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The Intersection Graph of a Finite Moufang Loop

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Abstract. The intersection graph $\Gamma_{SI}(G)$ of a group G with identity element e is the graph whose vertex set is the set $V(\Gamma_{SI}(G)) = G - e$ and two distinct vertices x and y are adjacent in $\Gamma_{SI}(G)$ if and only if $|\langle x\rangle \cap \langle y\rangle| > 1$, where $\langle x\rangle$ is the cyclic subgroup of G generated by x. In this paper, at first we obtain some results for this graph for any Moufang loop. More specially we observe non-isomorphic finite Moufang loops may have isomorphic intersection graphs.

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1. Introduction

A quasi-group is a non-empty set Q with a binary operation "$\cdot$" where, for any two elements $a, b \in Q$, there exist unique elements $x, y \in Q$ such that both equations $a.x = y$ and $y.a = b$ are hold. The quasi-group with an identity element is called a loop; that is, an element e such that $x.e = e.x = x$ for all $x \in Q$. A loop is a Moufang loop if any of the four following identities holds for every $x, y, z \in Q$:

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$$\begin{aligned} ((xy)x)z &= x(y(xz)), \quad (M1)\\ x(y(zy)) &= ((xy)z)y, \quad (M2)\\ (xy)(zx) &= x((yz)x), \quad (M3)\\ (xy)(zx) &= (x(yz))x. \quad (M4) \end{aligned}$$

In general, Moufang loops are non-associative, but they preserve many known properties of the groups. For parable, for every x, there exist two-sided inverse x^{-1} such that $xx^{-1} = x^{-1}x = 1$; also, any two elements of a Moufang loop generate a subgroup. The order of every elements in loops divides the order of the loop [1, 3]. The Sylow theorem and Hall theorem are hold in the finite Moufang loops.

The classification of the non-associative Moufang loops started by Chein in [4, 5]. Naghy and Vojtechovsky in [9] classified the non-associative non-isomorphic Moufang loops of order 64 and 81. In continue, Slattery and Zenisek in [10] completed the classification of Moufang loops of order 243. The interesting result is following table where $M(n)$ is the number of paitrwise non-isomorphic Moufang loops of order n:

n	12	16	20	24	28	32	36	40	42	44	48	52	54	56	60	64	81	243
M(n)	1	5	1	5	1	71	4	5	1	1	51	1	2	4	5	4262	5	72

For a finite group of order n and a new element $u, (u \notin G)$, Chein [4] defined the construction $M(G, 2) = G \cup Gu$ by the multiplication as follows:

$$\begin{cases} goh = gh, & \text{if } g, h \in G,\\ go(hu) = (hg)u, & \text{if } g \in G, \quad hu \in Gu,\\ (gu)oh = (gh^{-1})u, & \text{if } gu \in Gu, \quad h \in G,\\ (gu)o(hu) = h^{-1}g, & \text{if } gu, hu \in Gu. \end{cases}$$

and obtained that $M(G, 2)$ is a Moufang loop of order $2n$. It is obvious that $M(G, 2)$ is non-associative if and only if G is non-abelian. There is an another structure of loops that called Bol loop. A left Bol loop is a loop L which, for all x, y, and z in L, satisfies the left Bol relation

$$x(y(xz)) = (x(yx))z.$$

Similarly, loop L is a right Bol loop provided it satisfies the right Bol relation

$$((zx)y)x = z((xy)x).$$

Also, a loop which is both a left and right Bol loop is called a Moufang loop [3].

Theorem 1.1. *Theorem 6.2(Cauchys theorem)[11]. Let L be a Bol loop of odd order. For every prime p dividing L, there exists $x \in L$ of order p.*

In general, there is an intimate relation between the groups and graphs. Before starting we would like to introduce some necessary notation and definitions about the intersection graph. For any graph Γ, we denote the sets of the vertices by V and edges by E, denote it by $\Gamma = (V, E)$. For any vertex g in a graph Γ, $deg(g)$ is the number of edges incident to g. The neighbour set of a vertex g, is the set of the adjacent vertices with g and denoted by $N(g)$. A set $S \subseteq V$ in graph Γ is said to be dominating if $N(S) = V - S$, $(N(S) = \cup_{s \in S} N(s))$. A minimal dominating set is dominating set which no proper subset. The size of smallest minimal dominatig set is called dominating number and denoted by $\gamma(\Gamma)$.

Two graphs Γ_1 and Γ_2 are isomorphic (written $(\Gamma_1 \cong \Gamma_2)$ if there exists a bijective map $\psi : V(\Gamma_1) \longrightarrow V(\Gamma_2)$ such that any two elements x and y are adjacent in Γ_1 if and only if the elements $\psi(x)$ and $\psi(y)$ are adjacent in Γ_1. A path P is a sequence $v_0e_1v_1e_2 \dots e_kv_k$ whose terms are alternately distinct vertices and distinct edges and for any i, $\quad 1 \leqslant i \leqslant k$, the ends of e_i are v_{i-1} and v_i. The number k is called the *length* of path. If in the path P, the terms v_0 and v_k are adjacent by an edge e_{k+1}, then the path $v_0e_1v_1e_2 \dots e_kv_ke_{k+1}$ is called a cycle and the length is the number of its edges. A graph having no cycles is said to be a forest. A graph Γ is called connected if there is a path between each pair of the vertices of Γ. The number of connected components in graph Γ is denoted by $\omega(\Gamma)$. The vertex g in Γ is called cut-vertex if $\omega(\Gamma - g) > \omega(\Gamma)$[2].

The intersection graph $\Gamma_{SI}(G)$ of a group G with identity element e is the graph whose vertex set is the set $V(\Gamma_{SI}(G)) = G - e$ and two distinct vertices x and y are adjacent in $\Gamma_{SI}(G)$ if and only if $|\langle x \rangle \cap \langle y \rangle| > 1$, where $\langle x \rangle$ is the cyclic subgroup of G generated by x [7].

Our main results concerning finite Moufang loops and we will obtain some results about the intresection graph of these Moufang loops. More specially we observe non-isomorphic finite Moufang loops may have isomorphic intersection graphs.

In this paper we examine the intersecion graph of the finite Moufang loops. We prove that if ε is the number of edges in $\Gamma_{SI}(M)$, then

$$\varepsilon \geqslant \frac{1}{2} \sum_{x \in M-e} o(x) - 2.$$

And we show that if M is a Moufang loop of odd order, then $\Gamma_{SI}(M)$ is a complete graph if and only if M consist of unique subloop of order p and $o(M) = p^m$, where, p is a prime number and m is a positive integer. In fact, we prove that the following main Theorem.

Main Theorem. Let G be a finite group and t be the number of the connected components in the $\Gamma_{SI}(G)$. Then $\gamma(\Gamma_{SI}(G)) = t$.

2. Results

At first, we need to define the intersection graph of a Moufang loop M as follows:

Definition 2.1. *The intersection graph $\Gamma_{SI}(M)$ of a Moufang loop M with identity element e is the graph whose vertex set is the set $V(\Gamma_{SI}(M))$ $= M - e$ and two distinct vertices x and y are adjacent in $\Gamma_{SI}(M)$ if and only if $|\langle x\rangle \cap \langle y\rangle| > 1$, where $\langle x\rangle$ is the cyclic subloop of M generated by x.*

Proposition 2.2. *Let M be a finite Moufang loop and $x \in M - e$. Then $deg(x) \geqslant o(x) - 2$.*

Proof. Suppose that $x \in M - e$, then by definition of the subloop intersection graph, x adjacent with all elements x^i where $i = 2, \ldots, o(x) - 1$ and which yields $deg(x) \geqslant o(x) - 2$. □

Proposition 2.3. *For every finite Moufang loop M, isolated vertices of $\Gamma_{SI}(M)$ are of order 2.*

Proof. Let x be an isolated vertex of $\Gamma_{SI}(M)$ and $o(x) > 2$. Then by Proposition 2.2 x adjacent with all elements x^i $(i = 1, \ldots, o(x) - 1)$ and this is a contradiction. □

Remark 2.4. *The converse of Proposition 2.3 is not true. For example, in the Moufang loop* $M := M(D_8, 2)$*, the element* a^2 *is of order 2 but* $deg(a^2) = 2 \neq 0$.

Proposition 2.5. *Let* M *be a finite Moufang loop and* ε *be the number of edges in* $\Gamma_{SI}(M)$*. Then*

$$\varepsilon \geqslant \frac{1}{2} \sum_{x \in M-e} o(x) - 2.$$

Proof. We know that for any graph, [2], we have:

$$2\varepsilon = \sum_{x \in v(\Gamma_{SI})} deg(x)$$

and by Proposition 2.2, $deg(x) \geqslant o(x) - 2$. So,

$$\varepsilon \geqslant \frac{1}{2} \sum_{x \in M-e} o(x) - 2. \quad \square$$

Proposition 2.6. *Let* M *be a finite Moufang loop and* ε *be the number of edges in* $\Gamma_{SI}(M)$*. Then* $\varepsilon = \frac{1}{2} \sum_{x \in M-e} o(x) - 2$ *if and only if every element in the* M *is of prime order other than identity.*

Proof. Let $\Gamma_{SI}(M)$ be a graph with $\frac{1}{2} \sum_{x \in M-e} o(x) - 2$ edges. By Proposition 2.3, for all vertices $x \in M - e$, we have $deg(x) = o(x) - 2$. Assume $o(x)$ is not a prime, without loss of generality, we can consider $o(x) = pq$, where p is a prime and q is a positive integer. The subloop generated with element x will be a subgroup. Suppose $H =< x >$, then from $p \mid o(H)$, we get the subgroup H has an element of order p say that y, so, $o(y) = p$. By assumption $deg\Gamma_{SI}(y) = o(y) - 2 = p - 2$.

Also, $x \notin < y >$ and $y \in < x >$, y is adjacent to at least $x, y^2, \ldots, y^{p-1}$, which yields that $deg\Gamma_{SI}(y) > p - 2$, this is a contradiction and so every element in M is of prime order.

Conversely, suppose that every elements other than identity in the Moufang loop M is of prime order and there exists an element $x \in M - \{e\}$ such that $deg(x) > o(x) - 2$. Then there exists an element $y \in M - \{e, x\}$,

$y \notin < x >$ and y adjacent with x. So, for some i, j, $x^i = x^j$, on the other hand $o(x^i) = o(x)$ and $o(y^j) = o(y)$, because, $o(x)$ and $o(y)$ are prime. Then $o(x) = o(y)$. So, $< x >=< y >$, which yields to contradiction to $y \notin < x >$. Hence, $deg(x) = o(x) - 2$ for all $x \in M - e$ in the $\Gamma_{SI}(M)$. $\square$

Theorem 2.7. *Let M be a Moufang loop of odd order. Then $\Gamma_{SI}(M)$ is a complete graph if and only if M consist of unique subloop of order p and $o(M) = p^m$, where, p is a prime number and m is a positive integer.*

Proof. Let M be a Moufang loop of order n and let $\Gamma_{SI}(M)$ be a complete graph. If n is not a prime power, then there exists two prime dividers p and q of n, also the definition of Moufang loop M implies that M is a bol loop of odd order. By Theorem 1.1, M has two elements a and b such that $o(a) = p$ and $o(b) = q$. Clearly, $| < a > \cap < b > | = 1$, so, a and b are not adjacent in $\Gamma_{SI}(M)$ and which yields a contradiction with complete graph, hence $o(M) = p^m$. Now, suppose that the Moufang loop M has two distinct subloop of order p, then there exists two non-identify elements a and b such that $o(a) = o(b) = p$ and $| < a > \cap < b > | = 1$, so, a and b are non-adjacent in $\Gamma_{SI}(M)$ and this is a contradiction. Hence, M has unique subloop of order p.

Conversely, assume that $o(M) = p^m$ where p is prime number and m is a positive integer and M has unique subloop of order p namely H. Since every subloop of order p is cycle so there exists an element $a \in M$ such that $H =< a >$. Also, from $o(M) = p^m$, for any $b \in M - e$, there exists an integer k where $1 \leqslant k \leqslant m$ such that $o(b) = p^k$. Since $H =< a >$ is a unique subloop of order p, for all $b \in M - e$, we have $< a > \subseteq < b >$. Therefore $| < x > \cap < y > | \geqslant p \geqslant 1$ for all $x, y \in M - e$ and so all vertices are adjacent in $\Gamma_{SI}(M)$, hence $\Gamma_{SI}(M)$ is a complete graph. $\square$

Proposition 2.8. *Let M be a finite Moufang loop. Then $\Gamma_{SI}(M)$ is forest if and only if $o(M) = 2^\alpha \times 3^\beta$ where, α and β are positive integers and the order of all elements of M is equal to 2 or 3.*

Proof. Clearly, if $o(M) = 2^\alpha \times 3^\beta$ and order of all elements of M is equals 2 or 3, then $\Gamma_{SI}(M)$ is the union of the complete graphs with two vertices, $K_2^{,}s$ and isolated vertices, hence $\Gamma_{SI}(M)$ is a forest.

Conversely, assume that $\Gamma_{SI}(M)$ is a forest and a is an element of order more than 3 in M, then a adjacent with all the elements a^i where, $i = 2, \ldots, o(a) - 1$ and there exists integers i, j such that $| < a^i > \cap < a^j > | > 1$, also, a^i is adjacent with a^j, so, we get a cycle in graph and wich yields the contradiction. Hence, the order of all elements of M is small than 4 and hence $o(M) = 2^\alpha \times 3^\beta$. $\square$

Proposition 2.9. *Let $G = D_{2n}$ be the dihedral group of order $2n$. If $n = p^m$ where p is prime number and m is a positive integer, then*

$$\Gamma_{SI}(G) = \cup_{i=1}^{n} K_1 + K_{n-1}.$$

Proof. By the presentation

$$D_{2n} =< a, b \mid a^n = b^2 = (ab)^2 = 1 >$$

we get that all elements in the form $a^i b$ where $0 \leqslant i \leqslant n-1$ is of order 2 and $| < a^i b > \cap < a^j b > | = 1$ and $| < a^i > \cap < a^j b > | = 1$ for every integers i, j $(0 \leqslant i, j \leqslant n-1)$. So, all vertices in the form $a^i b$ are isolated and since $n = p^m$, then $| < a^i > \cap < a^j > | > 1$ for every integers i, j $(0 \leqslant i, j \leqslant n-1)$, hence all vertices in the form a^i's $1 \leqslant i \leqslant n$ are adjacent and we get a connected component with $n - 1$ vertices. $\square$

Proposition 2.10. *The intersection graph of Moufang loops $M(G, 2)$ where G is a finite group is not connected and the number of isolated vertices are equal or biggest than $|G|$.*

Proof. The number of elements of all Moufang loops $M(G, 2)$ is equal to $2|G|$ where, there exists $|G|-$number of elements of the form gu and $o(gu) = 2$ and $| < gu > \cap < hu > | = 1$, $| < gu > \cap < h > | = 1$ for every $g, h \in G$. Hence all elements gu are isolated vertices in $\Gamma_{SI}(M)$. $\square$

Remark 2.11. *In the intesection graph of the Moufang loops $M(D_{2n}, 2)$, the $3n$ vertices are isolated and other vertices format a complete connected component where $n = p^m$.*

Remark 2.12. *There are Moufang loops that intersection graph of them are isomorphism but they are not isomorphism. For example, $\Gamma_{SI}(M(16, 2)) \cong \Gamma_{SI}(M(16, 4))$, $\Gamma_{SI} \cong K_7 + 8K_1$ and $M(16, 2) \ncong M(16, 4)$.*

Theorem 2.13. *Let G be a finite group and t be the number of the connected components in the $\Gamma_{SI}(G)$. Then $\gamma(\Gamma_{SI}(G)) = t$.*

Proof. Clearly, for every connected component with the number of vertices equal or less than 3, any of the vertices will be a dominating set and the number of dominating set is 1. Now by using the induction over the number of the vertices if the number of the dominating in every connected component set with n vertices is 1, then we prove that the number of the dominating in every connected component set with $n+1$ vertices is 1, for this porpuse, suppose that the number of vertices in the connected component be $n+1$, x be a vertex with the maximum degree in the compnent and $o(x) = m$ and T be a connected component with $n+1$ vertices. Then every neighbours of x is not a cut-vertex. To prove it suppose that y be any vertex in $\Gamma_{SI}(G)$ such that adjacent with x and $o(y) = k$. If $deg(y) = 1$, then $\omega(T-y) = 1$ and the assertion is hold. Now suppose that $deg(y) > 1$ and y adjacent with another vertex $z(z \neq x)$, by defining there are positive integers i and j such that $x^i = y^j$ and also, there exist positive integers l and q such that $y^l = z^q$. If x adjacent with z, the proof is completed, otherwise we get the following cases:

(i) If k is a prime integer, then $(l, k) = 1$ and $(j, k) = 1$, so, $< y^j >=< y^l >=< y >$ and z^q will be adjacent with the x^i. Hence, xx^iz^qz is a path from x to z.

(ii) If k is not prime but $(l, k) = 1$ or $(j, k) = 1$. Then $< y >=< y^l >$ or $< y >=< y^j >$ and z^q will be adjacent with the x^i and xz^qz or xx^iz is a path from x to z.

(iii) If $(l, k) \neq 1$ and $(j, k) \neq 1$, then there exist prime integer $p(1 \leqslant p \leqslant k-1)$ such that $y^p \in < y >$ and $(p, l) = 1$, $(p, j) = 1$ and $< y^p >=< y >$, so, y^p is adjacent with y^l and y^j and $xy^jy^py^lz$ ia a path from x to z.

So, y is not a cut-vertex and $\omega(T-y) = 1$. By induction we have $T-y$ is a connected component with n vertex and x has a maximum degree in this component. Now, $\{x\}$ is a dominating set for $T-y$ and $\gamma(T-y) = 1$. In the component T, the vertex x is adjacent with y, so, $\{x\}$ is a dominating set for T and $\gamma(T) = 1$. $\square$

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