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Module Amenability of Semigroup Algebras under Certain Module Actions

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Abstract. In this paper we define a congruence $\sim$ on inverse semigroup S such that amenability of S is equivalent to amenability of $S/\sim$. We study module amenability of semigroup algebra $l^1(S/\sim)$ when S is an inverse semigroup with idempotents E and prove that it is equivalent to module amenability of $l^1(S)$. The main difference of this action with the more studied trivial action is that in this case the corresponding homomorphic image is a Clifford semigroup rather than a discrete group.

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1. Introduction

A Banach algebra $\mathcal{A}$ is amenable if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-module X^* is inner, for every Banach $\mathcal{A}$-module X. This concept was introduced and intensively studied in [4].
Author in [1] introduced the concept of module amenability for Banach algebras which are Banach modules on another Banach algebra with compatible actions. This could be considered as a generalization of the Johnson's amenability. He proved that for an inverse semigroup S with

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idempotents E, if $l^1(S)$ is considered as an $l^1(E)$-module with trivial left action and canonical right action (by multiplication), then its module amenability is equivalent to the amenablity of S. The main reason that the action in both [1] and [2] is assumed to be trivial from one side is that in this case, the corresponding equivalence relation $\approx$ leads a group homomorphic image $S/\approx$. Then it is shown that a quotient $l^1(S)/J$ of $l^1(S)$ is isomorphic to $l^1(S/\approx)$ to employ the amenability results on group algebras.

In this paper we replace the left trivial action with the canonical action (defined by multiplication from both sides). In the group case this is just trivial action. The main difference is that the corresponding equivalence relation $\sim$ leads a Clifford semigroup homomorphic image $S/\sim$.

Again the corresponding quotient $l^1(S)/J$ of $l^1(S)$ is isomorphic to $l^1(S/\sim)$. We prove for each inverse semigroup S if $l^1(S)$ is $l^1(E)$-module amenable with canonical action, then S is amenable, but converse is not true in general. We show that an inverse semigroup S is amenable if and only if $S/\sim$ is amenable. Finally we show that for a Brandt semigroup S over a group G (amenable or no) with index set I such that $|I| > 1$, $l^1(S)$ is module amenable with canonical actions.

2. Preliminaries

In this section, we recall some notions and define some basic concepts. Throughout this paper, $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a,\ b \in \mathcal{A},\ \alpha \in \mathcal{U}).$$

The Banach algebra $\mathcal{U}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathcal{U}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$ ($a \cdot \alpha = f(\alpha)a$), where f is a continuous linear functional on $\mathcal{U}$.

Let X be a Banach $\mathcal{A}$-module and a Banach $\mathcal{U}$-module with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$

$$(\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \ (a \in \mathcal{A},\ \alpha \in \mathcal{U},\ x \in X);$$

the same equalities valid for right actions. Then we say that X is a Banach $\mathcal{A}$-$\mathcal{U}$-module. If in addition,

$$\alpha \cdot x = x \cdot \alpha \qquad (\alpha \in \mathcal{U},\ x \in X)$$

then X is called a commutative $\mathcal{A}$-$\mathcal{U}$-module. If X is a (commutative) Banach $\mathcal{A}$-$\mathcal{U}$-module, then so is X^*, where the left actions of $\mathcal{A}$ and $\mathcal{U}$ on X^* are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha),\ \ (a \cdot f)(x) = f(x \cdot a)\ \ (a \in \mathcal{A},\ \alpha \in \mathcal{U},\ x \in X,\ f \in X^*)$$

and the same for the right actions.

We denote by J, the closed ideal of $\mathcal{A}$ generated by elements of the form $\alpha \cdot ab - ab \cdot \alpha$ for $\alpha \in \mathcal{U},\ a, b \in \mathcal{A}$.

A bounded map $D : \mathcal{A} \to X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b) \ \ , \ \ D(ab) = Da \cdot b + a \cdot Db \qquad (a, b \in \mathcal{A})$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a) \ \ , \ \ D(a \cdot \alpha) = D(a) \cdot \alpha \qquad (a \in \mathcal{A}, \alpha \in \mathcal{U}).$$

We note that D is not necessarily linear, but still its boundedness implies its norm continuity (since D preserves subtraction). When X is commutative, each $x \in X$ defines a module derivation

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

These are called inner module derivations.

Definition 2.1. *A Banach algebra $\mathcal{A}$ is called module amenable (as an $\mathcal{U}$-module) if for any commutative Banach $\mathcal{A}$-$\mathcal{U}$-module X, each module derivation $D : \mathcal{A} \to X^*$ is inner.*

Proposition 2.2. ([2, Proposition 3.2.]) *Let $\mathcal{A}$ be module amenable with trivial left action, and let J_0 be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has an identity, then $\mathcal{A}/J_0$ is amenable.*

Definition 2.3. *Let S be a semigroup and $x \in S$. An element $x^* \in S$ is said to be an inverse of x if $xx^*x = x, x^*xx^* = x^*$. If every element of S has a unique inverse, then S is called an inverse semigroup. An element $e \in S$ is called an idempotent if $e = e^2$.*

Throughout this paper S is an inverse semigroup with set of idempotents E. Then E is a commutative subsemigroup of S ([5, Theorem 5.1.1]) and $l^1(E)$ can be regarded as a subalgebra of $l^1(S)$. Thereby $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module by multiplication from right and trivially from left, that is,

$$\delta_e * \delta_s = \delta_s,\ \ \delta_s \cdot \delta_e = \delta_{se}\ (s \in S, e \in E). \tag{1}$$

In this case J is the closed ideal of $l^1(S)$ generated by element of the form

$$\{\delta_s - \delta_s \cdot \delta_e\}.$$

In this paper we denote by J_t the closed ideal J and if $l^1(S)$ is $l^1(E)$-module amenable with actions mentioned in (1), we say that $l^1(S)$ is module amenable with trivial left action.
Consider an equivalence relation on S as follows:

$$s \approx t \Leftrightarrow \delta_s - \delta_t \in J_t \quad (s, t \in S).$$

The following result restoring the Johnson's Theorem for the case of inverse semigroups.

Theorem 2.4. ([1, Theorem 3.1.]) *Let S be an inverse semigroup with idempotents E. Consider $l^1(S)$ as a Banach module over $l^1(E)$ with the multiplication as the right action and the trivial left action. Then $l^1(S)$ is module amenable with trivial left action if and only if S is amenable.*

3. Module Amenability of Semigroup Algebras

We may change the action of $l^1(E)$ on $l^1(S)$ to get different module amenability results. Here we consider $l^1(E)$ acts on $l^1(S)$ with compatible actions by canonical actions, that is

$$\delta_e \cdot \delta_s = \delta_{es},\ \delta_s \cdot \delta_e = \delta_{se} \quad (s \in S, e \in E).$$

Thus J is the closed ideal of $l^1(S)$ generated by

$$\{\delta_{es} - \delta_{se} :\ s, t \in S,\ e \in E\}.$$

All over this paper we consider $l^1(S)$ as $l^1(E)$-module with canonical actions and use notation X for arbitrary commutative $l^1(S)$-$l^1(E)$-module unless they are otherwise specified. In this case, if $l^1(S)$ is module amenable, then we say that $l^1(S)$ is module amenable with canonical actions.

In this case we denote by J_c, the closed ideal J. Then trivially $l^1(S)/J_c$ is a Banach $l^1(S)$-$l^1(E)$-module under the canonical module actions. We consider another equivalence relation on S as follows

$$s \sim t \Leftrightarrow \delta_s - \delta_t \in J_c \quad (s, t \in S).$$

Consider semigroup morphism $\varphi : S \to S/\sim$. It follows from Theorem 5.1.4 of [5] that $S/\sim$ is an inverse semigroup and by definition of J_c, $S/\sim$ is a clifford inverse semigroup.

Remark 3.1. *If X is a commutative $l^1(S)$-$l^1(E)$-module, then there exist two actions of $l^1(E)$ on X. $l^1(E)$ acts on X as a subalgebra of $l^1(S)$ and as Banach algebra $\mathcal{U}$. We show by $\bullet$, left and right actions of $l^1(E)$ as a Banach algebra $\mathcal{U}$ on X.*

Lemma 3.2. *Let X be a commutative $l^1(S)$-$l^1(E)$-module and $D : l^1(S) \to X$ be a module derivation. Then D is a derivation.*

Proof. Let $\lambda \in \mathbb{C}$. For each $s \in S$, we have

$$D(\lambda\delta_s) = D(\lambda\delta_{ss^*} \cdot \delta_s) = \lambda\delta_{ss^*} \bullet D(\delta_s) = \lambda D(\delta_s).$$

Thus D is linear. □

Proposition 3.3. *$l^1(S)$ is module amenable with canonical actions if and only if $l^1(S/\sim)$ is module amenable with canonical actions.*

Proof. The semigroup morphism $\varphi : S \to S/\sim$, $x \mapsto [x]$, extends linearly to a continuous epimorphism $\pi : l^1(S) \to l^1(S/\sim)$ and we have $l^1(S)/J_c \cong l^1(S/\sim)$. By Proposition 2.5. of [1], it is enough we show that $l^1(S)$ is module amenable with canonical actions if and only if $l^1(S)/J_c$ is module amenable with canonical actions. Let X be a commutative $l^1(S)$-$l^1(E)$-module and $D : l^1(S) \to X^*$ be a module derivation. We have

$$\delta_{se} \cdot x = \delta_s \cdot (\delta_e \bullet x) = \delta_s \cdot (x \bullet \delta_e) = (\delta_s \cdot x) \bullet \delta_e = \delta_e \bullet (\delta_s \cdot x) = \delta_{es} \cdot x.$$

Therefore $J_c \cdot X = 0$ and similarly $X \cdot J_c = 0$. So X is commutative $l^1(S)/J_c$-$l^1(E)$-module with the following module actions

$$(\delta_s + J_c) \cdot x := \delta_s \cdot x,\ x \cdot (\delta_s + J_c) := x \cdot \delta_s,$$

$$(\delta_s + J_c) \bullet x := \delta_s \bullet x,\ x \bullet (\delta_s + J_c) := x \bullet \delta_s.$$

Since $D(\delta_{es}) = \delta_e \bullet D(\delta_s)$ and $D(\delta_{se}) = D(\delta_s) \bullet \delta_e$, $D(\delta_{es} - \delta_{se}) = 0$. On the other hand,

$$D(\delta_t) \cdot (\delta_{es} - \delta_{se}) = 0 = (\delta_{es} - \delta_{se}) \cdot D(\delta_h),$$

for $t, h, s \in S, e \in E$ and so

$$\begin{aligned} D(\delta_t(\delta_{es} - \delta_{se})\delta_h) &= D(\delta_t) \cdot (\delta_{es} - \delta_{se})\delta_h \\ &+ \delta_t(\delta_{es} - \delta_{se}) \cdot D(\delta_h) \\ &= 0 \end{aligned}$$

Therefore $D|_{J_c} = 0$. Thus D induces a module derivation $\tilde{D} : l^1(S)/J_c \to X^*$ defined by $\tilde{D}(\delta_s + J_c) = D(\delta_s)$. If $l^1(S)/J_c$ is module amenable with canonical actions, then $\tilde{D}$ is inner and so is D.

For the converse, let X be a commutative $l^1(S)/J_c$-$l^1(E)$-module with canonical actions and $D : l^1(S)/J_c \to X^*$ be a module derivation. Then X is a $l^1(S)$-$l^1(E)$-module with module actions given by

$$\delta_s \cdot x := (\delta_s + J_c) \cdot x, \quad x \cdot \delta_s := x \cdot (\delta_s + J_c).$$

Now $\tilde{D} : l^1(S) \to X^*$ defined by $\tilde{D}(\delta_s) = D(\delta_s + J_c)$ is a module derivation. The module amenability of $l^1(S)/J_c$ implies that $\tilde{D}$ is inner and so is D. $\square$

Proposition 3.4. *If $l^1(S)$ is $l^1(E)$-module amenable with canonical actions, then $l^1(S)$ is $l^1(E)$-module amenable with trivial left action.*

Proof. Let $l^1(S)$ be $l^1(E)$-module with trivial left action and X be a commutative $l^1(S)$-$l^1(E)$-module. We have

$$\begin{aligned} \delta_{se} \cdot x &= \delta_s \cdot (\delta_e \bullet x) = \delta_s \cdot (x \bullet \delta_e) \\ &= (\delta_s \cdot x) \bullet \delta_e = \delta_e \bullet (\delta_s \cdot x) \\ &= (\delta_e \bullet \delta_s) \cdot x \\ &= \delta_s \cdot x. \end{aligned}$$

Thus $J_t \cdot X = 0$ and similarly $X \cdot J_t = 0$. Now since $S/\approx$ is a group, $\delta_{es} - \delta_s \in J_t$ and so $\delta_{es} - \delta_{se} \in J_t$. It follows that X is a commutative $l^1(S)$-$l^1(E)$-module with canonical actions. Suppose that $D : l^1(S) \to X^*$ is a module derivation. Thus

$$D(\delta_{se}) = D(\delta_s) \bullet \delta_e = \delta_e \bullet D(\delta_s) = D(\delta_s).$$

Therefore $D|_{J_t} = 0$ and so $D(\delta_{es} - \delta_{se}) = 0$. Now if $l^1(S)$ considered as $l^1(E)$-module with canonical actions, then D is a module derivation. So by assumption it is inner. $\square$

By the above proposition and Theorem 2.4., if $l^1(S)$ is $l^1(E)$-module amenable with canonical action, then S is amenable. But the following example shows in general that converse does not hold.

Example 3.5. Let $S = G \cup \{0\}$ such that G is a non amenable group. Since S has a zero, it is amenable but according to [6], $l^1(S)$ is not amenable. So, if possible let $l^1(S)$ be module amenable with canonical actions. Suppose that X is a $l^1(G)$-module and $D : l^1(G) \to X^*$ be a derivation. Since $l^1(S) = l^1(G) \bigoplus \mathbb{C}\delta_0$, by the following definition, X is a commutative $l^1(S)$-$l^1(E)$-module

$$x \cdot \delta_0 = \delta_0 \cdot x = 0,$$

$$x \bullet \delta_0 = \delta_0 \bullet x = 0.$$

Consider $\tilde{D} : l^1(S) \to X^*$ defined by $\tilde{D}(\delta_g) = D(\delta_g)(g \in G)$ and $D(\delta_0) = 0$. Clearly $\tilde{D}$ is a module derivation and so it is inner. Therefore D is an inner derivation and this contradicts the fact that $l^1(G)$ is not amenable. It is shown in [2] that if E is upward directed, then the quotient $S/\approx$ is a discrete group. But it is possible to ignore the condition that E is upward directed. If $e,\ f \in E$, then $\delta_e - \delta_{ef} = \delta_f * \delta_e - \delta_e \cdot \delta_f \in J_t$ and $\delta_f - \delta_{ef} = \delta_e * \delta_f - \delta_f \cdot \delta_e \in J_t$ and so $e \approx f$. Now the argument of [2] could be adapted to show that in this case $S/\approx$ is again a discrete group.

Theorem 3.6. *For every inverse semigroup S the following statement are equivalent:*

(i) *$S/\sim$ is amenable.*

(ii) *$S/\approx$ is amenable.*

(iii) *S is amenable.*

Proof. Suppose that $\widetilde{E}$ is the set of idempotent elements of $S/\sim$. Clearly $l^1(\tilde{E})$ is closed linear span of $\{\pi(\delta_e)|e \in E\}$. Suppose that $l^1(\widetilde{E})$ acts on $l^1(S/\sim)$ by multiplication from right and trivially from left. Let $\widetilde{J}$ be the closed ideal of $l^1(S/\sim)$ generated by elements of the form

$$\{\pi(\delta_s - \delta_{se})|s \in S, e \in E\}.$$

It is easy to show that $l^1(S/\approx) \cong l^1(S/\sim)/\widetilde{J}$. Now if $S/\sim$ is amenable, then $l^1(S/\sim)$ is module amenable with trivial left action. It follows from Proposition 2.2. that $l^1(S/\approx)$ is amenable, thus $S/\approx$ is amenable. Conversely, if $S/\approx$ is amenable, then $l^1(S/\approx)$ is amenable. We show that $l^1(S/\sim)$, $l^1(\tilde{E})$-module amenable with trivial left action. Suppose that $D : l^1(S/\sim) \to X^*$ is a module derivation and X is a commutative $l^1(S/\sim)$-$l^1(\tilde{E})$- module. We have

$$\begin{aligned}
\pi(\delta_s - \delta_{se}) \cdot x &= \pi(\delta_s) \cdot x - \pi(\delta_s)\pi(\delta_e) \cdot x \\
&= \pi(\delta_s) \cdot x - \pi(\delta_s)(\pi(\delta_e) \bullet x) \\
&= \pi(\delta_s) \cdot x - \pi(\delta_s)(x \bullet \pi(\delta_e)) \\
&= \pi(\delta_s) \cdot x - (\pi(\delta_s) \cdot x) \bullet \pi(\delta_e) \\
&= \pi(\delta_s) \cdot x - \pi(\delta_e) \bullet (\pi(\delta_s) \cdot x) \\
&= \pi(\delta_s) \cdot x - \pi(\delta_e) * \pi(\delta_s) \cdot x \\
&= 0.
\end{aligned}$$

The last equality follows from trivial left action of $l^1(\tilde{E})$ on $l^1(S/\sim)$. Thus $\tilde{J} \cdot X = X \cdot \tilde{J} = 0$ and X is $l^1(S/\sim)/\widetilde{J}$- module and so $l^1(S/\approx)$-module. Consider $\tilde{D} : l^1(S/\approx) \to X^*$ defined by $\tilde{D}(\delta_s + J_t) := D(\delta_s + J_c)$. By Lemma 3.2., we have $\tilde{D}$ is linear. Since $l^1(S/\approx)$ is amenable $\tilde{D}$ is inner and so is D. Thus $l^1(S/\sim)$ is module amenable with trivial left action. Now result follows from Theorem 2.4. Thus (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows from Theorem 1. of [3]. □

Corollary 3.7. *If $S/\sim$ is a group, then $l^1(S)$ is module amenable with canonical actions if and only if $S/\sim$ is amenable.*

Proof. If $S/\sim$ is amenable, by Johnson's Theorem $l^1(S/\sim)$ is amenable. It follows from Proposition 3.3. of [2] that $l^1(S)$ is module amenable with canonical action. The converse follows from Theorems 3.4. and 2.4. □

Remark 3.8. *It is easy to show that if $S/\sim$ is a group, then $S/\sim = S/\approx$. The next example asserts that there exists an amenable inverse semigroup S such that $S/\sim$ is not a group, but $l^1(S)$ is module amenable with canonical actions.*

Example 3.9. Let S be a symmetric inverse semigroup on the set X, where $X = \{1, 2\}$. If S has a zero, then it is amenable. Since $S/\sim$ is isomorphic to a semigroup $\{0, 1, x\}$ such that $x^2 = 1_S$, $S/\sim$ is a zero group. It follows from [6], $l^1(S/\sim)$ is amenable and so module amenable with canonical actions. By Proposition 3.3. $l^1(S)$ is also module amenable with canonical actions.

4. Module Amenability of Brandt Semigroup Algebras

Let G be a group and let I be a non-empty set.Set

$$\mathcal{M}^0(G,I) = \{(g)_{ij} : g \in G,\ i,\ j \in I\} \cup \{0\}$$

where $(g)_{ij}$ denotes the $I \times I$-matrix with entry $g \in G$ in the (i,j)-position and zero elsewhere. Then $\mathcal{M}^0(G,I)$ with the multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & if\ j = k \\ 0 & if\ j \neq k. \end{cases}$$

is an inverse semigroup with $(g)^*_{ij} = (g^{-1})_{ji}$, that is called the Brandt semigroup over G with index set I.

Theorem 4.1. *Let $S = \mathcal{M}^0(G,I)$. Then $l^1(S)$ is a module amenable with canonical actions when $|I| > 1$. When $|I| = 1$, $l^1(S)$ is module amenable with canonical actions if and only if $l^1(S)$ is amenable.*

Proof. When $|I| = 1$, in the example 3.5. we see that $l^1(S)$ is module amenable with canonical actions if and only if $l^1(G)$ is amenable. In the case $|I| > 1$ we have

$$(e)_{ii}(g)_{kl} = \begin{cases} (g)_{il} & if\ i = k \\ 0 & if\ i \neq k. \end{cases}$$

And

$$(g)_{kl}(e)_{ii} = \begin{cases} (g)_{ki} & if\ l = i \\ 0 & if\ l \neq i. \end{cases}$$

Therefore J_c is closed ideal generated by $\{\delta_{(g)_{il}} - \delta_0 | i, l \in I, i \neq l\}$. Since

$$\delta_{(g)_{ii}} - \delta_0 = (\delta_{(g)_{il}} - \delta_0) \cdot \delta_{(e)_{li}},$$

$(g)_{il} \sim 0$,for each $i,\ l \in I$. Thus $S/\sim$ is trivial group with one element and it is module amenable with canonical actions. $\square$

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