1}{3}$ *and* $\alpha = 1$, *then we get the following inequality*

$$\begin{aligned} & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leqslant \; & \frac{(b-a)^2}{81}\left[\frac{|f''(a)| + |f''(b)|}{2}\right] \end{aligned}$$

which is given by Sarikaya and Aktan in [23].

Remark 2.8. *Under the assumptions of Theorem 2.2 with* $\lambda = \frac{1}{2}$ *and* $\alpha = 1$, *then we get the following inequality*

$$\begin{aligned} & \left| \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] \right| \\ \leqslant \; & \frac{(b-a)^2}{48}\left[\frac{|f''(a)| + |f''(b)|}{2}\right] \end{aligned}$$

which is given by Sarikaya and Aktan in [23].

Theorem 2.9. *Let* $I \subset \mathbb{R}$ *be an open intervial,* $a, b \in I$ *with* $a < b$ *and* $f : I \to \mathbb{R}$ *be a twice differentiable mapping such that* f'' *is integrable and* $0 \leqslant \lambda \leqslant 1$, $\alpha \geqslant 1$. *If* $|f''|^q$ *is a convex on* $[a, b]$, $q \geqslant 1$ *then the following inequalities hold:*

$$\begin{aligned} & \left| \left(\lambda - \frac{\alpha+1}{2^\alpha}\right) f\left(\frac{a+b}{2}\right) - \lambda\left(\frac{f(a)+f(b)}{2}\right) \right. \\ & \left. + \frac{\Gamma(\alpha+2)}{2(b-a)^\alpha}\left(J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right) \right| \\ \leqslant \; & \frac{(b-a)^2}{2}\left(\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2} + \frac{1}{2^{\alpha+2}(\alpha+2)} - \frac{\lambda}{8} \right)^{1-\frac{1}{q}} \\ & \times \left\{ \left[C_1 |f''(a)|^q + C_2 |f''(b)|^q \right]^{\frac{1}{q}} + \left[C_2 |f''(a)|^q + C_1 |f''(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned} \tag{14}$$

for $0 \leqslant \lambda \leqslant \frac{1}{2}$ *and*

$$\begin{aligned}
&\left|\left(\lambda-\frac{\alpha+1}{2^{\alpha}}\right)f\left(\frac{a+b}{2}\right)-\lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \qquad (15)\\
&\left.+\frac{\Gamma(\alpha+2)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right)\right|\\
\leqslant\ &\frac{(b-a)^2}{2}\left(\frac{\lambda}{8}-\frac{1}{2^{\alpha+2}(\alpha+2)}\right)^{1-\frac{1}{q}}\\
&\times\left\{\left[C_3\left|f''(a)\right|^q+C_4\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\left[C_4\left|f''(a)\right|^q+C_3\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

for $\frac{1}{2}\leqslant\lambda\leqslant 1$ *where* $\frac{1}{p}+\frac{1}{q}=1,$

$$\begin{aligned}
C_1 &= \left(\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}-\frac{\lambda}{24}\right)\\
C_2 &= \left(\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}-\frac{\lambda}{12}\right)\\
C_3 &= \left(\frac{\lambda}{24}-\frac{1}{2^{\alpha+3}(\alpha+3)}\right)\\
C_4 &= \left(\frac{\lambda}{12}-\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\right).
\end{aligned}$$

Proof. Suppose that $q\geqslant 1$. From Lemma 2.1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left|\left(\lambda-\frac{\alpha+1}{2^{\alpha}}\right)f\left(\frac{a+b}{2}\right)-\lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \qquad (16)\\
&\left.+\frac{\Gamma(\alpha+2)}{2(b-a)^{\alpha}}\left(J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right)\right|\\
\leqslant\ &\frac{(b-a)^2}{2}\int_0^1|k(t)|\left|f''(ta+(1-t)b)\right|dt\\
\leqslant\ &\frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}}\left|t\left(t^{\alpha}-\lambda\right)\right|\left|f''(ta+(1-t)b)\right|dt\right.\\
&\left.+\int_{\frac{1}{2}}^{1}\left|(1-t)\left((1-t)^{\alpha}-\lambda\right)\right|\left|f''(ta+(1-t)b)\right|dt\right\}
\end{aligned}$$

$$= \frac{(b-a)^2}{2}\left\{\left(\int_0^{\frac{1}{2}}|t(t^\alpha-\lambda)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t(t^\alpha-\lambda)|\,|f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}|(1-t)((1-t)^\alpha-\lambda)|\,dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{1}{2}}^{1}|(1-t)((1-t)^\alpha-\lambda)|\,|f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right\}.$$

Let $0 \leqslant \lambda \leqslant \frac{1}{2}$. Then since $|f'|^q$ is convex on $[a,b]$,we know that for $t \in [0,1]$

$$\left|f'(ta+(1-t)b)\right|^q \leqslant t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q$$

hence, by simple computation

$$\int_0^{\frac{1}{2}}|t(t^\alpha-\lambda)|\,|f''(ta+(1-t)b)|^q\,dt \tag{17}$$

$$\leqslant \int_0^{\lambda^{\frac{1}{\alpha}}} t(\lambda-t^\alpha)\left[t|f''(a)|^q+(1-t)|f''(b)|^q\right]dt$$

$$+\int_{\lambda^{\frac{1}{\alpha}}}^{\frac{1}{2}} t(t^\alpha-\lambda)\left[t|f''(a)|^q+(1-t)|f''(b)|^q\right]dt$$

$$= |f''(a)|^q\left[\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}-\frac{\lambda}{24}\right]$$

$$+|f''(b)|^q\left[\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}-\frac{\lambda}{12}\right],$$

$$\int_{\frac{1}{2}}^{1}|(1-t)((1-t)^\alpha-\lambda)|\left|f''(ta+(1-t)b)\right|^q dt \tag{18}$$

$$\leqslant \int_{\frac{1}{2}}^{1-\lambda^{\frac{1}{\alpha}}}(1-t)((1-t)^\alpha-\lambda)\left[t\left|f''(a)\right|^q+(1-t)\left|f''(b)\right|^q\right]dt$$

$$+\int_{1-\lambda^{\frac{1}{\alpha}}}^{1}(1-t)\left(\lambda-(1-t)^{\alpha}\right)\left[t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right]dt$$

$$=\left|f''(a)\right|^{q}\left[\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}-\frac{\lambda}{12}\right]$$

$$+\left|f''(b)\right|^{q}\left[\frac{2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}-\frac{\lambda}{24}\right],$$

$$\begin{aligned}&\int_{0}^{\frac{1}{2}}\left|t\left(t^{\alpha}-\lambda\right)\right|dt \qquad (19)\\ =&\int_{0}^{\lambda^{\frac{1}{\alpha}}}t\left(\lambda-t^{\alpha}\right)dt+\int_{\lambda^{\frac{1}{\alpha}}}^{\frac{1}{2}}t\left(t^{\alpha}-\lambda\right)dt\\ =&\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}+\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{\lambda}{8}\end{aligned}$$

and

$$\int_{\frac{1}{2}}^{1}\left|(1-t)\left((1-t)^{\alpha}-\lambda\right)\right|dt \qquad (20)$$

$$=\int_{\frac{1}{2}}^{1-\lambda^{\frac{1}{\alpha}}}(1-t)\left((1-t)^{\alpha}-\lambda\right)dt+\int_{1-\lambda^{\frac{1}{\alpha}}}^{1}(1-t)\left(\lambda-(1-t)^{\alpha}\right)dt$$

$$=\frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}+\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{\lambda}{8}.$$

Substituting the equalities (17)-(20) in (16), the we obtain the inequality (14). One can prove the inequality (15) similar to (14). It is omitted. □

Remark 2.10. *Under the assumptions Theorem 2.9 with $\alpha=1$, then Theorem 2.9 reduces to Theorem 4 in [23].*

Remark 2.11. *Under the assumptions of Theorem 2.9 with $\lambda=\frac{1}{3}$ and $\alpha=1$, then we get the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leqslant \frac{(b-a)^2}{162}\left[\left(\frac{59\left|f''(a)\right|^q+133\left|f''(b)\right|^q}{2^6\times 3}\right)^{\frac{1}{q}}+\left(\frac{133\left|f''(a)\right|^q+59\left|f''(b)\right|^q}{2^6\times 3}\right)^{\frac{1}{q}}\right]$$

which is given by Sarikaya and Aktan in [23].

Corollary 2.12. *Under the assumptions Theorem 2.9 with $\lambda = 0$, then we get the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right)-f\left(\frac{a+b}{2}\right)\right| \tag{21}$$

$$\leqslant \frac{(b-a)^2\,2^{\alpha-1}}{\alpha+1}\left\{\left(\frac{1}{2^{\alpha+2}(\alpha+2)}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left|f''(a)\right|^q\frac{1}{2^{\alpha+3}(\alpha+3)}+\left|f''(b)\right|^q\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\right]^{\frac{1}{q}}$$

$$\left.+\left[\left|f''(a)\right|^q\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\left|f''(b)\right|^q\frac{1}{2^{\alpha+3}(\alpha+3)}\right]^{\frac{1}{q}}\right\}.$$

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Mehmet Zeki Sarikaya
Professor of Mathematics
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce, Turkey
E-mail: sarikayamz@gmail.com

Sümeyye Sönmezoğlu
Graduate Student of Mathematics
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce, Turkey
E-mail: smyy.mat@gmail.com