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# Norm Estimates of the Pre–Schwarzian Derivatives for Two Certain Subclasses of Starlike Functions

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**Abstract.** Let $\alpha \in [\pi/2, \pi)$ and $\gamma_1, \gamma_2 \in (0, 1]$. For a normalized analytic functions $f$ in the open unit disc $\Delta$ we consider

$$\mathcal{M}(\alpha) := \left\{ f \in \mathcal{A} : 1 + \frac{\alpha - \pi}{2\sin\alpha} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha}, \quad z \in \Delta \right\},$$

and

$$\mathcal{S}_t^*(\gamma_1, \gamma_2) := \left\{ f \in \mathcal{A} : -\frac{\pi\gamma_1}{2} < \arg\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{\pi\gamma_2}{2}, \quad z \in \Delta \right\}.$$

In the present paper, we establish a sharp norm estimate of the pre–Schwarzian derivative for functions $f$ belonging two these subclasses of analytic and normalized functions.

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**Keywords and Phrases:** Univalent, starlike, locally univalent, subordination, pre-Schwarzian norm.

## 1 Introduction

Let $\mathcal{H}$ be the class of functions $f$ analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be a subclass of $\mathcal{H}$ that its members are normalized

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by the condition $f(0) = 0 = f'(0) - 1$. Therefore each $f \in \mathcal{A}$ has the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta).$$

The subclass of all functions in $\mathcal{A}$ which are univalent (one–to–one) in $\Delta$ is denoted by $\mathcal{U}$.

Recently, Kargar *et al.* (see [7]) introduced and studied the class $\mathcal{M}(\alpha)$ as follows:

$$\mathcal{M}(\alpha) := \left\{ f \in \mathcal{A} : 1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \right\} \quad (z \in \Delta),$$

where $\alpha \in [\pi/2, \pi)$. The class $\mathcal{M}(\alpha)$ is a subclass of the starlike functions of order $\gamma$ where $\gamma \in [0.2146, 0.5)$, see [12]. We recall that a function $f \in \mathcal{A}$ is starlike of order $\gamma$ $(0 \le \gamma < 1)$ if, and only if,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in \Delta).$$

The class of the starlike functions of order $\gamma$ in $\Delta$ is denoted by $\mathcal{S}^*(\gamma)$. It is known that $\mathcal{S}^*(\gamma) \subset \mathcal{U}$ for each $\gamma \in [0, 1)$. According to the above we have $\mathcal{M}(\alpha) \subset \mathcal{U}$ where $\pi/2 \le \alpha < \pi$. The class of starlike functions is denoted by $\mathcal{S}^* \equiv \mathcal{S}^*(0)$. Also, we say that a function $f \in \mathcal{A}$ is strongly starlike of order $\beta$, where $0 < \beta \le 1$ if, and only if,

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi\beta}{2} \quad (z \in \Delta).$$

The class of strongly starlike functions of order $\beta$ is denoted by $\mathcal{SS}^*(\beta)$. We note that every strongly starlike function $f$ of order $\beta \in (0, 1)$ is bounded, see [3].

Another class that we are interested to study in this paper is the class $\mathcal{S}_t^*(\gamma_1, \gamma_2)$. We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_t^*(\gamma_1, \gamma_2)$, if $f$ satisfies the following two–sided inequality

$$-\frac{\pi\gamma_1}{2} < \arg\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{\pi\gamma_2}{2} \quad (z \in \Delta),$$

where $0 < \gamma_1, \gamma_2 \le 1$. The class $\mathcal{S}_t^*(\gamma_1, \gamma_2)$ was introduced by Takahashi and Nunokawa (see [18]). It is clear that $\mathcal{S}_t^*(\gamma_1, \gamma_2) \subset \mathcal{S}^*$ and that $\mathcal{S}_t^*(\gamma_1, \gamma_2)$ is a subclass of the class of strongly starlike functions of order $\beta = \max\{\gamma_1, \gamma_2\}$, i.e. $\mathcal{S}_t^*(\gamma_1, \gamma_2) \subset \mathcal{S}^*(\beta, \beta) \equiv \mathcal{SS}^*(\beta)$.

Now let $\mathcal{LU}$ denote the subclass of $\mathcal{H}$ consisting of all locally univalent functions, namely, $\mathcal{LU} := \{f \in \mathcal{H} : f'(z) \neq 0, \ \ z \in \Delta\}$. For a $f \in \mathcal{LU}$ the pre–Schwarzian and Schwarzian derivatives of $f$ are defined as follows

$$T_f(z) := \frac{f''(z)}{f'(z)} \quad \text{and} \quad S_f(z) := T_f'(z) - \frac{1}{2} T_f^2(z),$$

respectively. We note that the quantity $T_f$ (resp. $S_f$) is analytic on $\Delta$ precisely when $f$ is analytic (resp. meromorphic) and locally univalent on $\Delta$. Since $\mathcal{LU}$ is a vector space over $\mathbb{C}$ (see [6]), thus we can define the norm of $f \in \mathcal{LU}$ by

$$||f|| = \sup_{z \in \Delta} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right|.$$

This norm has significance in the theory of Teichmüller spaces, see [1]. It is known that $||f|| < \infty$ if and only if $f$ is uniformly locally univalent. This means that there exists a constant $\varepsilon := \varepsilon(f) > 0$ such that $f$ is univalent in each disk

$$\left\{ z \in \mathbb{C} : \left| \frac{z - \xi}{1 - \bar{\xi} z} \right| < \varepsilon, \quad |\xi| < 1 \right\},$$

see for example [19]. On the other hand, by the norm $||f||$, we can give univalence criteria for a non–constant meromorphic function $f$ on $\Delta$. Indeed, if $||f|| \le 1$, then $f$ is univalent in $\Delta$ and conversely, if $f$ univalent in $\Delta$, then $||f|| \le 6$ and the equality is attained for the Koebe function and its rotation. In fact, both of these bounds are sharp, see [2]. Also, if $f$ is starlike of order $\gamma \in [0, 1)$, then we have the sharp estimate $||f|| \le 6 - 4\gamma$ (see e.g. [20]). For more details on the geometric and analytic properties of the norm $||f||$ one can refer to [10]. Moreover, many authors have given norm estimates for classical subclasses of univalent functions, see for instance [4, 9, 11, 13, 14, 15, 16, 17, 20].

Let $f$ and $g$ belong to the class $\mathcal{H}$. We say that a function $f$ is subordinate to $g$, written as

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g,$$

if there exists a Schwarz function $\phi : \Delta \to \Delta$ with the following properties

$$\phi(0) = 0 \quad \text{and} \quad |\phi(z)| < 1 \quad (z \in \Delta),$$

such that $f(z) = g(\phi(z))$ for all $z \in \Delta$. In particular, if $g \in \mathcal{U}$, then we have the following geometric equivalence relation

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

We need the following lemmas.

**Lemma 1.1.** (Kargar et al. [7, Lemma 1.1]) *Let $f(z) \in \mathcal{A}$ and $\alpha \in [\pi/2, \pi)$. Then $f \in \mathcal{M}(\alpha)$ if, and only if,*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \mathcal{B}_\alpha(z) \quad (z \in \Delta),$$

*where*

$$\mathcal{B}_\alpha(z) := \frac{1}{2i \sin\alpha} \log\left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right) \quad (z \in \Delta). \tag{1}$$

We note that

$$\mathcal{B}_\alpha(\Delta) = \left\{\zeta \in \mathbb{C} : \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\,\{\zeta\} < \frac{\alpha}{2\sin\alpha}, \quad \frac{\pi}{2} \leq \alpha < \pi\right\} =: \Omega_\alpha.$$

**Lemma 1.2.** (Kargar et al. [8, Lemma 1.1]) *Let $f(z) \in \mathcal{A}$, $0 < \gamma_1, \gamma_2 \leq 1$, $c = e^{\pi i\theta}$ and $\theta = \frac{\gamma_2 - \gamma_1}{\gamma_2 + \gamma_1}$. Then $f \in \mathcal{S}^*_t(\gamma_1, \gamma_2)$ if, and only if,*

$$\frac{zf'(z)}{f(z)} \prec G(z) \quad (z \in \Delta),$$

*where*

$$G(z) := G(\gamma_1, \gamma_2, c)(z) = \left(\frac{1 + cz}{1 - z}\right)^{(\gamma_1+\gamma_2)/2} \quad (G(0) = 1, z \in \Delta). \tag{2}$$

We note that the function $G$ is convex univalent in $\Delta$ and maps $\Delta$ onto $\Omega_{\gamma_1,\gamma_2}$ where

$$\Omega_{\gamma_1,\gamma_2} := \left\{w \in \mathbb{C} : -\frac{\pi\gamma_1}{2} < \arg\{w\} < \frac{\pi\gamma_2}{2}\right\}.$$

In this paper, we give the best possible estimate of the norms of pre–Schwarzian derivatives for the functions $f$ belonging to the subclasses $\mathcal{M}(\alpha)$ and $\mathcal{S}^*_t(\gamma_1, \gamma_2)$.

## 2 Main Results

The first result is continued in the following form.

**Theorem 2.1.** *Let* $\alpha \in [\pi/2, \pi)$. *If a function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{M}(\alpha)$, *then*

$$||f|| \leq \frac{2}{\sin\alpha}\sqrt{\pi^2 + \sin^2\alpha} - \frac{2\sin\alpha}{\sqrt{\pi^2 + \sin^2\alpha}}.$$

*The result is sharp.*

**Proof.** Let $f \in \mathcal{A}$ belongs to the class $\mathcal{M}(\alpha)$ and $\alpha \in [\pi/2, \pi)$. Then by Lemma 1.1 we have

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right) \quad (z \in \Delta). \tag{3}$$

Now by definition of subordination, the relation (3) implies that there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2i\sin\alpha}\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right) \quad (z \in \Delta). \tag{4}$$

By taking the derivative of the logarithmic on both sides of (4), after simplification, we get

$$\begin{aligned}\frac{f''(z)}{f'(z)} &= \frac{1}{2iz\sin\alpha}\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right) \\ &\quad + \frac{2i\sin\alpha w'(z)}{\left(1+\phi(z)e^{i\alpha}\right)\left(1+\phi(z)e^{-i\alpha}\right)\left(2i\sin\alpha + \log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right)}.\end{aligned} \tag{5}$$

It is known that for each Schwarz function $\phi$ we have $|\phi(z)| \leq |z|$ (cf. [5]). Also by the Schwarz–Pick lemma, we have the following inequality for a Schwarz function $\phi$

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \quad (z \in \Delta). \tag{6}$$

On the other hand if log is the principal branch of the complex logarithm, then we have

$$\log z = \ln|z| + i\arg z \quad (z \in \Delta \setminus \{0\}, -\pi < \arg z \leq \pi). \tag{7}$$

Therefore, by the above equation (7), it is well–known that if $|z| \geq 1$, then

$$|\log z| \leq \sqrt{|z-1|^2 + \pi^2}, \tag{8}$$

while for $0 < |z| < 1$, we have

$$|\log z| \leq \sqrt{\left|\frac{z-1}{z}\right|^2 + \pi^2}. \tag{9}$$

Now we consider the two following cases.
**Case 1.** Let

$$\left|\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right| \geq 1.$$

Therefore, by (8) and the inequality $|\phi(z)| \leq |z|$ for all $z \in \Delta$, we obtain

$$\begin{aligned}\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right| &\leq \sqrt{\left|\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}} - 1\right|^2 + \pi^2} \\ &= \frac{\sqrt{4\sin^2\alpha|\phi(z)|^2 + \pi^2\left|1+\phi(z)e^{-i\alpha}\right|^2}}{|1+\phi(z)e^{-i\alpha}|} \\ &\leq \frac{\sqrt{\left(\pi^2 + 4\sin^2\alpha\right)|\phi(z)|^2 + \pi^2\left(1+2|\phi(z)|\right)}}{1-|\phi(z)|} \\ &\leq \frac{\sqrt{\left(\pi^2 + 4\sin^2\alpha\right)|z|^2 + \pi^2\left(1+2|z|\right)}}{1-|z|}. \end{aligned} \tag{10}$$

Using (6), (10) and the triangular inequality, (5) yields that

$$\begin{aligned}
&\left|\frac{f''(z)}{f'(z)}\right| \\
= &\left|\frac{1}{2iz\sin\alpha}\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right. \\
&\left.+\frac{2i\sin\alpha w'(z)}{\left(1+\phi(z)e^{i\alpha}\right)\left(1+\phi(z)e^{-i\alpha}\right)\left(2i\sin\alpha+\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right)}\right| \\
\leq &\frac{1}{2|z|\sin\alpha}\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right| \\
&+\frac{2\sin\alpha|\phi'(z)|}{(1-|\phi(z)|)^2\left(2\sin\alpha-\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right|\right)} \\
\leq &\frac{1}{2|z|\sin\alpha}\frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}{1-|z|} \\
&+\frac{2\sin\alpha}{(1-|\phi(z)|)^2\left(2\sin\alpha-\frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2(1+2|z|)}}{1-|z|}\right)}\cdot\frac{1-|\phi(z)|^2}{1-|z|^2} \\
\leq &\frac{1}{2|z|\sin\alpha}\frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}{1-|z|} \\
&+\frac{2\sin\alpha}{2(1-|z|)\sin\alpha-\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}\cdot\frac{1+|z|}{1-|z|^2}
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
||f|| &= \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \\
&\leq \sup_{z\in\Delta}\left\{\frac{1+|z|}{2|z|\sin\alpha}\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}\right. \\
&\quad\left.+\frac{2(1+|z|)\sin\alpha}{2(1-|z|)\sin\alpha-\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}\right\} \\
&= \frac{2}{\sin\alpha}\sqrt{\pi^2+\sin^2\alpha}-\frac{2\sin\alpha}{\sqrt{\pi^2+\sin^2\alpha}}.
\end{aligned}$$

**Case 2.** Let
$$\left|\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right|<1.$$
Using (9) and simple calculation, we have
$$\begin{aligned}\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right| &\le \sqrt{\left|\frac{\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}-1}{\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}}\right|^2+\pi^2}\\ &= \frac{\sqrt{4\sin^2\alpha|\phi(z)|^2+\pi^2\left|1+\phi(z)e^{i\alpha}\right|^2}}{|1+\phi(z)e^{i\alpha}|}\\ &\le \frac{\sqrt{4\sin^2\alpha|\phi(z)|^2+\pi^2\left(1+|\phi(z)|\right)^2}}{1-|\phi(z)|}\\ &\le \frac{\sqrt{\left(\pi^2+4\sin^2\alpha\right)|z|^2+\pi^2\left(1+2|z|\right)}}{1-|z|}.\end{aligned}$$
Therefore, in this case 2 we have the equal estimate for
$$\left|\log\left(\frac{1+\phi(z)e^{i\alpha}}{1+\phi(z)e^{-i\alpha}}\right)\right|,$$
too. Now, by the same argument, we get the desired result. Some calculations as above show that the result is sharp for the function
$$\begin{aligned}f_1(z) &:= z\exp\left\{\int_0^z \frac{\mathcal{B}_\alpha(t)}{t}\mathrm{d}t\right\} \quad (z\in\Delta)\\ &= z+z^2+\frac{1}{2}(1-\cos\alpha)z^3+\frac{1}{18}(1-9\cos\alpha+8\cos^2\alpha)z^4+\cdots,\end{aligned}$$
or one of its rotations, where $\mathcal{B}_\alpha$ is defined in (1) and $\alpha\in[\pi/2,\pi)$. This is the end of proof. □

**Corollary 2.2.** *As an application of Theorem 2.1 define*
$$h(\alpha):=\frac{2}{\sin\alpha}\sqrt{\pi^2+\sin^2\alpha}-\frac{2\sin\alpha}{\sqrt{\pi^2+\sin^2\alpha}} \quad (\alpha\in[\pi/2,\pi)).$$

*It is easy to see that* $\lim_{\alpha\to\pi^-} h(\alpha) = \infty$. *Also, it is a simple exercise that* $h$ *is an increasing function on* $[\pi/2, \pi)$ *and thus we have*

$$5.9871877\ldots \approx h(\pi/2) \leq h(\alpha) < \infty.$$

*The above estimate means that if* $f \in \mathcal{M}(\alpha)$, *then it is uniformly locally univalent.*

Now, we have the following.

**Theorem 2.3.** *Let* $\gamma_1 \in (0,1]$ *and* $\gamma_2 \in (0,1]$. *If* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{S}_t^*(\gamma_1,\gamma_2)$, *then*

$$||f|| \leq 2(\gamma_1+\gamma_2).$$

*The result is sharp.*

**Proof.** Let $\gamma_1 \in (0,1]$, $\gamma_2 \in (0,1]$ and $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_t^*(\gamma_1,\gamma_2)$. Then by Lemma 1.2 and by definition of subordination, there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\gamma} \quad (\gamma := (\gamma_1+\gamma_2)/2), \tag{11}$$

where $c = e^{\pi i\theta}$ and $\theta = \frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}$. If we take the derivative of logarithmic on both sides of (11), then we get

$$\frac{f''(z)}{f'(z)} = \frac{1}{z}\left[\left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\gamma} - 1\right] + \gamma\left(\frac{c\phi'(z)}{1+c\phi(z)} + \frac{\phi'(z)}{1-\phi(z)}\right).$$

By triangle inequality and (6) and since $|c| = 1$, we obtain

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{|z|}\left[\left(\frac{1+|\phi(z)|}{1-|\phi(z)|}\right)^{\gamma} + 1\right] + 2\gamma\frac{1+|\phi(z)|}{1-|z|^2}. \tag{12}$$

So, using this inequality $|\phi(z)| \leq |z|$, the last inequality (12) yields that

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{|z|}\left[\left(\frac{1+|z|}{1-|z|}\right)^{\gamma} + 1\right] + 2\gamma\frac{1+|z|}{1-|z|^2}. \tag{13}$$

Multiplying the last inequality (13) by $(1-|z|^2)$ gives us

$$\begin{aligned} ||f|| &= \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \\ &\le \sup_{z\in\Delta}\left\{\frac{1-|z|^2}{|z|}\left[\left(\frac{1+|z|}{1-|z|}\right)^{\gamma}+1\right]+2\gamma(1+|z|)\right\} = 4\gamma \end{aligned}$$

and concluding the proof.

For the sharpness, consider the function

$$f_2(z) := z\exp\left\{\int_0^z \frac{G(t)-1}{t}\mathrm{d}t\right\} = z + Az^2 + \cdots \quad (z\in\Delta), \tag{14}$$

where $G$ is defined in (2). It is easy to see that $f_2 \in \mathcal{A}$ and

$$\frac{zf_2'(z)}{f_2(z)} = G(z) \prec G(z) \quad (z\in\Delta)$$

and thus we have $f_2 \in \mathcal{S}_t^*(\gamma_1,\gamma_2)$. With the same argument we get the desired result and thus the details will be omitted. Here the proof ends. □

**Corollary 2.4.** *Let* $\gamma_1 \in (0,1]$, $\gamma_2 \in (0,1]$ *and* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{S}_t^*(\gamma_1,\gamma_2)$*. Since* $f$ *is univalent in* $\Delta$ *and* $2(\gamma_1+\gamma_2) \le 4$*, thus by Theorem 2.3 we have*

$$f \in \mathcal{S}_t^*(\gamma_1,\gamma_2) \Rightarrow ||f|| \le 4 \le 6.$$

*This shows that the Becker and Pommerenke criterion is established [2].*

**Remark 2.5.** It was proved in [8] that

$$\int_0^z \frac{G(t)-1}{t}\mathrm{d}t = \sum_{n=1}^{\infty}\frac{\lambda_n}{n}z^n,$$

is convex univalent in $\Delta$ where

$$\lambda_n := \lambda_n(\gamma_1,\gamma_2,c) = \sum_{k=1}^{n}\binom{n-1}{k-1}\binom{(\gamma_1+\gamma_2)/2}{k}(1+c)^k \quad (n\ge 1)$$

and $c = e^{\pi i\theta}$ and $\theta = \frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1}$. Therefore, we conclude that the function $f_2$ given by (14) is convex univalent in $\Delta$, too.

## References

[1] K. Astala and F.W. Gehring, Injectivity, the BMO norm and the universal Teichmüller space, *J. Anal. Math.* **46** (1986), 16–57.

[2] J. Becker and Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, *J. Reine Angew. Math.* **354** (1984), 74–94.

[3] D.A. Brannan and W.E. Kirwan, On some classes of bounded univalent functions, *J. London Math. Soc.* **1** (1969), 431–443.

[4] J.H. Choi, Y.C. Kim, S. Ponnusamy and T. Sugawa, Norm estimates for the Alexander transforms of convex functions of order alpha, *J. Math. Anal. Appl.* **303** (2005), 661–668.

[5] P.L. Duren, Univalent Functions, *Springer–Verlag*, New York, 1983.

[6] H. Hornich, Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktionen, *Monatsh. Math.* **73** (1969), 36–45.

[7] R. Kargar, A. Ebadian and J. Sokół, Radius problems for some subclasses of analytic functions, *Complex Anal. Oper. Theory* **11** (2017), 1639–1649.

[8] R. Kargar, J. Sokół and H. Mahzoon, On a certain subclass of strongly starlike functions, *arXiv:1811.01271 [math.CV]*

[9] Y.C. Kim, S. Ponnusamy and T. Sugawa, Geometric properties of nonlinear integral transforms of certain analytic functions, *Proc. Japan Acad. Ser. A Math. Sci.* **80** (2004) 57–60.

[10] Y.C. Kim and T. Sugawa, Growth and coefficient estimates for uniformly locally univalent functions on the unit disk, *Rocky Mountain J. Math.* **32** (2002), 179–200.

[11] Y.C. Kim and T. Sugawa, Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions, *Proc. Edinb. Math. Soc.* **49** (2006) 131–143.

[12] H. Mahzoon, Coefficient and Fekete–Szegö problem estimates for certain subclass of analytic and bi–univalent functions, *Filomat* **34** (2020), 4637–4647.

[13] Y. Okuyama, The norm estimates of pre–Schwarzian derivatives of spiral–like functions, *Complex Var. Theory Appl.* **42** (2000) 225–239.

[14] A. Orouji and R. Aghalary, The norm estimates of pre–Schwarzian derivatives of spirallike functions and uniformly convex $\alpha$–spirallike functions, *Sahand Commun. Math. Anal.* **12** (2018), 89–96

[15] R. Parvatham, S. Ponnusamy and S.K. Sahoo, Norm estimates for the Bernardi integral transforms of functions defined by subordination, *Hiroshima Math. J.* **38** (2008), 19–29.

[16] S. Ponnusamy and S.K. Sahoo, Norm estimates for convolution transforms of certain classes of analytic functions, *J. Math. Anal. Appl.* **342** (2008), 171–180.

[17] T. Sugawa, On the norm of the pre–Schwarzian derivatives of strongly starlike functions, *Ann. Univ. Mariae Curie–Skłodowska Sect. A* **52** (1998) 149–157.

[18] N. Takahashi and M. Nunokawa, A certain connection between starlike and convex functions, *Appl. Math. Lett.* **16** (2003), 653–655.

[19] S. Yamashita, Almost locally univalent functions, *Monatsh. Math.* **81** (1976), 235–240.

[20] S. Yamashita, Norm estimates for function starlike or convex of order alpha, *Hokkaido Math. J.* **28** (1999) 217–230.

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