Note on the Covariance Coset of the Moore-Penrose Inverses in $C^*$-Algebras

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Abstract. We will introduce and study several algebraic properties of the covariance cosets in $C^*$-algebras. Indeed, we will characterize the covariance coset in terms of commutators. Also, we will show that for an invertible element $b$, the covariance coset of $b^{-1}$ coincides with the covariance coset of $b^*$. Moreover, if $b$ is normal, then the covariance coset of $b$ coincides with the covariance coset of $b^*$. In addition, we will prove that the covariance coset is a cone.

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1. Introduction

The concept of a generalized inverse seems to have been first mentioned in 1903 by Fredholm and the class of all pseudoinverses was characterized in 1912 by Hurwitz [2]. Generalized inverses of differential and integral operators thus antedated the generalized inverses of matrices, whose existence was first introduced and studied by Moore [13, 14] during the years 1910-1920. This notion was rediscovered by Penrose [9] in 1955, and is nowadays called the Moore-Penrose inverse. In recent years, generalized inverses and their properties have received a lot of attention (see for example [2, 3, 4, 10], and the references therein). The notion of generalized inverses in $C^*$-algebra, was introduced in the seminal paper by Harte and Mbekhta [4]. Harte and Mbekhta have shown that many important properties of Moore-Penrose inverses in $C^*$-algebra. A part of

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literature has been dedicated to weighted Moore-Penrose inverse, spectral theory, closed range operators, linear preserver problems and numerical computations in the area of optimization, statistics and ill-posed problem, etc. (see [2, 5, 10, 14]). All these reasons have convinced many mathematicians, to start research in this rich and important branch of mathematics.

Throughout this paper \( \mathcal{A} \) be a unital \( C^\ast \)-algebra. An element \( a \in \mathcal{A} \) is called regular if it has a generalized inverse (in the sense of von Neumann) in \( \mathcal{A} \), i.e., there exists \( b \in \mathcal{A} \) such that

\[
aba = a.
\]

An element \( a \in \mathcal{A} \) is Moore-Penrose invertible if there exists \( b \in \mathcal{A} \) such that

\[
aba = a, \quad bab = b, \quad (ab)^\ast = ab \quad \text{and} \quad (ba)^\ast = ba.
\]

It is well known that each regular element in a \( C^\ast \)-algebra, has the Moore-Penrose inverse (denoted by MP– inverse from now on.). Generally MP– inverse is uniquely determined in \( \mathcal{A} \) if it exists. We will denote the MP–inverse of \( a \) by \( a^\dagger \).

In the following, we will denote by \( \mathcal{A}^{-1} \) and \( \mathcal{A}^\dagger \) the set of all invertible and MP– invertible elements of \( \mathcal{A} \), respectively. An element \( a \) in \( \mathcal{A} \) is called idempotent if \( a^2 = a \). A projection \( p \in \mathcal{A} \) satisfies \( p = p^\ast = p^2 \).

It should be noticed that if \( x \in \mathcal{A}^\dagger \), then \( xx^\dagger \) and \( x^\dagger x \) are projections. Moreover,

\[
(x x^\dagger)^\dagger = x x^\dagger \quad \text{and} \quad (x^\dagger x)^\dagger = x^\dagger x.
\]

The commutator of a pair of elements \( x \) and \( y \) in \( \mathcal{A} \) is defined by

\[
[x, y] = xy - yx.
\]

Obviously \( [x, y] = 0 \) if and only if \( x \) and \( y \) commute.

Assume that \( a \) is an element in \( \mathcal{A}^{-1} \). Its inverse \( a^{-1} \) is covariant with respect to \( \mathcal{A}^{-1} \), i.e., for all \( b \in \mathcal{A}^{-1} \) we have

\[
(ba b^{-1})^{-1} = ba^{-1} b^{-1}.
\]
In general, the elements of $A^\dagger$ are not covariant under $A^{-1}$ (see [1]). For a given element $a \in A^\dagger$ with MP–inverse $a^\dagger$, we will denote the covariance set by $C(a)$ and define

$$C(a) = \{ b \in A^{-1} : (bab^{-1})^\dagger = ba^\dagger b^{-1} \}. \quad (1)$$

Covariance set was studied by [1], [6], [11] and [13]. In this note we introduce the notion of covariance coset of the Moore-Penrose inverses in $C^*$-algebras. In fact we define this set by reversing the roles of $a$ and $b$ in $C(a)$ and denote it by $B(b)$. i.e.,

$$B(b) = \{ a \in A^\dagger : (bab^{-1})^\dagger = ba^\dagger b^{-1} \}. \quad (2)$$

The notion of covariance coset was introduced by Robinson in [12] for matrices. In this paper we will characterize the covariance coset in terms of commutators. Also, we will show that for an invertible element $b \in A$ we have $B(b^{-1}) = B(b^*)$. Moreover, if $b$ is normal, then $B(b) = B(b^{-1}) = B(b^*)$. In Proposition 4 and related corollaries we will describe the main properties of the covariance coset in $C^*$-algebras. We conclude the results by showing that for any non-zero scalar $\lambda$, $B(b) = B(\lambda b)$.

2. Main Results

In the following proposition we characterize $B(b)$ in terms of commutators.

**Proposition 2.1.** Assume $b \in A^{-1}$. Then the following statements are equivalent:

(i) $a \in B(b)$;

(ii) $[a^\dagger a, b^*b] = 0$ and $[aa^\dagger, b^*b] = 0$.

**Proof.** (i) $\Rightarrow$ (ii): Suppose $a \in B(b)$. Therefore $ba^\dagger b^{-1}$ is the MP–inverse of $bab^{-1}$. Thus, $(ba^\dagger ab^{-1})^* = ba^\dagger ab^{-1}$. Therefore, $(b^*)^{-1} a^\dagger ab^*b = ba^\dagger a$. From here one can conclude that $[a^\dagger a, b^*b] = 0$. In a similar manner from $(baa^\dagger b^{-1})^* = ba^\dagger ab^{-1}$ we get that $[aa^\dagger , b^*b] = 0$. 


(ii) $\Rightarrow$ (i): Since $a^\dagger$ is MP–inverse of $a$, it suffices to show that $(ba^\dagger ab^{-1})^* = ba^\dagger ab^{-1}$ and $(baa^\dagger b) = baa^\dagger b$. By the assumptions $[a^\dagger a, b^* b] = 0$. From this we obtain $(b^*)^{-1}a^\dagger ab^* b = ba^\dagger b$. Thus, $(ba^\dagger ab^{-1})^* = ba^\dagger ab^{-1}$. In a similar manner from $[aa^\dagger, b^* b] = 0$ we get $(baa^\dagger b^{-1})^* = baa^\dagger b^{-1}$.

Note that if $a \in A^\dagger$ with MP–inverse $a^\dagger$ and $b \in A^{-1}$, then from the above proposition and Lemma 2.1 in [1], we conclude that

$$b \in C(a) \text{ if and only if } a \in B(b).$$

Also, we remark that $C(a) \subset A^{-1} \subset B(b) \subset A^\dagger$ for all $a \in A^\dagger$ and for each $b \in A^{-1}$.

**Proposition 2.2.** Assume that $b \in A^{-1}$. Then $B(b^*) = B(b^{-1})$.

**Proof.** By Proposition 2.1.

$$a \in B(b^*) \text{ if and only if } [a^\dagger a, bb^*] = 0 \text{ and } [aa^\dagger, bb^*] = 0.$$  

This is equivalent to

$$a^\dagger abb^* = bb^* a^\dagger a \quad \text{and} \quad aa^\dagger bb^* = bb^* aa^\dagger. \quad (3)$$

Multiply (3) from left and right by $(bb^*)^{-1}$, we get

$$[a^\dagger a, (b^{-1})^* b^{-1}] = 0 \quad \text{and} \quad [aa^\dagger, (b^{-1})^* b^{-1}] = 0. \quad (4)$$

Again Proposition 1.1, shows that (4) holds if and only if $a \in B(b^{-1})$. □

**Proposition 2.3.** Assume that $b \in A^{-1}$ and $b$ is normal. Then $B(b) = B(b^{-1})$.

**Proof.** Since $b$ is normal, the equality is an immediate consequence of Proposition 2.2. □

We recall that a set $K \subset A$ is a cone if $x \in K$ implies $\lambda x \in K$ for each $\lambda \geq 0$. Also, an element $a \in A$ is called simply polar [4] if it has a commuting generalized inverse, that is, there exists a generalized inverse $c$ of $a$, such that $[a, c] = 0$. 
In the following proposition we collect some main properties of the covariance coset.

**Proposition 2.4.** Assume that \( b \in \mathcal{A}^{-1} \). Then, the following statements are equivalent:

(i) \( a \in \mathcal{B}(b) \);
(ii) \( a^\dagger \in \mathcal{B}(b) \);
(iii) \( a^* \in \mathcal{B}(b) \);
(iv) \( aa^\dagger \in \mathcal{B}(b) \) and \( a^\dagger a \in \mathcal{B}(b) \);
(v) \( \lambda a \in \mathcal{B}(b) \) for any non-zero scalar \( \lambda \).

**Proof.** First we show that (i) and (ii) are equivalent: By Proposition 1.1, \( a \in \mathcal{B}(b) \) if and only if

\[
\begin{bmatrix}
    a^\dagger a, b^* b
\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}
    aa^\dagger, b^* b
\end{bmatrix} = 0.
\]

(5)

Since \( (a^\dagger)^\dagger = a \). Thus (5) is equivalent to

\[
\begin{bmatrix}
    a^\dagger (a^\dagger)^\dagger, b^* b
\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}
    (a^\dagger)^\dagger a^\dagger, b^* b
\end{bmatrix} = 0.
\]

(6)

Again, Proposition 1.1, shows that (6) holds if and only if \( a^\dagger \in \mathcal{B}(b) \).

(i) \( \iff \) (iii): In a similar manner, since \( a^\dagger a \) and \( aa^\dagger \) are normal elements we infer that \( a^\dagger a = a^* (a^*)^\dagger \) and \( aa^\dagger = (a^*)^\dagger a^* \) by applying Proposition 1.1, we get \( a \in \mathcal{B}(b) \iff a^* \in \mathcal{B}(b) \).

(i) \( \implies \) (iv): \( a \in \mathcal{B}(b) \) if and only if (5) holds. Since \( aa^\dagger = aa^\dagger aa^\dagger \) and \( (aa^\dagger)^\dagger = aa^\dagger \). Thus (5) is equivalent to

\[
\begin{bmatrix}
    aa^\dagger (aa^\dagger)^\dagger, b^* b
\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}
    (aa^\dagger)^\dagger aa^\dagger, b^* b
\end{bmatrix} = 0.
\]

(7)

This implies that \( aa^\dagger \in \mathcal{B}(b) \). Similarly we get

\[
\begin{bmatrix}
    a^\dagger a (a^\dagger a)^\dagger, b^* b
\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}
    (a^\dagger a)^\dagger a^\dagger a, b^* b
\end{bmatrix} = 0.
\]

(8)

Thus, \( a^\dagger a \in \mathcal{B}(b) \).

For the proof of (iv) \( \implies \) (i); It is easy to verify that (iv) satisfies if and only if (5) and (8) hold. These together imply (5), that is, (i) holds.
(i) \(\iff\) (v): Since \(\lambda \neq 0\), \((\lambda a)^\dagger = \frac{1}{\lambda} a^\dagger = (a\lambda)^\dagger\). Now applying the Proposition 1.1, we obtain the result. \(\Box\)

**Corollary 2.5.** If \(b \in A^{-1}\), then \(B(b)\) is a cone.

It is well known that every normal element is simply polar. Hence

**Corollary 2.6.** If \(a\) is normal, then

\[ a \in B(b) \iff aa^\dagger \in B(b) \iff a^\dagger a \in B(b).\]

**Proposition 2.7.** Assume that \(b \in A^{-1}\) and \(\lambda \neq 0\) is any scalar. Then \(B(b) = B(\lambda b)\).

**Proof.** By Proposition 1.1, \(a \in B(b)\) if and only if (5) satisfies which is equivalent to

\[
\left[ a^\dagger a, (\lambda b)^* (\lambda b) \right] = 0 \quad \text{and} \quad \left[ aa^\dagger, (\lambda b)^* (\lambda b) \right] = 0.
\]

This holds if and only if \(a \in B(\lambda b)\). \(\Box\)

**References**


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