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Original Research Paper

Developing an Iterative Method to Solve Two- and Three-Dimensional Mixed Volterra-Fredholm Integral Equations

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Abstract. In this paper, an iterative method is extended to solve nonlinear two- and three-dimensional mixed Volterra-Fredholm integral equations. We consider a nonlinear operator of these integral equations and then develop the iterative method which was introduced in [J MATH ANAL APPL. 316 (2006) 753-763] to solve them. Convergence property of the suggested schemes are proved under some mild assumptions. In both cases, numerical examples are given to compare the performance of the proposed method with some existing methods.

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1 Introduction

Two- and three-dimensional nonlinear mixed Volterra-Fredholm integral equations (VFIE) of the second kind arise in the theory of the nonlinear parabolic boundary value problems and the mathematical model of the spatiotemporal development of various physical, mechanical, and biological problems [5, 15]. Several algorithms have been proposed for solving VFIE. In [7], a hybrid Legendre block-pulse method is used to solve mixed Volterra-Fredholm integral equations. In the case of two-dimensional VFIE, Hadizadeh et al. in [6] obtained a numerical solution of Volterra-Fredholm integral equations of mixed type using the bivariate Chebyshev collocation approach. A continuous time collocation method for the linear VFIE with discrete convergence properties was presented in [8]. Also, Banifatemi et al. [3] introduced a method for solving VFIE using two-dimensional Legendre wavelets. Two methods based on Adomian decomposition series for approximation of the solution of VFIE are presented in [10, 16]. The nonlinear systems of mixed Volterra-Fredholm integral equations are solved numerically by Maleknejad and Fadaei Yami [9]. Assari and Dehghan [2] introduced a numerical scheme to solve two-dimensional nonlinear Volterra integral equations of the second kind based on Galerkin method and moving least squares (MLS) approach. Furthermore, some researchers investigated numerical schemes to solve three-dimensional nonlinear mixed Volterra-Fredholm integral equations. Mirzaee et al. [12] obtained a numerical solution for three-dimensional nonlinear mixed Volterra-Fredholm integral equations via three-dimensional block-pulse functions. Also, Ziqan et al. [17] used the reduced differential transform method for solving the three-dimensional Volterra integral equations.

This paper is organized as follows: In Section 2, we review an iterative method for solving nonlinear functional equations. In Section 3, an extension of the iterative method is presented for two-dimensional nonlinear mixed Volterra-Fredholm integral equations. We use the iterative method to solve nonlinear three-dimensional Volterra-Fredholm integral equations in Section 4. Finally, some conclusions are given in Section 5.

2 Review of an iterative method

Daftardar-Gejji and Jafari [4] introduced an iterative technique to solve nonlinear functional equations. Consider the following general functional equation

$$y = N(y) + g, \tag{1}$$

where N is a nonlinear operator in Banach space and g is a known function. An iterative method for solving (1) is as follows [4]:

$$\begin{aligned} &y_0 = g, \\ &y_1 = N(y_0), \\ &\vdots \\ &y_{m+1} = N(y_0 + \cdots + y_m) - N(y_0 + \cdots + y_{m-1}), \qquad m = 1, 2, \cdots \end{aligned} \tag{2}$$

Definition 2.1. *Let $\mathbb{B}$ be Banach space. The nonlinear operator $N : \mathbb{B} \longrightarrow \mathbb{B}$ is a contraction, if for any $x, y \in \mathbb{B}$ there exists a constant $0 < L < 1$ such that*

$$\|N(x) - N(y)\| \leq L\|x - y\|.$$

The constant L is called the contraction coefficient [13].

Theorem 2.2 (Banach's Fixed Point Theorem [13])**.** *Let $\mathbb{B}$ be Banach space and N be a contraction on $\mathbb{B}$. Then, there exists a unique $x^* \in \mathbb{B}$ such that $N(x^*) = x^*$.*

Based on Theorem 2.2, Daftardar-Gejji and Jafari prove that

$$y = g + \sum_{i=1}^{\infty} y_i,$$

is convergent to the solution of functional equation (1).

3 Two-dimensional integral equations

Consider the nonlinear two-dimensional Volterra-Fredholm integral equation (`2DVFIE`)

$$f(x,t) = g(x,t) + \int_a^x \int_a^b \mathcal{H}\big(x, t, y, s, f(y,s)\big) ds dy, \quad x \in [a, b), \tag{3}$$

where $f(x,t) \in L^2\big(\Omega\big)$ is an unknown function, the function $g(x,t) \in L^2\big(\Omega\big)$ and $\mathcal{H}\big(x,t,y,s,f(y,s)\big) \in L^2\big(\Omega^2 \times \mathbb{R}\big)$ are given and $\Omega = [a,b) \times [a,b)$. From the iterative scheme (2), we get

$$
\begin{aligned}
f_0(x,t) &= g(x,t),\\
f_1(x,t) &= \int_a^x \int_a^b \mathcal{H}\big(x,t,y,s,f_0(y,s)\big)dsdy,\\
f_2(x,t) &= \int_a^x \int_a^b \mathcal{H}\big(x,t,y,s,f_0(y,s)+f_1(y,s)\big)dsdy\\
&\quad - \int_a^x \int_a^b \mathcal{H}\big(x,t,y,s,f_0(y,s)\big)dsdy,\\
&\;\;\vdots\\
f_{m+1}(x,t) &= \int_a^x \int_a^b \mathcal{H}\big(x,t,y,s,f_0(y,s)+\cdots+f_m(y,s)\big)dsdy\\
&\quad - \int_a^x \int_a^b \mathcal{H}\big(x,t,y,s,f_0(y,s)+\cdots+f_{m-1}(y,s)\big)dsdy, \qquad (4)\\
&\qquad\qquad\qquad\qquad m = 2,3,\ldots
\end{aligned}
$$

Hence, we can obtain the solution of the nonlinear two-dimensional Volterra-Fredholm integral equation (3) as

$$f(x,t) = g(x,t) + \sum_{i=1}^{\infty} f_i(x,t).$$

Now, to establish the uniformly convergent (4) to the solution (3), we consider the following standard assumptions.

(H1) The nonlinear function $\mathcal{H}$ is Lipschitz continuous, i.e., there exist a constant $0 < L_1 < 1$ such that

$$\Big|\mathcal{H}\big(x,t,y,s,\Phi_1\big) - \mathcal{H}\big(x,t,y,s,\Phi_2\big)\Big| \le L_1|\Phi_1 - \Phi_2|, \qquad \forall \Phi_1, \Phi_2 \in L^2(\mathbb{R}).$$

(**H2**) The function $\mathcal{H}\big(x,t,y,s,\Phi\big)$ is bounded, i.e., there exists a constant $B_1 > 0$ such that

$$\Big|\mathcal{H}\big(x,t,y,s,\Phi\big)\Big| \le B_1, \qquad \forall \Phi \in L^2(\mathbb{R}).$$

(**H3**) There exists a constant $\alpha > 0$ such that

$$|x-a| \le \alpha, \qquad \forall x \in [a,b).$$

Theorem 3.1. *Suppose that **(H1)**-**(H3)** hold. Then,*

$$\sum_{i=1}^{\infty} f_i(x,t),$$

is uniformly convergent to the solution of the nonlinear two-dimensional Volterra-Fredholm integral equation (3).

Proof. From **(H2)**–**(H3)** and (4), we get

$$\begin{aligned}
\big|f_1(x,t)\big| &= \left|\int_a^x \int_a^b \mathcal{H}\big(x,t,y,s,f_0(y,s)\big)dsdy\right| \\
&\le \int_a^x \int_a^b \Big|\mathcal{H}\big(x,t,y,s,g(y,s)\big)\Big|dsdy \\
&\le B_1 \int_a^x \int_a^b dsdy = B_1(b-a)(x-a) \\
&\le B_1\alpha(b-a) := M_1.
\end{aligned}$$

Also, **(H1)**–**(H3)** result in

$$\begin{aligned}
\big|f_2(x,t)\big| &= \Big|\int_a^x \int_a^b \Big[\mathcal{H}\big(x,t,y,s,f_0(y,s)+f_1(y,s)\big) \\
&\quad - \mathcal{H}\big(x,t,y,s,f_0(y,s)\big)\Big]dsdy\Big| \\
&\le \int_a^x \int_a^b \Big|\mathcal{H}\big(x,t,y,s,f_0(y,s)+f_1(y,s)\big) \\
&\quad - \mathcal{H}\big(x,t,y,s,f_0(y,s)\big)\Big|dsdy \\
&\le L_1 \int_a^x \int_a^b \big|f_1(y,s)\big|dsdy \\
&\le B_1L_1 \int_a^x \int_a^b (y-a)(b-a)dsdy \\
&= B_1L_1(b-a)^2\frac{(x-a)^2}{2} \\
&\le \frac{B_1}{L_1}\frac{\Big(\alpha L_1(b-a)\Big)^2}{2!} := M_2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\big|f_{m+1}(x,t)\big| &= \Big|\int_a^x\int_a^b \Big[\mathcal{H}\big(x,t,y,s,f_0(y,s)+\cdots+f_m(y,s)\big) \\
&\quad - \mathcal{H}\big(x,t,y,s,f_0(y,s)+\cdots+f_{m-1}(y,s)\big)\Big]dsdy\Big| \\
&\leq L_1\int_a^x\int_a^b \big|f_m(y,s)\big|dsdy \\
&\leq \frac{B_1}{L_1}\frac{\Big(\alpha L_1(b-a)\Big)^{m+1}}{(m+1)!} := M_{m+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{\infty} f_i(x,t) &\leq \frac{B_1}{L_1}\frac{\alpha L_1(b-a)}{1!} + \frac{B_1}{L_1}\frac{\Big(\alpha L_1(b-a)\Big)^2}{2!} \\
&\quad + \frac{B_1}{L_1}\frac{\Big(\alpha L_1(b-a)\Big)^3}{3!} + \cdots \\
&= \frac{B_1}{L_1}\Big(e^{\alpha L_1(b-a)}-1\Big) := M^*.
\end{aligned}$$

Therefore, $\sum_{i=1}^{\infty} f_i(x,t)$ is absolutely and uniformly convergent to the solution of 2DVFIE. □

3.1 Numerical experiments for 2DVFIE

To compare numerical results of the iterative method (4) with some existing methods to solve the two dimensional Volterra-Fredholm integral equations, we consider three examples in which their exact solutions are available. These examples have been solved using the proposed method and the results have been reported in Tables 1-3. In order to analyze the error of the new method, we introduce the absolute error at the selected points of the given interval as

$$e_{i+1}(x,t) = |f(x,t) - f_i(x,t)|, \qquad i = 0,1,2,\cdots \qquad \forall (x,t)\in\Omega,$$

where $f(x,t)$ denotes the exact solution and $f_i(x,t)$ is the approximate solution of 2DVFIE. Also, to show the accuracy and efficiency of the new method, we compare the absolute error obtained from our method with absolute error achieved from operational matrix method (OM)[11] and the method of multiquadric (MQs) radial basis functions [1]. Our codes are implemented in Matlab 2017 programming environment on a 2.3Hz Intel core i3 processor laptop and 4GB of RAM.

Example 3.2. Consider the following two-dimensional linear Volterra-Fredholm integral equation of the second kind [1, 11]

$$f(x,t) = g(x,t) - \int_0^x \int_0^1 t^2 e^{-s} f(y,s)\,ds\,dy, \qquad 0 \le x \le 1,$$

where

$$g(x,t) = \frac{1}{3}x^2(3e^t + xt^2).$$

The exact solution is $f(x,t) = x^2e^t$. The absolute error obtained by the new method, OM and MQs are reported in Table 1.

Table 1: Absolute error for Example 3.2

Nodes	OM [11] $m_1 = m_2 = 16$	MQs [1] $N = 5$, $c = 1.9$	New method		
(x,t)			$e_1(x,t)$	$e_2(x,t)$	$e_3(x,t)$
$(0,0)$	0	$2.4656E-05$	0	0	0
$(0.1,0.1)$	$8.5538E-04$	$1.4696E-05$	$3.3333E-06$	$1.3383E-08$	$4.2988E-11$
$(0.2,0.2)$	$6.5403E-04$	$3.3702E-04$	$1.0666E-04$	$8.5654E-07$	$5.5025E-09$
$(0.3,0.3)$	$7.2516E-04$	$2.4509E-03$	$8.1000E-04$	$9.7566E-06$	$9.4016E-08$
$(0.4,0.4)$	$1.0695E-03$	$1.0058E-02$	$3.4133E-03$	$5.4819E-05$	$7.0432E-07$
$(0.5,0.5)$	$3.5914E-04$	$3.0542E-02$	$1.0416E-02$	$2.0911E-04$	$3.3584E-06$
$(0.6,0.6)$	$1.2525E-03$	$7.5896E-02$	$2.5920E-02$	$6.2442E-04$	$1.2034E-05$
$(0.7,0.7)$	$1.0928E-03$	$1.6355E-01$	$5.6023E-02$	$1.5745E-03$	$3.5403E-05$
$(0.8,0.8)$	$1.2101E-03$	$3.1748E-01$	$1.0922E-01$	$3.5084E-03$	$9.0153E-05$
$(0.9,0.9)$	$1.6056E-03$	$5.6960E-01$	$1.9683E-01$	$7.1125E-03$	$2.0561E-04$

Example 3.3. Consider the following two-dimensional nonlinear Volterra-Fredholm integral equation of the second kind [1, 11]

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 t^2 e^{-4y} f^2(y,s)\,ds\,dy, \quad 0 \le x \le 1,$$

where

$$g(x,t) = x^2e^{2t} + \frac{e^4-1}{512}t^2e^{-4x}\Big(12x - 3e^{4x} + 24x^2 + 32x^3 + 32x^4 + 3\Big).$$

The exact solution is $f(x,t) = x^2e^{2t}$. The absolute error of desired methods are given in Table 2.

Table 2: Absolute error for Example 3.3

Nodes	OM [11] $m_1 = m_2 = 16$	MQs [1] $N = 5$, $c = 0.6$	New method		
(x,t)			$e_1(x,t)$	$e_2(x,t)$	$e_3(x,t)$
$(0,0)$	0	$6.4387E-03$	0	0	0
$(0.1,0.1)$	$9.3910E-04$	$6.9441E-03$	$1.9233E-07$	$5.5982E-11$	$5.5977E-11$
$(0.2,0.2)$	$6.3588E-04$	$4.3133E-02$	$1.7728E-05$	$3.0233E-08$	$3.0218E-08$
$(0.3,0.3)$	$6.6109E-04$	$9.8257E-02$	$2.1893E-04$	$9.2761E-07$	$9.2645E-07$
$(0.4,0.4)$	$1.0241E-03$	$1.6816E-01$	$1.1899E-03$	$8.8489E-06$	$8.8296E-06$
$(0.5,0.5)$	$1.6976E-04$	$2.6325E-01$	$4.1339E-03$	$4.4737E-05$	$4.4597E-05$
$(0.6,0.6)$	$1.2434E-03$	$3.8837E-01$	$1.0838E-02$	$1.5208E-04$	$1.5147E-04$
$(0.7,0.7)$	$1.1309E-03$	$5.3340E-01$	$2.3440E-02$	$3.9481E-04$	$3.9294E-04$
$(0.8,0.8)$	$1.4257E-03$	$6.9169E-01$	$4.4095E-02$	$8.4481E-04$	$8.4029E-04$
$(0.9,0.9)$	$2.1739E-03$	$8.8598E-01$	$7.4676E-02$	$1.5654E-03$	$1.5563E-03$

Example 3.4. Consider the following two-dimensional linear Volterra-Fredholm integral equation of the second kind [1, 11]

$$f(x,t) = g(x,t) + \int_0^x \int_0^1 (2s-1)e^y f(y,s)dsdy, \quad 0 \le x \le 1,$$

where

$$g(x,t) = \sin x + t - \frac{1}{6}e^x + \frac{1}{6}.$$

The exact solution is $f(x,t) = \sin x + t$. The absolute error obtained from the new method, OM and MQs are reported in Table 3.

The iterative method (4) for solving two dimensional mixed Volterra-Fredholm integral equations has many benefits some of which are listed as follows:

Table 3: Absolute error for Example 3.4

Nodes	OM [11] $m_1 = m_2 = 16$	MQs [1] $N = 5,\, c = 2$	New method		
(x,t)			$e_1(x,t)$	$e_2(x,t)$	$e_3(x,t)$
$(0,0)$	0	$1.9281E-05$	0	0	0
$(0.1,0.1)$	$1.9057E-02$	$3.5065E-02$	$1.7528E-02$	$1.4733E-3$	$1.4733E-03$
$(0.2,0.2)$	$5.0202E-02$	$7.3797E-02$	$3.6900E-02$	$6.2382E-03$	$6.2382E-03$
$(0.3,0.3)$	$7.7732E-02$	$1.1661E-01$	$5.8310E-02$	$1.4823E-02$	$1.4823E-02$
$(0.4,0.4)$	$2.0232E-01$	$1.6394E-01$	$8.1971E-02$	$2.7761E-02$	$2.7761E-02$
$(0.5,0.5)$	$5.2930E-01$	$2.1625E-01$	$1.0812E-01$	$4.5568E-02$	$4.5568E-02$
$(0.6,0.6)$	$3.9886E-01$	$2.7404E-01$	$1.3702E-01$	$6.8723E-02$	$6.8723E-02$
$(0.7,0.7)$	$3.8816E-01$	$3.3791E-01$	$1.6895E-01$	$9.7635E-02$	$9.7635E-02$
$(0.8,0.8)$	$4.6088E-01$	$4.0851E-01$	$2.0426E-01$	$1.3261E-01$	$1.3261E-01$
$(0.9,0.9)$	$9.3261E-01$	$4.8654E-01$	$2.4327E-01$	$1.7378E-01$	$1.7378E-01$

- It has a simple structure and new iteration can be obtained without computational complex operations.

- From Tables 1-3, after three iteration the approximate solutions approach to the exact answer with high accuracy.

4 Three-dimensional integral equations

In this section, we suppose the nonlinear three-dimensional Volterra-Fredholm integral equation (`3DVFIE`) as

$$u(x,y,z) = v(x,y,z) + \int_0^x \int_0^1 \int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u(s,t,r)\big)drdtds,$$

where $u(x,y,z)$ is an unknown function and the function $v(x,y,z)$ and $\mathcal{K}\big(x,y,z,s,t,r,u(s,t,r)\big)$ are known functions. Similar to (4), we intro-

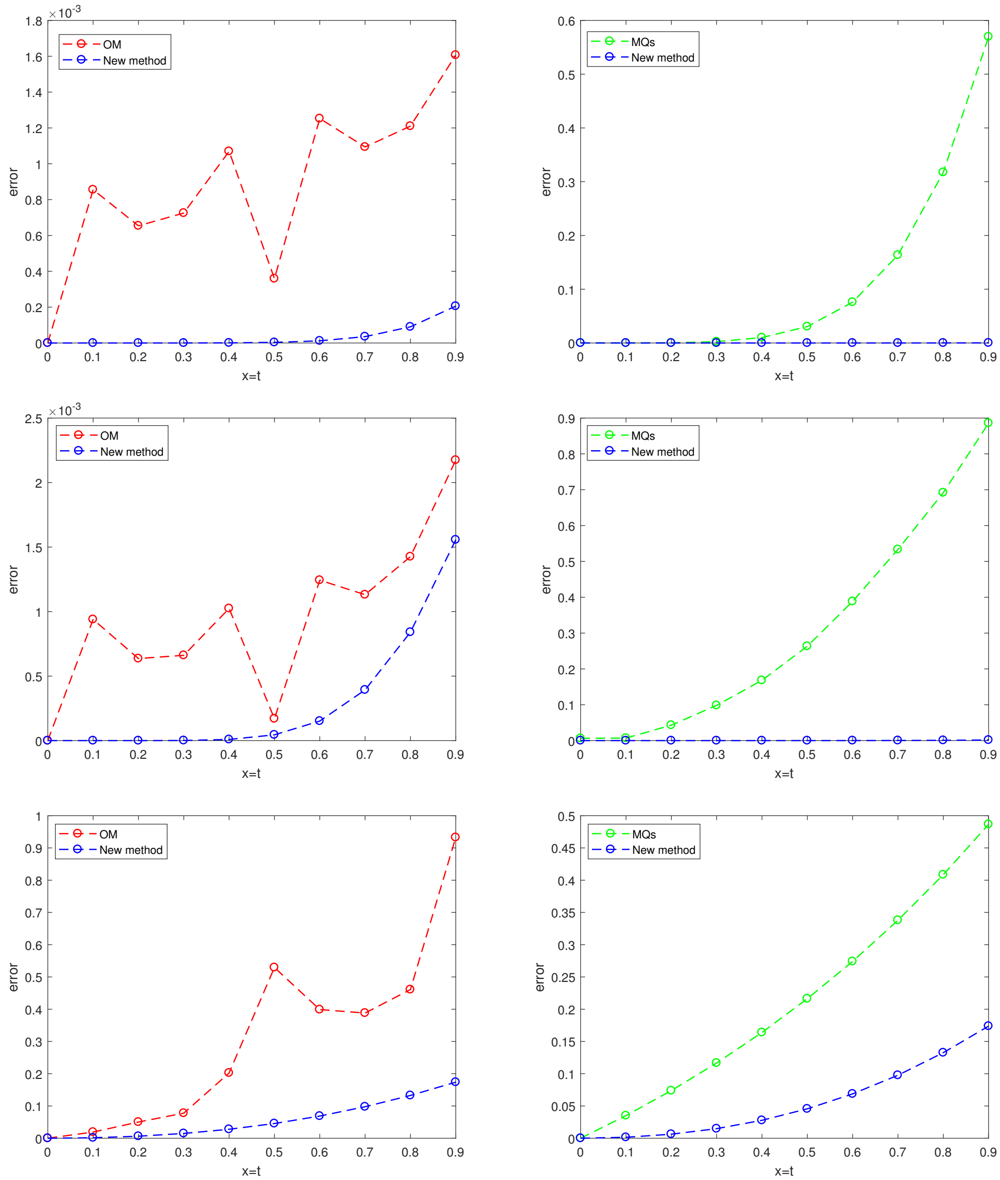

Figure 1: Comparing the absolute error of the new method, $e_3(x,t)$, with OM and MQs for Example 3.2 (top), Example 3.3 (middle), Example 3.4 (down).

duce an iterative method to solve `3DVFIE`. Take

$$
\begin{aligned}
u_0(x,y,z) &= v(x,y,z),\\
u_1(x,y,z) &= \int_0^x\int_0^1\int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)\big)drdtds,\\
u_2(x,y,z) &= \int_0^x\int_0^1\int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)+u_1(s,t,r)\big)drdtds\\
&\quad -\int_0^x\int_0^1\int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)\big)drdtds,\\
&\ \vdots\\
u_{m+1}(x,y,z) &= \int_0^x\int_0^1\int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)\\
&\quad +\cdots+u_m(s,t,r)\big)drdtds\\
&\quad -\int_0^x\int_0^1\int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)\\
&\quad +\cdots+u_{m-1}(s,t,r)\big)drdtds, \quad m=2,3,\dots
\end{aligned}
\tag{5}
$$

Hence, the approximate solution to the nonlinear three-dimensional Volterra-Fredholm integral equation is

$$u(x,y,z)=v(x,y,z)+\sum_{i=1}^{\infty}u_i(x,y,z).$$

Similar to Theorem 3.1, to establish the uniformly convergent (5) to the solution of `3DVFIE`, we consider the following assumptions.

(H4) The nonlinear function $\mathcal{K}$ is Lipschitz continuous, i.e., there exists a constant $0<L_2<1$ such that for all $\Psi_1,\Psi_2\in L^2(\mathbb{R})$

$$\Big|\mathcal{K}\big(x,y,z,s,t,r,\Psi_1\big)-\mathcal{K}\big(x,y,z,s,t,r,\Psi_2\big)\Big|\le L_2|\Psi_1-\Psi_2|.$$

(**H5**) The function $\mathcal{K}\big(x,y,z,s,t,r,\Psi\big)$ is bounded, i.e., there exists a constant $B_2>0$ such that

$$\Big|\mathcal{K}\big(x,y,z,s,t,r,\Psi\big)\Big|\le B_2,\qquad \forall\Psi\in L^2(\mathbb{R}).$$

Theorem 4.1. *Suppose that **(H3)**-**(H5)** hold. Then,*

$$\sum_{i=1}^{\infty} u_i(x,y,z),$$

is uniformly convergent to the solution of nonlinear `3DVFIE`*.*

Proof. The proof of this theorem is similar to Theorem 3.1. Hence, we omit some details. From **(H3)**, **(H5)** and (5), we have

$$\begin{aligned}\big|u_1(x,y,z)\big| &= \left|\int_0^x\int_0^1\int_0^1 \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)\big)drdtds\right|\\ &\le B_2\int_0^x\int_0^1\int_0^1 drdtds = B_2x\\ &\le B_2\alpha := D_1.\end{aligned}$$

Now, **(H4)** gives us

$$\begin{aligned}\big|u_2(x,y,z)\big| = \Big|&\int_0^x\int_0^1\int_0^1 \Big[\mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)+u_1(s,t,r)\big)\\ &- \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)\big)\Big]drdtds\Big|\\ &\le L_2\int_0^x\int_0^1\int_0^1 \Big|u_1(s,t,r)\Big|drdtds\\ &\le B_2L_2\int_0^x\int_0^1\int_0^1 sdrdtds\\ &\le \frac{B_2}{L_2}\frac{\Big(\alpha L_2\Big)^2}{2!} := D_2.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\big|u_{m+1}(x,y,z)\big| = \Big|&\int_0^x\int_0^1\int_0^1 \Big[\mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)+\cdots+u_m(s,t,r)\big)\\ &- \mathcal{K}\big(x,y,z,s,t,r,u_0(s,t,r)+\cdots+u_{m-1}(s,t,r)\big)\Big]drdtds\Big|\\ &\le B_2\int_0^x\int_0^1\int_0^1 \Big|u_m(s,t,r)\Big|drdtds\\ &\le \frac{B_2}{L_2}\frac{\Big(\alpha L_2\Big)^{m+1}}{(m+1)!} := D_{m+1}.\end{aligned}$$

So

$$\sum_{i=1}^{\infty} u_i(x,y,z) \leq \frac{B_2}{L_2}\frac{\alpha L_2}{1!} + \frac{B_2}{L_2}\frac{\left(\alpha L_2\right)^2}{2!} + \frac{B_2}{L_2}\frac{\left(\alpha L_2\right)^3}{3!} + \cdots$$
$$= \frac{B_2}{L_2}\left(e^{\alpha L_2} - 1\right) := D^*.$$

Therefore, (5) is absolutely and uniformly convergent to the solution of `3DVFIE`. □

4.1 Numerical experiments for `3DVFIE`

Now, we solve the nonlinear three-dimensional mixed Volterra-Fredholm integral equations. Numerical results are reported in Tables 4-5 which show that our proposed method has a higher accuracy than other methods. Note that to analyze the error of the new method for `3DVFIE`, we introduce the absolute error at the selected points of the given interval as

$$E_2(x,t) = |u(x,y,z) - u_1(x,y,z)|,$$

where $u(x,y,z)$ denotes the exact solution and $u_1(x,y,z)$ is the approximate solution.

Example 4.2. Consider the following three-dimensional nonlinear Volterra-Fredholm integral equation of the second kind [12, 14]

$$u(x,y,z) = v(x,y,z) + \frac{1}{4}\int_0^x \int_0^1 \int_0^1 (x+s)(y^2+r)ztu^2(s,t,r)drdtds,$$

where

$$v(x,y,z) = x^2yz - \frac{3}{32}x^6y^2z^3(1+2y^2).$$

The exact solution is $u(x,y,z) = x^2yz$. Table 4 illustrates the numerical results for this example.

Example 4.3. Consider the following three-dimensional nonlinear Volterra-Fredholm integral equation of the second kind [12, 14]

$$u(x,y,z) = v(x,y,z) + \frac{1}{2}\int_0^x \int_0^1 \int_0^1 x^2t(yz+sr)u^3(s,t,r)drdtds,$$

Table 4: Absolute errors for Example 4.2

Nodes	3DBPF [12] $m = 3$	3D-JCM [14] $N = 3$	New method
$(x, y, z) = 2^{-l}$			$E_2(x, t)$
$l = 1$	$2.2880E - 03$	$3.3951E - 05$	$9.4925E - 06$
$l = 2$	$2.8773E - 03$	$7.6182E - 06$	$3.5346E - 07$
$l = 3$	$7.848E - 04$	$1.0490E - 06$	$2.7774E - 09$
$l = 4$	$1.0137E - 03$	$1.1149E - 06$	$2.1448E - 11$
$l = 5$	$1.0280E - 03$	$1.1137E - 06$	$1.6695E - 13$
$l = 6$	$1.0289E - 03$	$7.1648E - 07$	$1.3030E - 15$

where

$$v(x, y, z) = yz \sin x - \frac{1}{16} x^3 y^3 z^3 \sin^3 x(x + 4yz).$$

The exact solution is $u(x, y, z) = yz \sin x$. Table 5 illustrates the numerical results for example 4.3.

Table 5: Absolute errors for Example 4.3

Nodes	3DBPF [12] $m = 3$	3D-JCM[14] $N = 3$	New method
$(x, y, z) = 2^{-l}$			$E_2(x, t)$
$l = 1$	$5.329E - 04$	$4.2354E - 05$	$3.0631E - 05$
$l = 2$	$1.594E - 04$	$5.3470E - 06$	$3.3029E - 07$
$l = 3$	$2.6389E - 03$	$1.2354E - 07$	$2.2671E - 09$
$l = 4$	$4.3430E - 03$	$2.5478E - 07$	$1.6333E - 11$
$l = 5$	$4.5565E - 03$	$2.3697E - 07$	$1.2205E - 13$
$l = 6$	$4.5832E - 03$	$8.2548E - 07$	$9.3161E - 16$

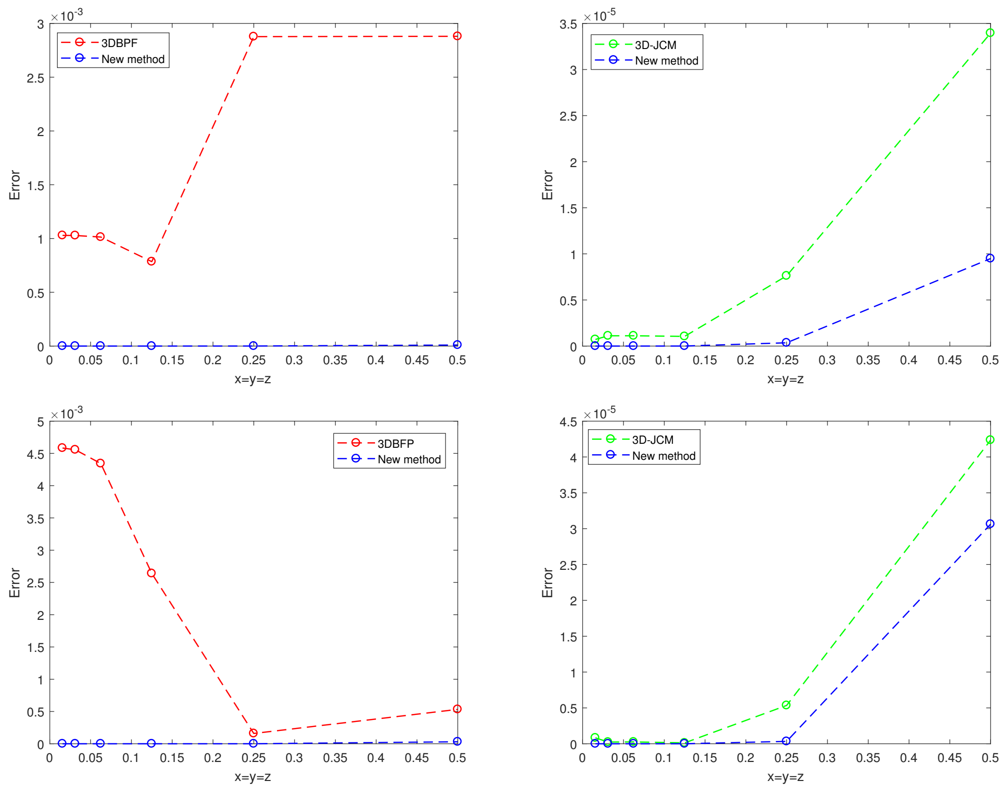

Figure 2: Comparing the absolute error of the new method with `3DBPF` and `3D-JCM` for Example 4.2 (top) and Example 4.3 (down).

5 Conclusion

In this paper, we used an iterative method to solve two- and three-dimensional nonlinear mixed Volterra-Fredholm integral equations of the second kind. Our approach is based on an iterative method for solving nonlinear functional equations. A simple recurrence formula was obtained for generating the sequence of approximate solutions where advantages of our method are the simple structure and cheap computational cost. The numerical results show that the accuracy of the obtained solutions of the new method is good in compared to some existing methods to solve `2DVFIE` and `3DVFIE`.

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