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Efficiency of two-stage systems in stochastic DEA

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Abstract. In the present world, there are many of two-stage systems that their information of inputs, outputs, and intermediate measures are imprecise (like stochastic, fuzzy, interval, etc). In these conditions, two-stage data envelopment analysis (two-stage DEA) cannot evaluate the efficiencies of these systems. In many two-stage systems, the simultaneous presence of the stages is necessary for the final product. Hence, in this paper, firstly we shall propose the stochastic multiplicative model and the deterministic equivalent to measure the efficiencies of these systems in presence of stochastic data under the constant returns to scale (CRS) assumption by using the non-compensatory property of the multiplication operator. Then, we will use the reparative property of additive operation to propose the additive models and the deterministic equivalents to calculate the efficiencies of two-stage systems in presence of stochastic data under the constant returns to scale (CRS) and variable returns to scale (VRS) assumptions that the simultaneous presence of the stages is not necessary for the final product and one stage compensates the another stage's shortcomings. Likewise, we shall convert each of these deterministic equivalents to quadratic programming problems. Based on the proposed stochastic models, the whole system is efficient if and only if the first and the second stages are efficient. At last, we will illustrate in the proposed multiplicative model by using the data of Taiwanese non-life insurance companies that extracted from the extant literature.

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1. Introduction

DEA is a non-parametric mathematical approach that evaluates the efficiency and the performance of decision making units (DMUs). The first time, DEA was presented by Charnes, Cooper, and Rhodes that their first proposed model was called CCR [2]. Afterward, many of models have proposed that measure the efficiency of DMUs by considering DMUs as the black box systems. In variety applications, data may not be precisely such as stochastic data. The stochastic DEA (SDEA) was presented to measure the efficiency of black box systems in presence of stochastic data by extending the classical DEA. In this field, some researchers are presented the Stochastic models (see, e.g, [4], [5], [10], [11]). These authors consider the envelopment form of DEA models and proposed the stochastic DEA models by using the chance constrained programming method. And also, Mirbolouki et al. [14] used the chance constrained programming method and presented a stochastic DEA model based on the multiplier form of DEA that measures the stochastic efficiency of the black box systems. To do this, they solved two problems (exist equally constraint and random variable in the objective function). In the real applications, there are systems with an internal structure such as network systems. Hence, a group of DEA models was presented in order to assess the efficiency of these systems. These models were called Network DEA (NDEA) models (see, e.g, [1], [3], [6], [7], [8], [9], [12], [13], [15]). The special case of network systems is their two-stage systems. Therefore, in this paper, we will combine SDEA and NDEA to propose the stochastic multiplicative and additive models that measure the stochastic efficiency of two-stage systems in presence of stochastic data. Note that in the proposed multiplicative model, the simultaneous presence of stages is necessary in the final product and the shortcoming (default) of one stage is not compensated by another stage. And also, the overall efficiency of the system is considered as a geometric average of the stages efficiencies under CRS assumption which is in the form of the overall efficiency of black box systems. Likewise, this model is not able to calculate the overall efficiency and efficiency of stages under VRS assumption. Hence, in the two-stage systems that the simultaneous presence of the stages is not necessary for the final product, the stochastic additive models can be measured the overall efficiency and efficiency of stages under CRS and VRS assumptions by using the weights that indicate the relative importance of stages. And also, in the proposed additive models the overall efficiency of the system is the arithmetic average of the stages efficiencies. In this case, the first and the second stages are present in evaluating the overall efficiency. This paper is organized as follows: In section 2, we briefly review the Kao and Hwang (2008) and Chen et al. (2009) models that measure the efficiency of two-stage systems. In section 3, firstly we propose the structure of stochastic efficiency of the two-stage systems in presence of stochastic data. Then, we apply the chance-constrained programming method on the Kao and Hwang (2008) model and determine corresponding deterministic equivalent form. And also, the stochastic versions of the Chen et al. (2009)'s models and the deterministic equivalents are presented. Finally, in section 4, the introduced stochastic models are illustrated by a case of 10 Taiwanese non-life insurance companies.

2. preliminaries

In this section, we briefly present the models to evaluate the CRS and VRS efficiency of two-stage systems with deterministic data that presented by Kao and Hwang (2008) and Chen et al. (2009). Suppose there are n DMUs with two-stage structure. Each DMU_j $(j = 1, \dots, n)$ in the stage 1 consumes m input $x_{ij}(i = 1, \dots, m)$ to produce D intermediate measure $z_{dj}(d = 1, \dots, n)$. Then, stage 2, uses D intermediate measure $z_{dj}(d = 1, \dots, n)$ to generate s output $y_{rj}(r = 1, \dots, s)$. The structure of a two-stage system is shown in figure 1.

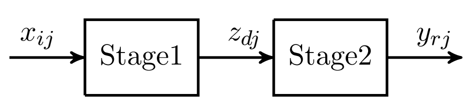

Figure 1. Two-stage system

Kao and Hwang (2008) presented the following model that measures the overall efficiency of the system and the efficiency of stages under CRS assumption, simultaneusly:

$$
\begin{aligned}
E_o^s = \max & \sum_{r=1}^{s} u_r y_{ro} \\
s.t \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \qquad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned}
\tag{1}
$$

If (u^*, v^*, w^*) be an optimal solution of this model, we have:

$$
E_o^s = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad E_o^I = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad E_o^{II} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}.
$$

That E_o^s, E_o^I, E_o^{II} indicate the overall efficiency of the system and efficiency of the first and second stages respectively.

Theorem 2.1. *DMU_o is overall efficient if and only if $E_o^I = E_o^{II} = 1$.*

Proof. Refer to [8] □

Their proposed model cannot measures the VRS efficiency of two-stage systems. Chen et al. (2009) proposed the models that calculate the overall efficiency of the system and efficiency of the stages under CRS and VRS assumptions. The following model is presented to measure the

CRS efficiency of two-stage systems by Chen et al. (2009):

$$
\begin{aligned}
E_o^{(chen-CRS)s} = \max \ & w_1 \frac{\sum_{d=1}^{D} w_d z_{do}}{\sum_{i=1}^{m} v_i x_{io}} + w_2 \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{d=1}^{D} w_d z_{do}} \\
s.t \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{2}
$$

Note that w_1, w_2 are defined as

$$
\begin{aligned}
w_1 &= (\sum_{i=1}^{m} v_i x_i o)/(\sum_{i=1}^{m} v_i x_i o + \sum_{d=1}^{D} w_d z_d o), \\
w_2 &= (\sum_{d=1}^{D} w_d z_d o)/(\sum_{i=1}^{m} v_i x_i o + \sum_{d=1}^{D} w_d z_d o)
\end{aligned}
$$

That demonstrate the relative importance of the stages. Therefore, model (2) can be converted the following form:

$$
\begin{aligned}
E_o^{(chen-CRS)s} = \max \ & \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} \\
s.t \quad & \sum_{i=1}^{m} v_i x_{io} - \sum_{d=1}^{D} w_d z_{do} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{3}
$$

If (u^*, v^*, w^*) be an optimal solution of this model, we have:

$$E_o^{(chen-CRS)s} = \frac{\sum_{r=1}^{s} u_r^* y_{ro} + \sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{d=1}^{D} w_d^* z_{do}}, \quad E_o^I = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}},$$

$$E_o^{II} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}.$$

$E_o^{(chen-CRS)s}$, E_o^I, E_o^{II} indicate the overall efficiency of the system and efficiency of the first and second stages respectively. Also, we have $E_o^{(chen-CRS)s} = w_1 E_o^I + w_2 E_o^{II}$. And also, Chen et al. (2009) proposed a model to compute the efficiency of two-stage system under VRS assumption.
Their proposed model is as follows:

$$\begin{aligned}
E_o^{(chen-CRS)s} = \max \ & w_1 \frac{\sum_{d=1}^{D} w_d z_{do} + u_{01}}{\sum_{i=1}^{m} v_i x_{io}} + w_2 \frac{\sum_{r=1}^{s} u_r y_{ro} + u_{02}}{\sum_{d=1}^{D} w_d z_{do}} \\
s.t \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& u_{01}, u_{02} \ \ free
\end{aligned} \tag{4}$$

By applying w_1, w_2 in this model, the following model is obtained:

$$E_o^{(chen-CRS)s} = \max \ \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02}$$

$$
\begin{aligned}
s.t \quad & \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& u_{01}, u_{02} \ free
\end{aligned} \tag{5}
$$

After solving this model, the overall efficiency of the system and efficiency of the stage 1, 2 ($E_o^{(chen-CRS)s}, E_o^I, E_o^{II}$) can be determined as follows:

$$
E_o^{(chen-CRS)s} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro} + \sum\limits_{d=1}^{D} w_d^* z_{do} + u_{01} + u_{02}}{\sum\limits_{i=1}^{m} v_i^* x_{io} + \sum\limits_{d=1}^{D} w_d^* z_{do}}, \quad E_o^I = \frac{\sum\limits_{d=1}^{D} w_d^* z_{do} + u_{01}}{\sum\limits_{i=1}^{m} v_i^* x_{io}},
$$

$$
E_o^{II} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro} + u_{02}}{\sum\limits_{d=1}^{D} w_d^* z_{do}}.
$$

Therefore, the relationship between $E_o^{(chen-CRS)s}$, E_o^I, E_o^{II} can be defined as follows: $E_o^{(chen-CRS)s} = w_1 E_o^I + w_2 E_o^{II}$.

3. Stochastic efficiency of two-stage systems

In many stitutions, the input, intermediate product and output vectors might be stochastic variables. Therefore, in this case, providing a stochastic model is necessary in order to measure the efficiency of two-stage systems under CRS and VRS assumptions. Suppose we have n DMUs with two-stage structure. Corresponding to the first stage of $DMU_j (j = 1, \ldots, n)$, $\tilde{x}_j$, $\tilde{z}_j$ are the random inputs and intermediate measures vectors. Then, the second stage, consumes these intermediate measures to produce the random output vector $\tilde{y}_j$. With no loss of generality, we suppose that all components of inputs, intermediate measures and output s have normal distribution:

$$\tilde{x}_{ij} \sim N(x_{ij}, \sigma_{ij}^2), \quad \tilde{y}_{rj} \sim N(y_{rj}, \sigma_{rj}^2), \quad \tilde{z}_{dj} \sim N(z_{dj}, \sigma_{dj}^2)$$

Wherein, x_{ij}, y_{rj}, z_{dj} $(i = 1, \ldots, m \quad r = 1, \ldots, s \quad d = 1, \ldots, D)$ are vectors of the expected values of inputs, intermediate measures and outputs of $DMU_j(j = 1, \ldots, n)$.

3.1. Stochastic efficiency of multiplicative model

In this subsection, firstly, we will propose a stochastic model of by using the multiplicative model. Then, the deterministic equivalent of the proposed model will be provided. The stochastic model that measures the efficiencies of the two-stage systems under CRS can be described as follows:

$$\begin{aligned}
E_o^s = \max \ & \sum_{r=1}^{s} u_r \tilde{y}_{ro} \\
s.t \quad & p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1\right\} \geq (1-\alpha) \\
& p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0\right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& p\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0\right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{6}$$

In this model, p means probability. And also, α indicates the level of error that is predetermined. In the objective function of model (6), there is random variable and also, we have the following wrong expression

$$p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1\right\} = 0 \geq (1-\alpha) \Rightarrow \alpha \geq 1$$

for the first constraint of model (6). Thus we consider an alternative form of model (1) and also we replace this constraint by (1) and also we replace this constraint by $p\{\sum_{i=1}^{m} v_i\tilde{x}_{io} \leq 1\} \geq (1-\alpha)$ (refer to Mirbolouki et al. [14]). Therefore, the following model can be constructed as an alternative form of the proposed model [6]:

$$
\begin{aligned}
&\tilde{E}_o^{s'} = \max\ k \\
&s.t \quad p\left\{\sum_{r=1}^{s} u_r\tilde{y}_{ro} \geq k\right\} \geq (1-\alpha) \\
&\qquad p\left\{\sum_{i=1}^{m} v_i\tilde{x}_{io} \leq 1\right\} \geq (1-\alpha) \\
&\qquad p\left\{\sum_{r=1}^{s} u_r\tilde{y}_{rj} - \sum_{d=1}^{D} w_d\tilde{z}_{dj} \leq 0\right\} \geq (1-\alpha), \quad j=1,\dots,n \\
&\qquad p\left\{\sum_{d=1}^{D} w_d\tilde{z}_{dj} - \sum_{i=1}^{m} v_i\tilde{x}_{ij} \leq 0\right\} \geq (1-\alpha), \quad j=1,\dots,n \\
&\qquad u_r, w_d, v_i \geq 0 \quad r=1,\dots,s \quad d=1,\dots,D \quad i=1,\dots,m
\end{aligned}
\tag{7}
$$

Note that the model (6) and model (7) , have equal objective values. Hence, we have the following theorem:

Theorem 3.1. *In the model* (6) *and model* (7)*, we have:* $\tilde{E}_o^{s} = \tilde{E}_o^{s'}$

Proof. Suppose S, S' indicate the feasible regions related to the models (6) and (7) respectively. Note that $S \subseteq S'$ and $k \leq \sum_{i=1}^{m} v_i\tilde{x}_{io} \leq 1$. Thus, according to the theorem 2.1 1 of Mirbolouki et al. [14], the proof is obvious. □

3.1.1. Deterministic equivalent of model (7)

In this subsection, we will exhibit a deterministic equivalent of model (7) using Cooper et al. (2004). Firstly, consider the following constraint:

$$p\left\{\sum_{i=1}^{m} v_i\tilde{x}_{io} \leq 1\right\} \geq (1-\alpha).$$

In order to achieve the equality constraint, we define $\zeta_1 > 0$ as external slack:

$$p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} \geq k\right\} = (1-\alpha) + \zeta_1.$$

Thus, there is $S_1 > 0$ such that

$$p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k \geq s_1\right\} = (1-\alpha).$$

Note that $\zeta_1 = 0$ if and only if $s_1 = 0$. And also, by defining $\zeta_2 > 0$ as an external slack, we have

$$p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1\right\} = (1-\alpha) + \zeta_2.$$

Hence there is $s_2 > 0$ such that

$$p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1 + s_2\right\} = (1-\alpha).$$

Corresponding to other constraints, we suppose there are $s_{3j}, s_{4j} > 0$ such that

$$p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq s_{3j}\right\} = (1-\alpha), \quad j = 1, \ldots, n$$
$$p\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq s_{4j}\right\} = (1-\alpha), \quad j = 1, \ldots, n$$

Now, we set:

$$E(\tilde{x}_{ij}) = x_{ij}, \quad E(\tilde{y}_{rj}) = y_{rj}, \quad E(\tilde{z}_{dj}) = z_{dj},$$
$$E(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k) = \sum_{r=1}^{s} u_r y_{ro} - k$$

Hence: $p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k \geq s_1\right\} = (1-\alpha)$ conclude that

$$p\left\{\sum_{r=1}^{s} u_r^* \tilde{y}_{ro} - k\right\} \leq s_1 = \alpha$$

Thus,

$$p\left\{\frac{(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k) - (\sum_{r=1}^{s} u_r y_{ro} - k)}{\sqrt{\operatorname{var}(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k)}} \leq \frac{s_1 - \left(\sum_{r=1}^{s} u_r y_{ro} - k\right)}{\sqrt{\operatorname{var}\left(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k\right)}}\right\} = \alpha \tag{8}$$

By considering Φ as standard normal distribution function, we recall that $p(\tilde{Z} \leq z) = \alpha \Rightarrow \Phi(z) = \alpha \Rightarrow \Phi^{-1}(\alpha) = z$ Hence, (8) can be converted to

$$\frac{s_1 - (\sum_{r=1}^{s} u_r y_{ro} - k)}{\sqrt{\operatorname{var}(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k)}} = \Phi^{-1}(\alpha).$$

In order to simplify, we denote

$$(\sigma^o(k,u))^2 = \operatorname{var}(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k) = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \operatorname{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o})$$

$$(\sigma^I(k,v))^2 = \operatorname{var}(1 - \sum_{i=1}^{m} v_i \tilde{x}_{io}) = \operatorname{var}(\sum_{i=1}^{m} v_i \tilde{x}_{io})$$
$$= \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \operatorname{cov}(\tilde{x}_{io}, \tilde{x}_{i'o})$$

$$(\sigma_j(w,u))^2 = \operatorname{var}(\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{r=1}^{s} u_r \tilde{y}_{rj}) = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \operatorname{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j})$$
$$+ \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \operatorname{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) - 2\operatorname{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj})$$

$$(\sigma'_j(v,w))^2 = \mathrm{var}(\sum_{i=1}^{m} v_i \tilde{x}_{ij} - \sum_{d=1}^{D} w_d \tilde{z}_{dj}) = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j})$$
$$+ \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j}) - 2\mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj})$$

Therefore, $\frac{s_1 - (\sum_{r=1}^{s} u_r y_{ro} - k)}{\sigma^o(k,u)} = \Phi^{-1}(\alpha)$. By applying the same approach for other constraints, the deterministic equivalent form of model (7) will be as follows:

$$\begin{aligned}
&\tilde{E}_o^{s'} = \max\ k \\
&s.t \quad \sum_{r=1}^{s} u_r y_{ro} - k + \Phi^{-1}(\alpha)\sigma^o(k,u) = s_1 \\
&\qquad \sum_{i=1}^{m} v_i x_{io} - \Phi^{-1}(\alpha)\sigma^I(k,v) + s_2 = 1 \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma_j(w,u) + s_{3j} = 0 \quad j = 1,\dots,n \\
&\qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\sigma'_j(v,w) + s_{4j} = 0 \quad j = 1,\dots,n \\
&\qquad u_r, w_d, v_i \geq 0 \quad r = 1,\dots,s \quad d = 1,\dots,D \quad i = 1,\dots,m \\
&\qquad s_{3j}, s_{4j} \geq 0 \ \ j = 1,\dots,n \\
&\qquad s_1, s_2 \geq 0
\end{aligned} \tag{9}$$

Note that this model is a nonlinear programming. Thus, follow Cooper et al. (2004), we transform this model to a quadratic programming problem. For this purpose we use the non-negative variables λ, λ', λ_j, λ'_j and obtain the quadratic programming problem as follows:

$$\begin{aligned}
&\tilde{E}_o^{s'} = \max\ k \\
&s.t \quad \sum_{r=1}^{s} u_r y_{ro} - k + \Phi^{-1}(\alpha)\lambda - s_1 = 0
\end{aligned}$$

$$
\begin{aligned}
&\sum_{i=1}^{m} v_i x_{io} - \Phi^{-1}(\alpha)\lambda' + s_2 = 1 \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\lambda_j + s_{3j} = 0 \quad j = 1, \ldots, n \\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\lambda'_j + s_{4j} = 0 \quad j = 1, \ldots, n \\
\lambda^2 &= \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) \\
{\lambda'}^2 &= \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{io}, \tilde{x}_{i'o}) \\
\lambda_j^2 &= \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) \\
&\quad - 2\mathrm{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj}) \\
{\lambda'_j}^2 &= \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j}) \\
&\quad - 2\mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj}) \\
&u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
&\lambda, \lambda', \lambda_j, \lambda'_j, s_{3j}, s_{4j} \geq 0 \;\; j = 1, \ldots, n \\
&s_1, s_2 \geq 0
\end{aligned} \tag{10}
$$

Theorem 3.2. *For $\alpha \in (0, 0.5]$ and any optimal solution $(u_r^*, w_d^*, v_i^*, \lambda^*, {\lambda'}^*, s_1^*, s_2^*, s_{3j}^*, s_{4j}^*)$ be an optimal solution, we have $0 < \tilde{E}^{{s'}^*} \leq 1$.*

Proof. If $\alpha \in (0, 0.5]$, then $\Phi^{-1}(\alpha) \leq 0$. In each optimal solution, we have:

$$\begin{cases} \sum\limits_{r=1}^{s} u_r^* y_{rj} - \sum\limits_{d=1}^{D} w_d^* z_{dj} \leq 0 \\ \sum\limits_{d=1}^{D} w_d^* z_{dj} - \sum\limits_{i=1}^{m} v_i^* x_{ij} \leq 0 \end{cases} \Rightarrow \sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} \leq 0$$

And also, based on the constraints $\sum\limits_{r=1}^{s} u_r^* y_{ro} \geq k$, $\sum\limits_{i=1}^{m} v_i^* x_{io} \leq 1$ of model (10) the proof is complete. □

Now for $\alpha \in (0, 0.5]$ and any optimal solution $(u_r^*, w_d^*, v_i^*, \lambda^*, \lambda'^*, s_1^*, s_2^*, s_{3j}^*, s_{4j}^*)$ of model (10), the overall efficiency and efficiency of the first and the second stages are defined as

$$\tilde{E}_o^{s'} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro}}{\sum\limits_{i=1}^{m} v_i^* x_{io}}, \quad \tilde{E}_o^{I} = \frac{\sum\limits_{d=1}^{D} w_d^* z_{do}}{\sum\limits_{i=1}^{m} v_i^* x_{io}}, \quad \tilde{E}_o^{II} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro}}{\sum\limits_{d=1}^{D} w_d^* z_{do}}.$$

Thus, we have: $\tilde{E}_o^{s'} = \tilde{E}_o^{I} \cdot \tilde{E}_o^{II}$.

Lemma 3.3. *For $\alpha \in (0, 0.5]$ and each DMU_o, we have: $0 < \tilde{E}_o^{I} \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$.*

Proof. In any optimal solution $(u_r^*, w_d^*, v_i^*, \lambda^*, \lambda'^*, s_1^*, s_2^*, s_{3j}^*, s_{4j}^*)$ of the model (10), for $j = o$, we have

$$\sum_{r=1}^{s} u_r^* y_{ro} - \sum_{d=1}^{D} w_d^* z_{do} - \Phi^{(-1)}(\alpha)\lambda_o^* + s_{3o}^* = 0,$$
$$\sum_{d=1}^{D} w_d^* z_{do} - \sum_{i=1}^{m} v_i^* x_{io} - \Phi^{(-1)}(\alpha)\lambda_o'^* + s_{4o}^* = 0.$$
$$\Phi^{(-1)}(\alpha), \lambda_o^*, \lambda_o'^* \geq 0, S_{3o}^*, s_{4o}^* \geq 0$$

Thus, in any optimal solution, we have:

$$\sum_{r=1}^{s} u_r^* y_{ro} - \sum_{d=1}^{D} w_d^* z_{do} \leq 0, \quad \sum_{d=1}^{D} w_d^* z_{do} - \sum_{i=1}^{m} v_i^* x_{io} \leq 0$$

These constraints mean that $0 < \tilde{E}_o^I \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$ and the proof is complete. $\square$

Lemma 3.4. *For $\alpha \in (0, 0.5]$, DMU_o, is stochastic overall efficient under the model (10) if and only if the first and the second stages are stochastic efficient, i.e. $\tilde{E}_o^{s'} = 1$ if and only if $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$.*

Proof. Suppose $\tilde{E}_o^{s'} = 1$, i.e $\sum_{r=1}^{s} u_r^* y_{ro} = \sum_{i=1}^{m} v_i^* x_{io}$. And also, for $\alpha \in (0, 0.5]$, we have

$$\sum_{r=1}^{s} u_r^* y_{ro} - \sum_{d=1}^{D} w_d^* z_{do} \leq 0, \quad \sum_{d=1}^{D} w_d^* z_{do} - \sum_{i=1}^{m} v_i^* x_{io} \leq 0$$

Therefore, we conclude that $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$. Conversely, if $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$, the proof is obviuse. $\square$

3.2. Stochastic efficiency of additive models

In this subsection, the stochastic versions of the additive models will be presented in presence of stochastic data. Then, the deterministic equivalent forms of these stochastic models are obtained. The proposed model of the previous section, cannot able to calculate the efficiency of two-stage system under VRS assumption. Thereby, follow Chen et al. (2009) we provide the stochastic models that measure the efficiency of the two-stage systems under CRS and VRS assumptions respectively. Our proposed models are as follows:

$$\begin{aligned}
\tilde{E}_o^{s(CRS)} &= \max \ \sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \\
s.t \quad & P\left\{ \sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} = 1 \right\} \geq (1-\alpha) \\
& P\left\{ \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0 \right\} \geq (1-\alpha), \quad j = 1, \ldots, n
\end{aligned} \tag{11}$$

$$P\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0\right\} \geq (1-\alpha), \quad j=1,\dots,n$$
$$u_r, w_d, v_i \geq 0 \quad r=1,\dots,s \quad d=1,\dots,D \quad i=1,\dots,m$$

$$\begin{aligned}
\tilde{E}_o^{s(VRS)} &= \max \ \sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} + u_{01} + u_{02} \\
s.t \quad & P\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} = 1\right\} \geq (1-\alpha) \\
& P\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} + u_{02} \leq 0\right\} \geq (1-\alpha), \quad j=1,\dots,n \\
& P\left\{\sum_{d=1}^{d} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} + u_{01} \leq 0\right\} \geq (1-\alpha), \quad j=1,\dots,n \\
& u_r, w_d, v_i \geq 0 \quad r=1,\dots,s \quad d=1,\dots,D \quad i=1,\dots,m \\
& u_{01}, u_{02} \ free
\end{aligned} \tag{12}$$

Note that in these models, p means probability. And also, amount of α is predetermined that determines the level of error. In the he objective functions of model (11) and model (12), there is random variable and also, we have the following wrong expression

$$p\left\{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{d=1}^{D} w_d^* z_{do} = 1\right\} = 0 \geq (1-\alpha) \Rightarrow \alpha \geq 1$$

(refer to Mirbolouki et al. [14]). Thus we obtain an alternative form of model (11) as follows:

$$\begin{aligned}
\tilde{E}_o^{s(CRS)'} &= \max \ k \\
s.t \quad & P\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \geq k\right\} \geq (1-\alpha) \\
& P\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \leq 1\right\} \geq (1-\alpha)
\end{aligned}$$

$$P\left\{\sum_{r=1}^{s} u_r\tilde{y}_{rj} - \sum_{d=1}^{D} w_d\tilde{z}_{dj} \le 0\right\} \ge (1-\alpha), \quad j=1,\ldots,n$$
$$P\left\{\sum_{d=1}^{D} w_d\tilde{z}_{dj} - \sum_{i=1}^{m} v_i\tilde{x}_{ij} \le 0\right\} \ge (1-\alpha), \quad j=1,\ldots,n$$
$$u_r, w_d, v_i \ge 0, \quad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m \tag{13}$$

Theorem 3.5. *For model (11) and model (13), we have* $\tilde{E}_o^{s(CRS)} = \tilde{E}_o^{s(CRS)'}$

Proof. The proof is similar to the proof of theorem (3.1). □

3.2.1. Deterministic equivalent of model (13)

In this section, the deterministic equivalent of model (13) can be obtained as deterministic equivalent of model (7). Therefore, by using the similar procedure, the deterministic equivalent form of model (13) is obtained as follows:

$$\tilde{E}_o^{s(CRS)'} = \max \ k$$
$$s.t \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{dj} - k + \Phi^{-1}(\alpha)\sigma^o(k,u,w) = s'_1$$
$$\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma^I(k,v,w) + s'_2 = 1$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma_j(w,u) + s'_{3j} = 0 \quad j=1,\ldots,n$$
$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\sigma'_j(v,w) + s'_{4j} = 0 \quad j=1,\ldots,n \tag{14}$$
$$u_r, w_d, v_i \ge 0, \quad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m$$
$$s'_{3j}, s'_{4j} \ge 0 \ \ j=1,\ldots,n$$
$$s'_1, s'_2 \ge 0$$

That:

$$(\sigma^o(k,u,w))^2 = \text{var}(\sum_{r=1}^{s} u_r\tilde{y}_{ro} + \sum_{d=1}^{D} w_d\tilde{z}_{do} - k) = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'}\text{cov}(\tilde{y}_{ro},\tilde{y}_{r'o})$$
$$+\sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\text{cov}(\tilde{z}_{do},\tilde{z}_{d'o}) + 2\text{cov}((\sum_{r=1}^{s} u_r\tilde{y}_{ro}),(\sum_{d=1}^{D} w_d\tilde{z}_{do} - k))$$
$$(\sigma^I(k,v,w))^2 = \text{var}(1-(\sum_{i=1}^{m} v_i\tilde{x}_{io} + \sum_{d=1}^{D} w_d\tilde{z}_{do})) = \text{var}(\sum_{i=1}^{m} v_i\tilde{x}_{io} + \sum_{d=1}^{D} w_d\tilde{z}_{do}) =$$
$$\sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'}\text{cov}(\tilde{x}_{io},\tilde{x}_{i'o})\sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\text{cov}(\tilde{z}_{do},\tilde{z}_{d'o}) + \text{cov}(\sum_{i=1}^{m} v_i\tilde{x}_{io},\sum_{d=1}^{D} w_d\tilde{z}_{do})$$
$$(\sigma_j(w,u))^2 = \text{var}(\sum_{d=1}^{D} w_d\tilde{z}_{dj} - \sum_{r=1}^{s} u_r y_{rj}) = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'}\text{cov}(\tilde{y}_{rj},\tilde{y}_{r'j})$$
$$+\sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\text{cov}(\tilde{z}_{dj},\tilde{z}_{d'j}) - 2\text{cov}(\sum_{d=1}^{D} w_d\tilde{z}_{dj},\sum_{r=1}^{s} u_r\tilde{y}_{ro})$$
$$(\sigma'_j(w,u))^2 = \text{var}(\sum_{i=1}^{m} v_i\tilde{x}_{ij} - \sum_{d=1}^{D} w_d\tilde{z}_{dj}) = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\text{cov}(\tilde{z}_{dj},\tilde{z}_{d'j})$$
$$+\sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'}\text{cov}(\tilde{x}_{ij},\tilde{x}_{i'j}) - 2\text{cov}(\sum_{i=1}^{m} v_i\tilde{x}_{ij},\sum_{d=1}^{D} w_d\tilde{z}_{dj})$$

This model is a nonlinear programming. In order to convert this model to a quadratic programming problem, the non-negative variables γ, γ', γ_j, γ'_j are introduced. Therefore the following quadratic programming problem is obtained:

$$\tilde{E}_o^{(chen-CRS)} = \max\ k$$
$$s.t \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} - k + \Phi^{-1}(\alpha)\gamma - s'_1 = 0$$
$$\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} - \Phi^{-1}(\alpha)\gamma' + s'_2 = 1$$

$$
\begin{aligned}
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\gamma_j + s'_{3j} = 0 \quad j = 1, \dots, n \\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\gamma'_j + s'_{4j} = 0 \quad j = 1, \dots, n \\
\gamma^2 &= \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o}) \\
&+ 2\mathrm{cov}((\sum_{r=1}^{s} u_r \tilde{y}_{ro}), (\sum_{d=1}^{D} w_d \tilde{z}_{do} - k)) \\
{\gamma'}^2 &= \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{io}, \tilde{x}_{i'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o}) \\
&+ \mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{io}, \sum_{d=1}^{D} w_d \tilde{z}_{do}) \qquad (15) \\
\gamma_j^2 &= \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) \\
&- 2\mathrm{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj}) \\
{\gamma'_j}^2 &= \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j}) \\
&- 2\mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj}) \\
&u_r, w_d, v_i \geq 0 \quad r = 1, \dots, s \quad d = 1, \dots, D \quad i = 1, \dots, m \\
&\gamma_j, \gamma'_j, s'_{3j}, s'_{4j} \geq 0 \;\; j = 1, \dots, n \\
&s'_1, s'_2, \gamma, \gamma' \geq 0
\end{aligned}
$$

Theorem 3.6. *For* $\alpha \in (0, 0.5]$, *if* $(u_r^*, w_d^*, v_i^*, \gamma^*, {\gamma'}^*, \gamma_j^*, {\gamma'_j}^*, {s'_1}^*, {s'_2}^*, {s'_{3j}}^*, {s'_{4j}}^*)$ *be an optimal solution of model* (15), *we have* $0 < \tilde{E}_o^{(chen-CRS)^*} \leq 1$.

Proof. The proof is similar to the proof of theorem 3.2. $\square$
Now, if $(u_r^*, w_d^*, v_i^*, \gamma^*, \gamma'^*, \gamma_j^*, \gamma_j'^*, s_1'^*, s_2'^*, s_{3j}'^*, s_{4j}'^*)$ be an optimal solution of model (15), for, the efficiency of the first and the second stages are defined:

$$\tilde{E}_o^I = \sum_{d=1}^{D} w_d^* z_{do} / \sum_{i=1}^{m} v_i^* x_{io}, \tilde{E}_o^{II} = \sum_{r=1}^{s} u_r^* y_{ro} / \sum_{d=1}^{D} w_d^* z_{do}.$$

Therefore, there is $\lambda \in (0,1)$ that $\tilde{E}_o^{(chen-CRS)} = \lambda \tilde{E}_o^I + (1-\lambda)\tilde{E}_o^{II}$.

Lemma 3.7. *For $\alpha \in (0, 0.5]$ and each DMU_o, we have: $0 < \tilde{E}_o^I \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$.*

Proof. The proof is similar to the proof of lemma 3.3. $\square$

Lemma 3.8. *For $\alpha \in (0, 0.5]$, DMU_o is stochastic overall efficient under the model (15) if and only if the first and the second stages are stochastic efficient, i.e. $\tilde{E}_o^{(chen-CRS)} = 1$ if and only if $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$.*

Proof. The proof is similar to the proof of lemma 3.4. $\square$
By applying the aforementioned manner to the model (12), the deterministic equivalent form of this model can be obtained as follows that is a nonlinear programming:

$$\begin{aligned}
&\tilde{E}_o^{(chen-VRS)} = \max\ k \\
&s.t \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} - k + \Phi^{-1}(\alpha)\sigma^o(k, u, w, u_{01}, u_{02}) = s_1'' \\
&\qquad \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} - \Phi^{-1}(\alpha)\sigma^I(k, v, w) + s_2'' = 1 \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma_j(w, u, u_{02}) + s_{3j}'' = 0 \quad j = 1, \ldots, n \\
&\qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} - \Phi^{-1}(\alpha)\sigma_j'(v, w, u_{01}) + s_{4j}'' = 0 \quad j = 1, \ldots, n \\
&\qquad u_r, w_d, v_i \geq 0, \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{16}$$

$$s''_{3j}, s''_{4j} \geq 0, \quad j = 1, \ldots, n$$
$$s''_1, s''_2 \geq 0$$
$$u_{01}, u_{02} \; free$$

We use the non-negative variables $\eta, \eta', \eta_j, \eta'_j \geq 0$ in order to achieve a quadratic programming problem. Therefore the quadratic programming problem is as follows:

$$\tilde{E}_o^{(chen-VRS)} = \max \; k$$

$$s.t \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} - k + \Phi^{-1}(\alpha)\eta = s''_1$$

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} - \Phi^{-1}(\alpha)\eta' + s''_2 = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} - \Phi^{-1}(\alpha)\eta_j + s''_{3j} = 0 \quad j = 1, \ldots, n$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} - \Phi^{-1}(\alpha)\eta'_j + s''_{4j} = 0 \quad j = 1, \ldots, n$$

$$\eta^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o})$$
$$+ 2\mathrm{cov}((\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_{01} + u_{02}), (\sum_{d=1}^{D} w_d \tilde{z}_{do} - k))$$

$$\eta'^2 = \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{io}, \tilde{x}_{i'o})$$
$$+ \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o}) + \mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{io}, \sum_{d=1}^{D} w_d \tilde{z}_{do}) \tag{17}$$

$$\eta_j^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'=1} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j})$$
$$- 2\mathrm{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj} + u_{02})$$

$$\eta_j'^2 = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j})$$
$$- 2\mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj} - u_{01})$$
$$u_r, w_d, v_i \geq 0 \quad r = 1, \dots, s \quad d = 1, \dots, D \quad i = 1, \dots, m$$
$$\eta, \eta', \eta_j, \eta_j', s_{3j}'', s_{4j}'' \geq 0 \;\; j = 1, \dots, n$$
$$s_1'', s_2'', \geq 0$$
$$u_{01}, u_{02} \; free$$

After solving this model for $\alpha \in (0, 0.5]$, we define the efficiencies of the system as follows:

$$\tilde{E}_o^{(chen-VRS)} = \frac{\sum_{r=1}^{s} u_r^* y_{ro} + \sum_{d=1}^{D} w_d^* z_{do} + u_{01}^* + u_{02}^*}{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{d=1}^{D} w_d^* z_{do}}$$

$$\tilde{E}_o^{I} = \frac{\sum_{d=1}^{D} w_d^* z_{do} + u_{02}^*}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad \tilde{E}_o^{II} = \frac{\sum_{r=1}^{s} u_r^* y_{ro} + u_{01}^*}{\sum_{d=1}^{D} w_d^* z_{do}}.$$

Wherein $\tilde{E}_o^{(chen-VRS)}$, $\tilde{E}_o^{I}$, $\tilde{E}_o^{II}$ indicate the stochastic overall efficiency and the stochastic efficiency of stage 1, 2 respectively. Therefore, there is $\lambda \in (0, 1)$ that $\tilde{E}_o^{(chen-VRS)} = \lambda \tilde{E}_o^{I} + (1-\lambda)\tilde{E}_o^{II}$.

Lemma 3.9. *For $\alpha \in (0, 0.5]$ and each DMU_o, we have: $0 < \tilde{E}_o^{I} \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$.*

Proof. The proof is similar to the proof of lemma 3.3. □

Lemma 3.10. *For $\alpha \in (0, 0.5]$, DMU_o is stochastic overall efficient under the model (15) if and only if the first and the second stages are stochastic efficient, i.e. $\tilde{E}_o^{(chen-VRS)} = 1$ if and only if $\tilde{E}_o^{I} = \tilde{E}_o^{II} = 1$.*

Proof. The proof is similar to the proof of lemma 3.8. □

Finally, we note that the results of the efficiency of the system and stages are in range $(0, 1]$ in all of the proposed stochastic models for $\alpha \in (0, 0.5]$. If $\alpha \in (0.5, 1)$, may be efficiencies are negative or greater than 1. It must be noted that in many cases the models (10), (15), (17) have multiple optimal solutions. Thus, in these models, the overall efficiency decomposition will not be individual. Hence, we cannot able to compare the stages efficiency of different DMUs together in each model. Therefore, follow Kao and Hwang (2008) approach, we suppose the efficiency of stage1, is the most important stage form the point of view of the decision maker (DM) and compute the maximum efficiency of stage 1, while the overall efficiency of system is unchanged. Then, we calculate the maximum efficiency of stage 2, while the efficiency of stage 1 and the overall efficiency of system are unchanged.

4. Case study

In this section, we will illustrate the deterministic equivalent form of the proposed stochastic model (7) on 10 Taiwanese non-life insurance companies with data in 2000, 2001 and 2002 years (that extracted from Kao and Hwang (2014)). Each company has a two-stage structure. Table 1, shows Inputs Intermediate measures and outputs that we use to illustrate the proposed models:

Table 1. The data of case study

Inputs	Intermediate measures	outputs
Operating expenses (X1)	Direct written premiums (Z1)	Underwriting profit (Y1)
Insurance expenses (X2)	Reinsurance premiums (Z2)	Investment profit (Y2)

4.1. Results of model (8)

Table 2 show the obtained efficiencies of model (8).

Table 2. Stochastic efficiency obtained from model (8)

DMU_s	Efficiency of stage 1	Efficiency of stage 2	Overall efficiency
1	0.94	0.4	0.37
2	0.72	0.59	0.43
3	0.54	1	0.54
4	0.8	0.24	0.19
5	0.49	0.23	0.11
6	0.37	0.59	0.22
7	0.46	0.26	0.12
8	0.38	0.65	0.25
9	0.47	0.34	0.16
10	0.55	0.46	0.25

The results computed by GAMS software and have been summarized in Table 2, by assuming $\alpha = 0.45$. In Table 2, first column is the number of each DMU. The stochastic efficiency of stage 1 and stage 2 and the overall efficiency are listed in the Columns 2 and 3 and 4 of Table 2, respectively. As it is clear in 2, all of DMU_s are inefficient. Between the inefficient DMU_s, DMU_3, DMU_5 with scores 0.54, 0.11 have the best and the lowest overall efficiency. Also, DMU_3 is efficient in stage 2. The highest efficiency belongs to DMU_1 in stage 1 and DMU_5 in stage 2 with scores 0.94, 0.65, respectively. DMU_6 in stage 1 and DMU_5 in stage 2 with efficiency scores 0.37, 0.23 have the lowest efficiency.

5. Conclusion

In the practice, there are many systems with internal structures such as network systems. NDEA is employed to evaluate the performance of the network systems in presence of deterministic data. The special case of network systems are their two-stage systems that the first stage consumes the inputs to produce the intermediate measures, then these intermediate measures consume to generate the outputs of the second stage. In practice, the observations of inputs, intermediate measures, and outputs are imprecise and they can be considered as stochastic data. Hence, SDEA is a useful method for measuring the efficiency of black box systems with stochastic data. Mirbolouki et al. [14] proposed a stochastic model that evaluates the efficiency of a black box system based on multiplier form of DEA. Therefore, in this paper, by using the non-compensatory property of the multiplication operator and the compensatory property of the additive operator, we extended NDEA and SDEA models and proposed the SNDEA models for computing the stochastic efficiencies of the two-stage systems in presence of stochastic data based on the multiplicative and additive models. Then, for our proposed stochastic models, we obtained the deterministic equivalent forms and converted these deterministic forms into the quadratic programming problems. Likewise, we showed that the obtained efficiencies of these models are positive for $\alpha \in (0, 0.5]$. Also, the proposed stochastic model (7) is illustrated on a set of data from 10 Taiwanese non-life insurance companies in 2000, 2001 and 2002 years that studied in Kao and Hwang (2008) by using GAMS software. For future study, this work can be extended to non- radial DEA models for measuring the efficiency of a two-stage system in presence of stochastic data and ranking them that has not this weakness of efficiencies for $\alpha \in (0.5, 1)$.

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