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Norm estimates of the pre–schwarzian derivatives for two certain subclasses of starlike functions

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Abstract. Let $\alpha \in [\pi/2, \pi)$ and $\gamma_1, \gamma_2 \in (0, 1]$. For a normalized analytic functions f in the open unit disc Δ we consider

$$\mathcal{M}(\alpha) := \left\{ f \in \mathcal{A} : 1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\alpha}{2\sin\alpha}, \quad z \in \Delta \right\},$$

and

$$\mathcal{S}_t^*(\gamma_1, \gamma_2) := \left\{ f \in \mathcal{A} : -\frac{\pi\gamma_1}{2} < \arg\left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{\pi\gamma_2}{2}, \quad z \in \Delta \right\}.$$

In the present paper, we establish a sharp norm estimate of the pre–Schwarzian derivative for functions f belonging two these subclasses of analytic and normalized functions.

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1 Introduction

Let $\mathcal{H}$ be the class of functions f analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be a subclass of $\mathcal{H}$ that its members are normalized

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by the condition $f(0) = 0 = f'(0) - 1$. Therefore each $f \in \mathcal{A}$ has the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta).$$

The subclass of all functions in $\mathcal{A}$ which are univalent (one–to–one) in Δ is denoted by $\mathcal{U}$.

Recently, Kargar *et al.* (see [7]) introduced and studied the class $\mathcal{M}(\alpha)$ as follows:

$$\mathcal{M}(\alpha) := \left\{ f \in \mathcal{A} : 1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\alpha}{2\sin\alpha} \right\} \quad (z \in \Delta),$$

where $\alpha \in [\pi/2, \pi)$. The class $\mathcal{M}(\alpha)$ is a subclass of the starlike functions of order γ where $\gamma \in [0.2146, 0.5)$, see [12]. We recall that a function $f \in \mathcal{A}$ is starlike of order γ $(0 \leq \gamma < 1)$ if, and only if,

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma \quad (z \in \Delta).$$

The class of the starlike functions of order γ in Δ is denoted by $\mathcal{S}^*(\gamma)$. It is known that $\mathcal{S}^*(\gamma) \subset \mathcal{U}$ for each $\gamma \in [0, 1)$. According to the above we have $\mathcal{M}(\alpha) \subset \mathcal{U}$ where $\pi/2 \leq \alpha < \pi$. The class of starlike functions is denoted by $\mathcal{S}^* \equiv \mathcal{S}^*(0)$. Also, we say that a function $f \in \mathcal{A}$ is strongly starlike of order β, where $0 < \beta \leq 1$ if, and only if,

$$\left| \arg\left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi\beta}{2} \quad (z \in \Delta).$$

The class of strongly starlike functions of order β is denoted by $\mathcal{SS}^*(\beta)$. We note that every strongly starlike function f of order $\beta \in (0, 1)$ is bounded, see [3].

Another class that we are interested to study in this paper is the class $\mathcal{S}^*_t(\gamma_1, \gamma_2)$. We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_t(\gamma_1, \gamma_2)$, if f satisfies the following two–sided inequality

$$-\frac{\pi\gamma_1}{2} < \arg\left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{\pi\gamma_2}{2} \quad (z \in \Delta),$$

where $0 < \gamma_1, \gamma_2 \leq 1$. The class $\mathcal{S}_t^*(\gamma_1, \gamma_2)$ was introduced by Takahashi and Nunokawa (see [18]). It is clear that $\mathcal{S}_t^*(\gamma_1, \gamma_2) \subset \mathcal{S}^*$ and that $\mathcal{S}_t^*(\gamma_1, \gamma_2)$ is a subclass of the class of strongly starlike functions of order $\beta = \max\{\gamma_1, \gamma_2\}$, i.e. $\mathcal{S}_t^*(\gamma_1, \gamma_2) \subset \mathcal{S}^*(\beta, \beta) \equiv \mathcal{SS}^*(\beta)$.

Now let $\mathcal{LU}$ denote the subclass of $\mathcal{H}$ consisting of all locally univalent functions, namely, $\mathcal{LU} := \{f \in \mathcal{H} : f'(z) \neq 0, \ \ z \in \Delta\}$. For a $f \in \mathcal{LU}$ the pre–Schwarzian and Schwarzian derivatives of f are defined as follows

$$T_f(z) := \frac{f''(z)}{f'(z)} \quad \text{and} \quad S_f(z) := T_f'(z) - \frac{1}{2}T_f^2(z),$$

respectively. We note that the quantity T_f (resp. S_f) is analytic on Δ precisely when f is analytic (resp. meromorphic) and locally univalent on Δ. Since $\mathcal{LU}$ is a vector space over $\mathbb{C}$ (see [6]), thus we can define the norm of $f \in \mathcal{LU}$ by

$$||f|| = \sup_{z\in\Delta}(1 - |z|^2) \left|\frac{f''(z)}{f'(z)}\right|.$$

This norm has significance in the theory of Teichmüller spaces, see [1]. It is known that $||f|| < \infty$ if and only if f is uniformly locally univalent. This means that there exists a constant $\varepsilon := \varepsilon(f) > 0$ such that f is univalent in each disk

$$\left\{z \in \mathbb{C} : \left|\frac{z - \xi}{1 - \bar{\xi}z}\right| < \varepsilon, \quad |\xi| < 1\right\},$$

see for example [19]. On the other hand, by the norm $||f||$, we can give univalence criteria for a non–constant meromorphic function f on Δ. Indeed, if $||f|| \leq 1$, then f is univalent in Δ and conversely, if f univalent in Δ, then $||f|| \leq 6$ and the equality is attained for the Koebe function and its rotation. In fact, both of these bounds are sharp, see [2]. Also, if f is starlike of order $\gamma \in [0, 1)$, then we have the sharp estimate $||f|| \leq 6 - 4\gamma$ (see e.g. [20]). For more details on the geometric and analytic properties of the norm $||f||$ one can refer to [10]. Moreover, many authors have given norm estimates for classical subclasses of univalent functions, see for instance [4, 9, 11, 13, 14, 15, 16, 17, 20].

Let f and g belong to the class $\mathcal{H}$. We say that a function f is subordinate to g, written as

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g,$$

if there exists a Schwarz function $\phi : \Delta \to \Delta$ with the following properties

$$\phi(0) = 0 \quad \text{and} \quad |\phi(z)| < 1 \quad (z \in \Delta),$$

such that $f(z) = g(\phi(z))$ for all $z \in \Delta$. In particular, if $g \in \mathcal{U}$, then we have the following geometric equivalence relation

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

We need the following lemmas.

Lemma 1.1. (Kargar et al. [7, Lemma 1.1]) *Let $f(z) \in \mathcal{A}$ and $\alpha \in [\pi/2, \pi)$. Then $f \in \mathcal{M}(\alpha)$ if, and only if,*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \mathcal{B}_\alpha(z) \quad (z \in \Delta),$$

where

$$\mathcal{B}_\alpha(z) := \frac{1}{2i \sin \alpha} \log \left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right) \quad (z \in \Delta). \tag{1}$$

We note that

$$\mathcal{B}_\alpha(\{z : z \in \Delta\}) = \left\{\zeta \in \mathbb{C} : \frac{\alpha - \pi}{2 \sin \alpha} < \mathrm{Re}\{\zeta\} < \frac{\alpha}{2 \sin \alpha}, \quad \frac{\pi}{2} \leq \alpha < \pi\right\} =: \Omega_\alpha.$$

Lemma 1.2. (Kargar et al. [8, Lemma 1.1]) *Let $f(z) \in \mathcal{A}$, $0 < \gamma_1, \gamma_2 \leq 1$, $c = e^{\pi i \theta}$ and $\theta = \frac{\gamma_2 - \gamma_1}{\gamma_2 + \gamma_1}$. Then $f \in \mathcal{S}^*_t(\gamma_1, \gamma_2)$ if, and only if,*

$$\frac{zf'(z)}{f(z)} \prec G(z) \quad (z \in \Delta),$$

where

$$G(z) := G(\gamma_1, \gamma_2, c)(z) = \left(\frac{1 + cz}{1 - z}\right)^{(\gamma_1 + \gamma_2)/2} \quad (G(0) = 1, z \in \Delta). \tag{2}$$

We note that the function G is convex univalent in Δ and maps Δ onto $\Omega_{\gamma_1, \gamma_2}$ where

$$\Omega_{\gamma_1, \gamma_2} := \left\{w \in \mathbb{C} : -\frac{\pi \gamma_1}{2} < \arg\{w\} < \frac{\pi \gamma_2}{2}\right\}.$$

In this paper, we give the best possible estimate of the norms of pre–Schwarzian derivatives for the functions f belonging to the subclasses $\mathcal{M}(\alpha)$ and $\mathcal{S}^*_t(\gamma_1, \gamma_2)$.

2 Main Results

The first result is continued in the following form.

Theorem 2.1. *Let $\alpha \in [\pi/2, \pi)$. If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{M}(\alpha)$, then*

$$||f|| \leq \frac{2}{\sin\alpha}\sqrt{\pi^2+\sin^2\alpha} - \frac{2\sin\alpha}{\sqrt{\pi^2+\sin^2\alpha}}.$$

The result is sharp.

Proof. Let $f \in \mathcal{A}$ belongs to the class $\mathcal{M}(\alpha)$ and $\alpha \in [\pi/2, \pi)$. Then by Lemma 1.1 we have

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right) \quad (z \in \Delta). \tag{3}$$

Now by definition of subordination, the relation (3) implies that there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2i\sin\alpha}\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right) \quad (z \in \Delta). \tag{4}$$

By taking the derivative of the logarithmic on both sides of (4), after simplification, we get

$$\begin{aligned}\frac{f''(z)}{f'(z)} &= \frac{1}{2iz\sin\alpha}\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right) \\ &\quad + \frac{2i\sin\alpha w'(z)}{\left(1+\phi(z)e^{i\alpha}\right)\left(1+\phi(z)e^{-i\alpha}\right)\left(2i\sin\alpha + \log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right)}.\end{aligned} \tag{5}$$

It is known that for each Schwarz function ϕ we have $|\phi(z)| \leq |z|$ (cf. [5]). Also by the Schwarz–Pick lemma, we have the following inequality for a Schwarz function ϕ

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \quad (z \in \Delta). \tag{6}$$

On the other hand if log is the principal branch of the complex logarithm, then we have

$$\log z = \ln|z| + i \arg z \quad (z \in \Delta \setminus \{0\}, -\pi < \arg z \leq \pi). \tag{7}$$

Therefore, by the above equation (7), it is well–known that if $|z| \geq 1$, then

$$|\log z| \leq \sqrt{|z-1|^2 + \pi^2}, \tag{8}$$

while for $0 < |z| < 1$, we have

$$|\log z| \leq \sqrt{\left|\frac{z-1}{z}\right|^2 + \pi^2}. \tag{9}$$

Now we consider the two following cases.
Case 1. Let

$$\left|\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right| \geq 1.$$

Therefore, by (8) and the inequality $|\phi(z)| \leq |z|$ for all $z \in \Delta$, we obtain

$$\begin{aligned} \left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right| &\leq \sqrt{\left|\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}} - 1\right|^2 + \pi^2} \\ &= \frac{\sqrt{4\sin^2\alpha|\phi(z)|^2 + \pi^2\left|1+\phi(z)e^{-i\alpha}\right|^2}}{|1+\phi(z)e^{-i\alpha}|} \\ &\leq \frac{\sqrt{\left(\pi^2 + 4\sin^2\alpha\right)|\phi(z)|^2 + \pi^2\left(1+2|\phi(z)|\right)}}{1-|\phi(z)|} \\ &\leq \frac{\sqrt{\left(\pi^2 + 4\sin^2\alpha\right)|z|^2 + \pi^2\left(1+2|z|\right)}}{1-|z|}. \end{aligned} \tag{10}$$

Using (6), (10) and the triangular inequality, (5) yields that

$$
\begin{aligned}
&\left|\frac{f''(z)}{f'(z)}\right| \\
= &\left|\frac{1}{2iz\sin\alpha}\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right. \\
&\left.+\frac{2i\sin\alpha w'(z)}{\left(1+\phi(z)e^{i\alpha}\right)\left(1+\phi(z)e^{-i\alpha}\right)\left(2i\sin\alpha+\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right)}\right| \\
\leq &\frac{1}{2|z|\sin\alpha}\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right| \\
&+\frac{2\sin\alpha|\phi'(z)|}{(1-|\phi(z)|)^2\left(2\sin\alpha-\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right|\right)} \\
\leq &\frac{1}{2|z|\sin\alpha}\frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}{1-|z|} \\
&+\frac{2\sin\alpha}{(1-|\phi(z)|)^2\left(2\sin\alpha-\frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2(1+2|z|)}}{1-|z|}\right)}\cdot\frac{1-|\phi(z)|^2}{1-|z|^2} \\
\leq &\frac{1}{2|z|\sin\alpha}\frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}{1-|z|} \\
&+\frac{2\sin\alpha}{2(1-|z|)\sin\alpha-\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}\cdot\frac{1+|z|}{1-|z|^2}
\end{aligned}
$$

Thus, we conclude that

$$
\begin{aligned}
||f|| &= \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \\
&\leq \sup_{z\in\Delta}\left\{\frac{1+|z|}{2|z|\sin\alpha}\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}\right. \\
&\quad \left. + \frac{2(1+|z|)\sin\alpha}{2(1-|z|)\sin\alpha-\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}\right\} \\
&= \frac{2}{\sin\alpha}\sqrt{\pi^2+\sin^2\alpha}-\frac{2\sin\alpha}{\sqrt{\pi^2+\sin^2\alpha}}.
\end{aligned}
$$

Case 2. Let

$$
\left|\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right|<1.
$$

Using (9) and simple calculation, we have

$$
\begin{aligned}
\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right| &\leq \sqrt{\left|\frac{\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}-1}{\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}}\right|^2+\pi^2} \\
&= \frac{\sqrt{4\sin^2\alpha|\phi(z)|^2+\pi^2\left|1+\phi(z)e^{i\alpha}\right|^2}}{|1+\phi(z)e^{i\alpha}|} \\
&\leq \frac{\sqrt{4\sin^2\alpha|\phi(z)|^2+\pi^2\left(1+|\phi(z)|\right)^2}}{1-|\phi(z)|} \\
&\leq \frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}{1-|z|}.
\end{aligned}
$$

Therefore, in this case 2 we have the equal estimate for

$$
\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right|,
$$

too. Now, by the same argument, we get the desired result. Some calculations as above show that the result is sharp for the function

$$\begin{aligned} f_1(z) &:= z\exp\left\{\int_0^z \frac{\mathcal{B}_\alpha(t)}{t}\mathrm{d}t\right\} \quad (z\in\Delta) \\ &= z + z^2 + \frac{1}{2}(1-\cos\alpha)z^3 + \frac{1}{18}(1-9\cos\alpha+8\cos^2\alpha)z^4 + \cdots, \end{aligned}$$

or one of its rotations, where $\mathcal{B}_\alpha$ is defined in (1) and $\alpha\in[\pi/2,\pi)$. This is the end of proof. □

Corollary 2.2. *As an application of Theorem 2.1 define*

$$h(\alpha) := \frac{2}{\sin\alpha}\sqrt{\pi^2+\sin^2\alpha} - \frac{2\sin\alpha}{\sqrt{\pi^2+\sin^2\alpha}} \quad (\alpha\in[\pi/2,\pi)).$$

It is easy to see that $\lim_{\alpha\to\pi^-} h(\alpha)=\infty$. *Also, it is a simple exercise that* h *is an increasing function on* $[\pi/2,\pi)$ *and thus we have*

$$5.9871877\ldots \approx h(\pi/2) \le h(\alpha) < \infty.$$

The above estimate means that if $f\in\mathcal{M}(\alpha)$, *then it is uniformly locally univalent.*

Now, we have the following.

Theorem 2.3. *Let* $\gamma_1\in(0,1]$ *and* $\gamma_2\in(0,1]$. *If* $f\in\mathcal{A}$ *belongs to the class* $\mathcal{S}_t^*(\gamma_1,\gamma_2)$, *then*

$$||f|| \le 2(\gamma_1+\gamma_2).$$

The result is sharp.

Proof. Let $\gamma_1\in(0,1]$, $\gamma_2\in(0,1]$ and $f\in\mathcal{A}$ belongs to the class $\mathcal{S}_t^*(\gamma_1,\gamma_2)$. Then by Lemma 1.2 and by definition of subordination, there exists a Schwarz function $\phi:\Delta\to\Delta$ with $\phi(0)=0$ and $|\phi(z)|<1$ such that

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\gamma} \quad (\gamma := (\gamma_1+\gamma_2)/2), \tag{11}$$

where $c = e^{\pi i\theta}$ and $\theta = \frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}$. If we take the derivative of logarithmic on both sides of (11), then we get

$$\frac{f''(z)}{f'(z)} = \frac{1}{z}\left[\left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\gamma} - 1\right] + \gamma\left(\frac{c\phi'(z)}{1+c\phi(z)} + \frac{\phi'(z)}{1-\phi(z)}\right).$$

By triangle inequality and (6) and since $|c| = 1$, we obtain

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{|z|}\left[\left(\frac{1+|\phi(z)|}{1-|\phi(z)|}\right)^{\gamma} + 1\right] + 2\gamma\frac{1+|\phi(z)|}{1-|z|^2}. \tag{12}$$

So, using this inequality $|\phi(z)| \leq |z|$, the last inequality (12) yields that

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{|z|}\left[\left(\frac{1+|z|}{1-|z|}\right)^{\gamma} + 1\right] + 2\gamma\frac{1+|z|}{1-|z|^2}. \tag{13}$$

Multiplying the last inequality (13) by $(1-|z|^2)$ gives us

$$\begin{aligned}||f|| &= \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \\ &\leq \sup_{z\in\Delta}\left\{\frac{1-|z|^2}{|z|}\left[\left(\frac{1+|z|}{1-|z|}\right)^{\gamma} + 1\right] + 2\gamma(1+|z|)\right\} = 4\gamma\end{aligned}$$

and concluding the proof.

For the sharpness, consider the function

$$f_2(z) := z\exp\left\{\int_0^z \frac{G(t)-1}{t}\mathrm{d}t\right\} = z + Az^2 + \cdots \quad (z \in \Delta), \tag{14}$$

where G is defined in (2). It is easy to see that $f_2 \in \mathcal{A}$ and

$$\frac{zf_2'(z)}{f_2(z)} = G(z) \prec G(z) \quad (z \in \Delta)$$

and thus we have $f_2 \in \mathcal{S}_t^*(\gamma_1,\gamma_2)$. With the same argument we get the desired result and thus the details will be omitted. Here the proof ends. □

Corollary 2.4. *Let $\gamma_1 \in (0,1]$, $\gamma_2 \in (0,1]$ and $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_t^*(\gamma_1,\gamma_2)$. Since f is univalent in Δ and $2(\gamma_1+\gamma_2) \leq 4$, thus by Theorem 2.3 we have*

$$f \in \mathcal{S}_t^*(\gamma_1,\gamma_2) \Rightarrow ||f|| \leq 4 \leq 6.$$

This shows that the Becker and Pommerenke criterion is established [2].

Remark 2.5. It was proved in [8] that

$$\int_0^z \frac{G(t)-1}{t}\mathrm{d}t = \sum_{n=1}^{\infty} \frac{\lambda_n}{n} z^n,$$

is convex univalent in Δ where

$$\lambda_n := \lambda_n(\gamma_1, \gamma_2, c) = \sum_{k=1}^{n} \binom{n-1}{k-1} \binom{(\gamma_1+\gamma_2)/2}{k} (1+c)^k \quad (n \geq 1)$$

and $c = e^{\pi i \theta}$ and $\theta = \frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}$. Therefore, we conclude that the function f_2 given by (14) is convex univalent in Δ, too.

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