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Hermite-Hadamard Type Inequalities for h-Convex Functions Via Generalized Fractional Integrals

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Abstract. The purpose of this paper is to establish some Hermite-Hadamard type inequalities for h-convex functions utilizing generalized fractional integrals. We also obtain some generalized trapezoid and midpoint type inequalities for the mapping whose first derivatives absolutely value are h-convex. The results proved in this paper generalize the several inequalities obtained earlier works

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1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[4], [10], [26, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The overall structure of the study takes the form of five sections including introduction. The remainder of this work is organized as follows: we first mention some works which focus on Hermite-Hadamard inequality. In Section 2, we present h-convexity and generalized fractional integrals defined by Sarikaya and Ertugral along with the very first results. In section 3 we proved new Hermite-Hadamard type inequalities for h-convex function by using generalized fractional integrals given Section 3. In Section 4 and Section 5 midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are h-convex via generalized fractional integrals are presented, respectively. Moreover many corollaries and remarks are given for the special cases of the functions h and φ.

In [5], Dragomir and Agarwal establish the following identity and using this identiy, present some bounds for the right hand side of the inequality (1).

Lemma 1.1. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is interior of I) with $a < b$. If $f' \in L[a,b]$, then we following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{2}\left[\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right]. \tag{2}$$

In [19], U. S. Kırmacı give the following identity and using this identity, obtain some bounds for the left hand side of the inequality (1)

Lemma 1.2. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is* *interior of I) with $a < b$. If $f' \in L[a,b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \tag{3}$$
$$= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (1-t)\, f'(ta+(1-t)b)dt\right].$$

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1). For some examples, please refer to ([6, 20, 23, 25, 27, 28, 33]).

On the other hand, Sarikaya et al. obtain the Hermite-Hadamard inequality for the Riemann-Lioville fractional integrals in [31]. Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractianal integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([2, 3, 7, 8, 11, 17, 21, 22, 30, 32, 34, 38, 40]). For more information about fraction calculus please refer to ([9, 18]).

In this paper, we obtain the new generalized Hermite-Hadamard type inequality for the generalized fractional integrals mentioned in next section..

2. h-Convexity and New Generalized Fractional Integral Operators

First we give some definitions:

Definition 2.1. *A non negative function $f : [a,b] \to \mathbb{R}$ is said to be convex function, if the following inequality holds:*

$$f(ta+(1-t)b) \leqslant tf(a) + (1-t)f(b),$$

for all t in $[0,1]$.

Definition 2.2. [1] *A non negative function $f : [a,b] \to \mathbb{R}$ is said to be $s-$convex function in second sense, if the following inequality holds:*

$$f(ta+(1-t)b) \leqslant t^s f(a) + (1-t)^s f(b),$$

for all t in $[0,1]$ and s in $(0,1]$.

In [39], Varošanec defined generalization of convex, $s-$convex, Godunova-Levin functions and $P-$convex functions as follows:

Definition 2.3. [39] *Let $h : J \to \mathbb{R}$ be non negative function and $h \neq 0$. A non negative function $f : I = [a, b] \to \mathbb{R}$ is called $h-$convex function, if the following inequalities holds:*

$$f(ta + (1-t)b) \leqslant h(t)f(a) + h(1-t)f(b),$$

for all t in $[0, 1]$. For more discussion about the h-convex function see [39]. we denote by $SX(h, I)$ the class of h-convex functions on I.

Now we summarize the generalized fractional integrals defined by Sarikaya and Ertugral in [29].

Let's define a function $\varphi : [0, \infty) \to [0, \infty)$ satisfiying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a, \tag{4}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b. \tag{5}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.

Sarıkaya and Ertugral also establish the following Hermite-Hadamard inequality and Lemmas for the generalized fractional integral operators:

Theorem 2.4. [29] *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ with $a < b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2\Lambda(1)} \left[_{a+}I_\varphi f(b) +_{b-} I_\varphi f(a)\right] \leqslant \frac{f(a)+f(b)}{2} \tag{6}$$

where the mapping $\Lambda : [0, 1] \to \mathbb{R}$ is defined by

$$\Lambda(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt.$$

Lemma 2.5. [29] *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equalities for generalized fractional integrals hold:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right] \\ = & \frac{(b-a)}{2\Lambda(1)}\int_0^1 \left[\Lambda(1-t) - \Lambda(t)\right] f'(ta+(1-t)b)\,dt \end{aligned} \tag{7}$$

Lemma 2.6. [29] *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equalities for generalized fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right] = \frac{b-a}{2\Lambda(1)}\sum_{k=1}^{4} J_k \tag{8}$$

where

$$\begin{array}{rr} J_1 = \int_0^{\frac{1}{2}} \Lambda(t) f'(tb+(1-t)a)\,dt, & J_2 = \int_0^{\frac{1}{2}} (-\Lambda(t))\, f'(ta+(1-t)b)\,dt, \\ J_3 = \int_{\frac{1}{2}}^1 (-\Delta(t))\, f'(tb+(1-t)a)\,dt, & J_4 = \int_{\frac{1}{2}}^1 \Delta(t) f'(ta+(1-t)b)\,dt. \end{array}$$

and

$$\Delta(y) = \int_y^1 \frac{\varphi((b-a)u)}{u}\,du.$$

3. Hermite-Hadamard Type Inequalities for h-Convex Functions

In this Section, we proved Hermite-Hadamard inequality for h-convex functions via generalized fractional integrals summarized above.

Theorem 3.1. *Let $f \in SX(h,I)$, $a,b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then we have the following inequalities for h-convex functions via generalized fractional integrals:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) & \leqslant \frac{h\left(\frac{1}{2}\right)}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \\ & \leqslant \frac{h\left(\frac{1}{2}\right)\left[f(a)+f(b)\right]}{\Lambda(1)}\int_0^1 \frac{\varphi((b-a)t)}{t}\left[h(t)+h(1-t)\right]dt. \end{aligned} \tag{9}$$

Proof. Since $f \in SX(h, I)$, we have the following inequality

$$f\left(\frac{x+y}{2}\right) \leqslant h\left(\frac{1}{2}\right)[f(x)+f(y)]. \tag{10}$$

For $x = ta + (1-t)b$ and $y = tb + (1-t)a$ with $t \in [0,1]$, (10) becomes

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leqslant f(ta+(1-t)b) + f(tb+(1-t)a). \tag{11}$$

Multiplying both sides of the inequality (11) by $\frac{\varphi((b-a)t)}{t}$ and integrating the resultant inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} &\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_0^1 \frac{\varphi((b-a)t)}{t} dt \\ \leqslant &\int_0^1 \frac{\varphi((b-a)t)}{t} f(ta+(1-t)b)dt + \int_0^1 \frac{\varphi((b-a)t)}{t} f(tb+(1-t)a)dt. \end{aligned}$$

As consequence, we get

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{h\left(\frac{1}{2}\right)}{\Lambda(1)} \left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right]$$

and thus the first inequality is proved.

To obtain second inequality in (9), since $f \in SX(h, I)$, we have

$$f(ta+(1-t)b) + f(tb+(1-t)a) \leqslant [f(a)+f(b)]\,[h(t)+h(1-t)]. \tag{12}$$

Then multiplying both sides of (12) by $\frac{\varphi((b-a)t)}{t}$ and integrating the resultant inequality with respect to t on $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 \frac{\varphi((b-a)t)}{t} f(ta+(1-t)b)dt + \int_0^1 \frac{\varphi((b-a)t)}{t} f(tb+(1-t)a)dt \\ \leqslant \quad &[f(a)+f(b)] \int_0^1 \frac{\varphi((b-a)t)}{t} [h(t)+h(1-t)]\, dt. \end{aligned}$$

As sequel, we obtain

$$\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right] \leqslant [f(a)+f(b)] \int_0^1 \frac{\varphi((b-a)t)}{t} [h(t)+h(1-t)]\, dt \tag{13}$$

and the second inequality is proved.

The proof is completed. $\square$

Remark 3.2. *If we choose $h(t) = t$ in Theorem 3.1, then we obtain the inequality (6).*

Remark 3.3. *Under the assumptions of Theorem 3.1 with $\varphi(t) = t$, then we have the following inequalities for h-convex functions via classical integrals:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{2h\left(\frac{1}{2}\right)}{b-a}\int_a^b f(x)dx \leqslant h\left(\frac{1}{2}\right)[f(a)+f(b)]\int_0^1 (h(t)+h(1-t))dt,$$

which proved by Sarikaya et al in [33].

Corollary 3.4. *Under the assumptions of Theorem 3.1 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we have the following inequalities for h-convex functions via Rieman-Liouville fractional integrals:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{\Gamma(\alpha+1)h\left(\frac{1}{2}\right)}{(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] \\ &\leqslant \alpha h\left(\frac{1}{2}\right)[f(a)+f(b)]\int_0^1 t^{\alpha-1}\left[h(t)+h(1-t)\right]dt \end{aligned}$$

which was given by Budak et al. in [2].

Remark 3.5. *If we choose $h(t) = t$ in Corollary 3.4, then we have the following inequalities for classical convex functions:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] \leqslant \frac{f(a)+f(b)}{2},$$

which proved by Sarikaya et al. in [31].

Corollary 3.6. *If we suppose $h(t) = t^s$ in Corollary 3.4, we obtain following inequalities for s-convex functions in second sense:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{\Gamma(\alpha+1)}{2^s(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] \\ &\leqslant \frac{\alpha[f(a)+f(b)]}{2^s}\left[\frac{1}{s+\alpha} + \beta(\alpha, s+1)\right]. \end{aligned}$$

Remark 3.7. *If we suppose $\alpha = 1$ in Corollary 3.6, then we obtain the inequality*

$$2^{1-s} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{s+1},$$

which was proved by Dragomir and Fitzpatrick in [6].

Corollary 3.8. *Under the assumptions of Theorem 3.1 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then the following inequalities for h-convex functions via k-Riemann-Liouville fractional integral holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{h\left(\frac{1}{2}\right)\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right] \\ &\leqslant \alpha h\left(\frac{1}{2}\right)[f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}\left[h(t)+h(1-t)\right]dt. \end{aligned}$$

Remark 3.9. *In Corollary 3.8,*

i) *if we choose $h(t) = t$, then we get following inequalities for the classical convex functions:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,\ k}^{\alpha}f(b)+I_{b-,\ k}^{\alpha}f(a)\right] \leqslant \frac{[f(a)+f(b)]}{2},$$

which was proved by Farid et al. in [8].

ii) *if we take $h(t) = t^s$ in Corollary 3.8, then we have the following inequalities for s-convex functions in second sense:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{\Gamma_k(\alpha+k)}{2^s(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,\ k}^{\alpha}f(b)+I_{b-,\ k}^{\alpha}f(a)\right] \\ &\leqslant \frac{\alpha\left[f(a)+f(b)\right]\left[\beta\left(\frac{\alpha}{k},s+1\right)+\frac{\alpha}{k}\beta\left(\frac{\alpha}{k},s+1\right)+1\right]}{2^s}. \end{aligned}$$

4. Trapezoid Type Inequalities for h-Convex Functions

In this Section, we obtain several trapezoid type inequalities for the functions whose first derivatives absolutely value are h-convex.

Theorem 4.1. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be positive function with $0 \leqslant a < b$ and $h \in L_1[0, 1]$, $f \in L_1[a, b]$. If $\left|f'\right|$ is an h-convex function on*

$[a,b]$, *then the following inequality for generalized fractional integral holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \tag{14}$$
$$\leqslant \frac{b-a}{\Lambda(1)}\frac{\left|f^{'}(a)\right|+\left|f^{'}(b)\right|}{2}\int_0^1 h(t)\left|\Lambda(1-t)-\Lambda(t)\right|dt.$$

Proof. From Lemma 2.5 and $\left|f^{'}\right|\in SX(h,I)$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\int_0^1 \left|\Lambda(1-t)-\Lambda(t)\right|\left|f^{'}(ta+(1-t)b)\right|dt \\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\int_0^1 \left|\Lambda(1-t)-\Lambda(t)\right|(h(t)\left|f^{'}(a)\right|+h(1-t)\left|f^{'}(b)\right|)dt \\
=\ & \frac{(b-a)}{2\Lambda(1)}\left|f^{'}(a)\right|\int_0^1 h(t)\left|\Lambda(1-t)-\Lambda(t)\right|dt \\
&+\frac{(b-a)}{2\Lambda(1)}\left|f^{'}(b)\right|\int_0^1 h(1-t)\left|\Lambda(1-t)-\Lambda(t)\right|dt \\
=\ & \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2\Lambda(1)}\int_0^1 h(t)\left|\Lambda(1-t)-\Lambda(t)\right|dt.
\end{aligned}$$

This completes the proof. $\square$

Corollary 4.2. *Under the assumptions of Theorem 4.1 with* $\varphi(t)=t$, *then the following inequality holds for h-convex functions via classical integrals:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2}\int_0^1 h(t)\left|1-2t\right|dt.$$

Remark 4.3. *In Corollary 4.2,*

i) *if we take* $h(t)=t$, *then we have the following inequality for the classical convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2},$$

which was proved by Dragomir and Agarwal in [5].

ii) *if we assume $h(t) = t^s$, then we have the following inequality for the classical convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)(s2^s+1)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)}{2^{s+1}(s+1)(s+2)},$$

which was proved by Kırmacı et al in [20].

Corollary 4.4. *Under the assumptions of Theorem 4.1 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then the following inequality for h-convex functions via Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right]\right| \\ &\leqslant \frac{(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)}{2}\int_0^1 h(t)\left|(1-t)^\alpha - t^\alpha\right| dt.\end{aligned}$$

Remark 4.5. *In Corollary 4.4,*

i) *if we suppose $h(t) = t$, we have the following inequality for the classical convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right]\right| \\ &\leqslant \frac{(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right),\end{aligned}$$

which was proved by Sarikaya et al. in [31].

ii) *if we suppose $h(t) = t^s$, we have the following inequality for the s-convex functions in second sense:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right]\right| \\ \leqslant\ &\frac{b-a}{2}\left[\beta\left(\frac{1}{2}, s+1, \alpha+1\right) - \beta\left(\frac{1}{2}, \alpha+1, s+1\right) + \frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\right] \\ &\times\left(\left|f'(a)\right| + \left|f'(b)\right|\right).\end{aligned}$$

which was proved by Özdemir et. al. in [24].

Corollary 4.6. *Under assumption of Theorem 4.1 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then the following inequality for h-convex functions via k-Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b) + I^{\alpha}_{b-,\ k}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{2}\int_0^1 h(t)\left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right| dt.\end{aligned}$$

Corollary 4.7. *In Corollary 4.6,*

i) *if we choose* $h(t) = t$ *in Corollary 4.6, then the following inequality holds for classical convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b) + I^{\alpha}_{b-,\ k}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{2(\frac{\alpha}{k}+1)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right),\end{aligned}$$

which was proved by Farid et al. in [8].

ii) *if we choose* $h(t) = t^s$ *in Corollary 4.6, then the following inequality holds for s-convex functions in second sense:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b) + I^{\alpha}_{b-,\ k}f(a)\right]\right| \\ \leqslant\ & \frac{b-a}{2}\left[\beta\left(\frac{1}{2}, s+1, \frac{\alpha}{k}+1\right) - \beta\left(\frac{1}{2}, \frac{\alpha}{k}+1, s+1\right) + \frac{2^{\frac{\alpha}{k}+s} - 1}{(\frac{\alpha}{k}+s+1)2^{\frac{\alpha}{k}+s}}\right] \\ & \times\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right).\end{aligned}$$

Theorem 4.8. *Let* $h : J \subset \mathbb{R} \to \mathbb{R}$ *and* $f : [a, b] \to \mathbb{R}$ *be positive function with* $0 \leqslant a < b$ *and* $h^q \in L_1[0, 1]$, $f \in L_1[a, b]$. *If* $\left|f^{'}\right|^q$ *is an h-convex function on* $[a, b]$, *then the following inequality for generalized fractional integral holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right]\right| \qquad (15)\\ \leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 \left|\Lambda(1-t) - \Lambda(t)\right|^p dt\right)^{\frac{1}{p}} \\ & \times\left[\left(\int_0^1 \left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right] dt\right)^{\frac{1}{q}}\right],\end{aligned}$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. From Lemma 2.5 and well known Hölder inequality, we have the following inequality

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)}{2\Lambda(1)}\int_0^1 |\Lambda(1-t)-\Lambda(t)|\left|f^{'}(ta+(1-t)b)\right|dt \\ \leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-t)-\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{'}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{16}$$

Since $f^{'}\in SX(h,I)$, then (16) becomes

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-t)-\Lambda(t)|^p\,dt\right)^{\frac{1}{p}} \\ & \times\left[\left(\int_0^1\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \end{aligned}$$

and the proof is completed. □

Corollary 4.9. *Under the assumptions of Theorem 4.8 with $\varphi(t)=t$, we obtain following inequality for h-convex functions via classical integrals:*

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ & \leqslant\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\left(\int_0^1\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Remark 4.10. *In Corollary 4.9,*

i) *if we take $h(t)=t$, then we get following inequality for classical convex functions*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

which was proved by Dragomir and Agarwal in [5].

ii) *if we choose* $h(t) = t^s$, *then we have the following inequality for* s-*convex functions in second sense:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{s+1}\right)^{\frac{1}{q}}.$$

Corollary 4.11. *If we suppose* $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ *in Theorem 4.8, we have the following inequality for* h-*convex functions via Riemann Liouville fractional integrals:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I^\alpha_{a+}f(b) + I^\alpha_{b-}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\left[\left(\int_0^1 \left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 4.12. *In Corollary 4.11,*

i) *if we suppose* $h(t) = t$, *then we have the following inequality for the classical convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I^\alpha_{a+}f(b) + I^\alpha_{b-}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

which was given by Özdemir et. al. in [25].

ii) *if we suppose* $h(t) = t^s$, *then we have the following inequality for the* s-*convex functions in second sense:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I^\alpha_{a+}f(b) + I^\alpha_{b-}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{s+1}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 4.13. *Under the assumptions of Theorem 4.8 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *we obtain following inequality for* h-*convex functions via* k-*Riemann-Liouville* *fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{2(\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left[\left(\int_0^1\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right].$$

Corollary 4.14. *In Corollary 4.13,*

i) *if we choose $h(t)=t$, then we have the following inequality for classical convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{2(\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

ii) *if we suppose $h(t)=t^s$, then we have the following inequality for the s-convex functions in second sense:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{2(\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{s+1}\right)^{\frac{1}{q}}.$$

Theorem 4.15. *Let $h:J\subset\mathbb{R}\to\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ be positive function with $0\leqslant a<b$ and $h^q\in L_1[0,1]$, $f\in L_1[a,b]$. If $|f'|^q$, $q\geqslant 1$ is an h-convex function on $[a,b]$, then the following inequality for generalized fractional integral holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \tag{17}$$
$$\leqslant \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1|\Lambda(1-t)-\Lambda(t)|\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1|\Lambda(1-t)-\Lambda(t)|\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

Proof. From Lemma 2.5 and well known power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \qquad (18)\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\int_0^1 |\Lambda(1-t)-\Lambda(t)|\left|f^{'}(ta+(1-t)b)\right|dt\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-t)-\Lambda(t)|\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 |\Lambda(1-t)-\Lambda(t)|\left|f^{'}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $f\in SX(h,I)$, then (18) becomes

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-t)-\Lambda(t)|\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 |\Lambda(1-t)-\Lambda(t)|\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}.
\end{aligned}$$

which completes the proof. □

Corollary 4.16. *Under the assumptions of Theorem 4.15 with $\varphi(t)=t$, then we obtain the following inequality for h-convex functions via classical integrals:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leqslant \frac{(b-a)}{2^{2-\frac{1}{q}}}\left(\int_0^1 |1-2t|\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 4.17. *If we take $h(t)=t$ in Corollary 4.16, we have the following established inequality for classical convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{(b-a)}{4}\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{\frac{1}{q}},$$

which was given by Pearce and Pecaric in [27].

Remark 4.18. *If we suppose $h(t) = t^s$ in Corollary 4.16, then we have the following inequality for s-convex functions in second sense:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

which was given by Kirmaci et. al. in [20].

Corollary 4.19. *Under the assumptions of Theorem 4.15 with $\frac{t^\alpha}{\Gamma(\alpha)}$, then we have the following inequality for the h-convex functions via Riemann-Liouville fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{2(1+\alpha)^{1-\frac{1}{q}}}\left(\int_0^1 |(1-t)^\alpha - t^\alpha|\left[h(t)\left|f'(a)\right|^q + h(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

Remark 4.20. *In Corollary 4.19,*

i) *if we take $h(t) = t$, then we obtain the following inequality for the classical convex functions*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}},$$

which was given by Özdemir et. al. in [25].

ii) *if we choose $h(t) = t^s$, we get following inequality for s-convex functions in second sense:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{2}\left[\frac{2}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\beta\left(\frac{1}{2}, s+1, \alpha+1\right) - \beta\left(\frac{1}{2}, \alpha+1, s+1\right) + \frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\right]^{\frac{1}{q}}$$
$$\times\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

which was given by Set et. al. in [35, 24].

Corollary 4.21. *Under the assumptions of Theorem 4.15 with* $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then we have the following inequality for the h-convex functions via k-Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right| \\
\leqslant\ &\frac{(b-a)}{2(1+\frac{\alpha}{k})^{1-\frac{1}{q}}}\left(\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 4.22. *In Corollary 4.21*

i) *if we take* $h(t)=t$ *in, then we have the following inequality for the classical convex functions:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right| \\
\leqslant\ &\frac{(b-a)}{(\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

ii) *if we choose* $h(t)=t^s$, *then we obtain the following inequality for the s-convex functions in second sense:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right| \\
\leqslant\ &\frac{(b-a)}{2}\left[\frac{2}{\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \\
&\times\left[\beta\left(\frac{1}{2},s+1,\frac{\alpha}{k}+1\right)-\beta\left(\frac{1}{2},\frac{\alpha}{k}+1,s+1\right)+\frac{2^{\frac{\alpha}{k}+s}-1}{(\frac{\alpha}{k}+s+1)2^{\frac{\alpha}{k}+s}}\right]^{\frac{1}{q}} \\
&\times\left[\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4.23. *Let* $h: J\subset\mathbb{R}\to\mathbb{R}$ *and* $f:[a,b]\to\mathbb{R}$ *be positive function with* $0\leqslant a<b$ *and* $h^q\in L_1[0,1]$, $f\in L_1[a,b]$. *If* $\left|f^{'}\right|$ *is an h-convex function*

on $[a,b]$, then the following inequality for generalized fractional integral holds:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \qquad (19)\\
\leqslant\ & \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2\Lambda(1)}\\
&\times\left[\left(\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p dt\right)^{\frac{1}{p}}-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p dt\right)^{\frac{1}{p}}\right)\right.\\
&\left.\times\left(\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}\right)\right],
\end{aligned}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.5, we have the following inequality

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \qquad (20)\\
\leqslant\ & \frac{b-a}{2\Lambda(1)}\int_0^1 \left|[\Lambda(1-t)-\Lambda(t)]\right|\left|f^{'}(ta+(1-t)b)\right|dt.
\end{aligned}$$

Since $\left|f^{'}\right|$ is h-convex on $[a,b]$, then (20) becomes

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \qquad (21)\\
\leqslant\ & \frac{b-a}{2\Lambda(1)}\left\{\int_0^{\frac{1}{2}}[\Lambda(1-t)-\Lambda(t)]\left[h(t)\left|f^{'}(a)\right|+h(1-t)\left|f^{'}(b)\right|\right]dt\right.\\
&\left.-\int_{\frac{1}{2}}^{1}[\Lambda(t)-\Lambda(1-t)]\left[h(t)\left|f^{'}(a)\right|+h(1-t)\left|f^{'}(b)\right|\right]dt\right\}\\
=\ & \frac{b-a}{2\Lambda(1)}\left\{\left|f^{'}(a)\right|\int_0^{\frac{1}{2}}\Lambda(1-t)h(t)dt-\left|f^{'}(a)\right|\int_0^{\frac{1}{2}}\Lambda(t)h(t)dt\right.\\
&+\left|f^{'}(a)\right|\int_{\frac{1}{2}}^{1}\Lambda(t)h(t)dt-\left|f^{'}(a)\right|\int_{\frac{1}{2}}^{1}\Lambda(1-t)h(t)dt\\
&+\left|f^{'}(b)\right|\int_0^{\frac{1}{2}}\Lambda(1-t)h(1-t)dt-\left|f^{'}(b)\right|\int_0^{\frac{1}{2}}\Lambda(t)h(1-t)dt\\
&\left.+\left|f^{'}(b)\right|\int_{\frac{1}{2}}^{1}\Lambda(t)h(1-t)dt-\left|f^{'}(b)\right|\int_{\frac{1}{2}}^{1}\Lambda(1-t)h(1-t)dt\right\}.
\end{aligned}$$

Now applying Hölder inequality in right hand side of (21), we have following inequalities

$$\int_0^{\frac{1}{2}} \Lambda(1-t)h(t)dt = \int_{\frac{1}{2}}^1 \Lambda(t)h(1-t) \tag{22}$$
$$\leqslant \left(\int_0^{\frac{1}{2}} (\Lambda(1-t))^p\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} (h(t))^q dt\right)^{\frac{1}{q}},$$

$$\int_0^{\frac{1}{2}} \Lambda(1-t)h(1-t)dt = \int_{\frac{1}{2}}^1 \Lambda(t)h(t)dt \tag{23}$$
$$\leqslant \left(\int_0^{\frac{1}{2}} (\Lambda(1-t))^p\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}},$$

$$\int_0^{\frac{1}{2}} \Lambda(t)h(t)dt = \int_{\frac{1}{2}}^1 \Lambda(1-t)h(1-t)dt \tag{24}$$
$$\leqslant \left(\int_0^{\frac{1}{2}} (\Lambda(t))^p\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} (h(t))^q dt\right)^{\frac{1}{q}},$$

$$\int_0^{\frac{1}{2}} \Lambda(t)h(1-t)dt = \int_{\frac{1}{2}}^1 \Lambda(1-t)h(t)dt \tag{25}$$
$$\leqslant \left(\int_0^{\frac{1}{2}} (\Lambda(t))^p\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}.$$

Using (22)-(25) in (21), we obtain

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\
\leqslant\ & \frac{b-a}{2\Lambda(1)}\left[\left|f^{'}(a)\right|\left\{\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}\right.\right. \\
&-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}} \\
&\left.-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}\right\} \\
&+\left|f^{'}(b)\right|\left\{\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}\right. \\
&-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}-\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}} \\
&\left.\left.+\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}\right\}\right] \\
=\ & \frac{b-a}{2\Lambda(1)} \\
&\times\left[\left|f^{'}(a)\right|\left\{\left(\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\right)\left(\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}\right)\right.\right. \\
&\left.+\left(\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\right)\left(\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}\right)\right\} \\
&+\left|f^{'}(b)\right|\left\{\left(\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\right)\left(\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}\right)\right. \\
&\left.\left.+\left(\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\right)\left(\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}\right)\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
=\ & \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2\Lambda(1)} \\
&\times\left[\left(\left(\int_0^{\frac{1}{2}}(\Lambda(1-t))^p\right)^{\frac{1}{p}}-\left(\int_0^{\frac{1}{2}}(\Lambda(t))^p\right)^{\frac{1}{p}}\right)\right. \\
&\left.\times\left(\left(\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}\right)+\left(\left(\int_{\frac{1}{2}}^{1}(h(t))^q dt\right)^{\frac{1}{q}}\right)\right)\right].
\end{aligned}
$$

Hence proof is completed. □

Corollary 4.24. *Under the assumptions of Theorem 4.23 with $\varphi(t) = t$, we have the following inequality for h-convex functions via classical integrals:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leqslant\ &\frac{(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2}\left[\left(\frac{2^{p+1}-1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}-\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\right]\\ &\times\left[\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 4.25. *In Corollary 4.24,*

i) *if we take $h(t) = t$, then we obtain following inequality for the classical convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leqslant\ &\frac{(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2}\left[\left(\frac{2^{p+1}-1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}-\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\right]\\ &\times\left[\left(\frac{1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}\right].\end{aligned}$$

ii) *if we take $h(t) = t^s$, then we obtain following inequality for the classical convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leqslant\ &\frac{(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2}\left[\left(\frac{2^{p+1}-1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}-\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\right]\\ &\times\left[\left(\frac{1}{2^{sq+1}(sq+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{sq+1}-1}{2^{sq+1}(sq+1)}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Remark 4.26. *Under the assumptions of Theorem 4.23 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then we have the following inequality for the Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2}\left[\left(\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}-\left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\right] \\
&\times\left[\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(h(t))^q dt\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which was given by Tunc in [36].

Corollary 4.27. *Under the assumptions of Theorem 4.23 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the following inequality for h-convex functions via k-Riemann-Liouville fractional integral holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,\ k}^{\alpha}f(b)+I_{b-,\ k}^{\alpha}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2} \\
&\times\left[\left(\frac{2^{\frac{\alpha p}{k}+1}-1}{2^{\frac{\alpha p}{k}+1}(\frac{\alpha p}{k}+1)}\right)^{\frac{1}{p}}-\left(\frac{1}{2^{\frac{\alpha p}{k}+1}(\frac{\alpha p}{k}+1)}\right)^{\frac{1}{p}}\right] \\
&\times\left[\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(h(t))^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 4.28. *In Corollary 4.27,*

i) *if we choose $h(t)=t$, then we obtain following inequality for the classical convex functions:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,\ k}^{\alpha}f(b)+I_{b-,\ k}^{\alpha}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{2} \\
&\times\left[\left(\frac{2^{\frac{\alpha p}{k}+1}-1}{2^{\frac{\alpha p}{k}+1}(\frac{\alpha p}{k}+1)}\right)^{\frac{1}{p}}-\left(\frac{1}{2^{\frac{\alpha p}{k}+1}(\frac{\alpha p}{k}+1)}\right)^{\frac{1}{p}}\right] \\
&\times\left[\left(\frac{1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

ii) *if we take $h(t) = t^s$, then we have the following inequality for the s-convex functions in second sense:*

$$\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I^{\alpha}_{a+,\ k} f(b) + I^{\alpha}_{b-,\ k} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)\left[\left| f^{'}(a) \right| + \left| f^{'}(b) \right| \right]}{2} \\
& \times \left[\left(\frac{2^{\frac{\alpha p}{k}+1}-1}{2^{\frac{\alpha p}{k}+1}(\frac{\alpha p}{k}+1)} \right)^{\frac{1}{p}} - \left(\frac{1}{2^{\frac{\alpha p}{k}+1}(\frac{\alpha p}{k}+1)} \right)^{\frac{1}{p}} \right] \\
& \times \left[\left(\frac{1}{2^{sq+1}(sq+1)} \right)^{\frac{1}{q}} + \left(\frac{2^{sq+1}-1}{2^{sq+1}(sq+1)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Theorem 4.29. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be positive function with $0 \leqslant a < b$ and $h^q \in L_1[0,1]$, $f \in L_1[a,b]$. If $\left| f^{'} \right|$ is an h-convex function on $[a,b]$, then the following inequality for generalized fractional integral holds:*

$$\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)\left(\left| f^{'}(a) \right| + \left| f^{'}(b) \right| \right)}{2\Lambda(1)} \left[B_1 - B_2 \right],
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
B_1 &= \left(\int_0^{\frac{1}{2}} \Lambda(1-t)dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\left(\int_0^{\frac{1}{2}} \Lambda(1-t)[h(t)]^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 \Lambda(t)[h(t)]^q dt \right)^{\frac{1}{q}} \right], \\
B_2 &= \left(\int_0^{\frac{1}{2}} \Lambda(t)dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left[\left(\int_0^{\frac{1}{2}} \Lambda(t)[h(t)]^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 \Lambda(1-t)[h(t)]^q dt \right)^{\frac{1}{q}} \right],
\end{aligned}$$

and $q > 1$.

Proof. From Lemma 2.5, we obtain the following inequality

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\ \leqslant\ &\frac{b-a}{2\Lambda(1)}\int_0^1 |[\Lambda(1-t)-\Lambda(t)]|\left|f^{'}(ta+(1-t)b)\right|dt. \end{aligned}\tag{27}$$

Since $\left|f^{'}\right|$ is h-convex on $[a,b]$, then (27) becomes

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\ \leqslant\ &\frac{b-a}{2\Lambda(1)}\left\{\int_0^{\frac{1}{2}}[\Lambda(1-t)-\Lambda(t)]\left[h(t)\left|f^{'}(a)\right|+h(1-t)\left|f^{'}(b)\right|\right]dt\right. \\ &\left.-\int_{\frac{1}{2}}^1[\Lambda(t)-\Lambda(1-t)]\left[h(t)\left|f^{'}(a)\right|+h(1-t)\left|f^{'}(b)\right|\right]dt\right\} \\ =\ &\frac{b-a}{2\Lambda(1)}\left\{\left|f^{'}(a)\right|\int_0^{\frac{1}{2}}\Lambda(1-t)h(t)dt-\left|f^{'}(a)\right|\int_0^{\frac{1}{2}}\Lambda(t)h(t)dt\right. \\ &+\left|f^{'}(a)\right|\int_{\frac{1}{2}}^1\Lambda(t)h(t)dt-\left|f^{'}(a)\right|\int_{\frac{1}{2}}^1\Lambda(1-t)h(t)dt \\ &+\left|f^{'}(b)\right|\int_0^{\frac{1}{2}}\Lambda(1-t)h(1-t)dt-\left|f^{'}(b)\right|\int_0^{\frac{1}{2}}\Lambda(t)h(1-t)dt \\ &\left.+\left|f^{'}(b)\right|\int_{\frac{1}{2}}^1\Lambda(t)h(1-t)dt-\left|f^{'}(b)\right|\int_{\frac{1}{2}}^1\Lambda(1-t)h(1-t)dt\right\}. \end{aligned}\tag{28}$$

Applying well known power mean inequality in right hand side of (28), we get following inequalities

$$\begin{aligned} &\int_0^{\frac{1}{2}}\Lambda(1-t)h(t)dt=\int_{\frac{1}{2}}^1\Lambda(t)h(1-t) \\ \leqslant\ &\left(\int_0^{\frac{1}{2}}\Lambda(1-t)\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\Lambda(1-t)(h(t))^q dt\right)^{\frac{1}{q}}, \end{aligned}\tag{29}$$

$$\begin{aligned} &\int_0^{\frac{1}{2}}\Lambda(1-t)h(1-t)dt=\int_{\frac{1}{2}}^1\Lambda(t)h(t)dt \\ \leqslant\ &\left(\int_0^{\frac{1}{2}}\Lambda(1-t)\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\Lambda(t)(h(t))^q dt\right)^{\frac{1}{q}}, \end{aligned}\tag{30}$$

$$\int_0^{\frac{1}{2}} \Lambda(t)h(t)dt = \int_{\frac{1}{2}}^1 \Lambda(1-t)h(1-t)dt \tag{31}$$
$$\leqslant \left(\int_0^{\frac{1}{2}} \Lambda(t)\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \Lambda(t)(h(t))^q dt\right)^{\frac{1}{q}},$$

$$\int_0^{\frac{1}{2}} \Lambda(t)h(1-t)dt = \int_{\frac{1}{2}}^1 \Lambda(1-t)h(t)dt \tag{32}$$
$$\leqslant \left(\int_0^{\frac{1}{2}} \Lambda(t)\right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \Lambda(1-t)(h(t))^q dt\right)^{\frac{1}{q}}.$$

Using (29)-(32) in (28), then we have our required inequality (26). □

Corollary 4.30. *Under the assumptions of Theorem 4.29 with $\varphi(t) = t$, we have the following new inequality for h-convex functions via classical integrals:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)}{2}\left[B_1 - B_2\right],$$

where

$$B_1 = \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left[\left(\int_0^{\frac{1}{2}} (1-t)[h(t)]^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 t[h(t)]^q dt\right)^{\frac{1}{q}}\right],$$
$$B_2 = \left(\frac{3}{8}\right)^{1-\frac{1}{q}} \left[\left(\int_0^{\frac{1}{2}} t[h(t)]^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (1-t)[h(t)]^q dt\right)^{\frac{1}{q}}\right].$$

Corollary 4.31. *In Corollary 4.30,*

i) *if we assume $h(t) = t$, then we have the following inequality for the classical convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)}{2}\left[B_1 - B_2\right],$$

where

$$B_1 = \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left[\left(\frac{q+3}{(q+1)(4q+8)2^q}\right)^{\frac{1}{q}} + \left(\frac{2^{q+2}-1}{(4q+8)2^q}\right)^{\frac{1}{q}}\right],$$
$$B_2 = \left(\frac{3}{8}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{2^{q+2}(q+2)}\right)^{\frac{1}{q}} + \left(\frac{2^{q+2}-q-3}{(q+1)(4q+8)2^q}\right)^{\frac{1}{q}}\right].$$

ii) *if we take $h(t) = t^s$, then we have the following inequality for s-convex functions in second sense:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2}\left[B_1-B_2\right],$$

where

$$\begin{aligned}
B_1 &= \left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{qs+3}{(qs+1)(4qs+8)2^{qs}}\right)^{\frac{1}{q}}+\left(\frac{2^{qs+2}-1}{(4qs+8)2^{qs}}\right)^{\frac{1}{q}}\right],\\
B_2 &= \left(\frac{3}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{qs+2}(qs+2)}\right)^{\frac{1}{q}}+\left(\frac{2^{qs+2}-qs-3}{(qs+1)(4qs+8)2^{qs}}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 4.32. *Under the assumptions of Theorem 4.29,*
i) *if we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we obtain following new inequality for h-convex functions via Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]\right|\\
&\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2}\left[B_1-B_2\right],
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^{\frac{1}{2}}(1-t)^\alpha[h(t)]^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 t^\alpha[h(t)]^q dt\right)^{\frac{1}{q}}\right],\\
B_2 &= \left(\frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^{\frac{1}{2}} t^\alpha[h(t)]^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(1-t)^\alpha[h(t)]^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

ii) *if we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we obtain following new inequality for h-*

convex functions via k-Riemann-Liouville fractional integrals:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2}\left[B_1-B_2\right],$$

where

$$\begin{aligned} B_1 &= \left(\frac{1}{2^{\frac{\alpha}{k}+1}(\frac{\alpha}{k}+1)}\right)^{1-\frac{1}{q}} \\ &\times\left[\left(\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}[h(t)]^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}[h(t)]^q dt\right)^{\frac{1}{q}}\right], \\ B_2 &= \left(\frac{2^{\frac{\alpha}{k}+1}-1}{2^{\frac{\alpha}{k}+1}(\frac{\alpha}{k}+1)}\right)^{1-\frac{1}{q}} \\ &\times\left[\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}[h(t)]^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}}[h(t)]^q dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

5. Midpoint Type Inequalities for h-Convex Functions

In this section, we obtain some generalized midpoint type inequalities for the functions whose first derivatives absolutely value are h-convex.

Theorem 5.1. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be positive function with $0 \leqslant a < b$ and $h \in L_1[0, 1]$, $f \in L_1[a, b]$. If $\left|f^{'}\right|$ is an h-convex function on $[a, b]$, then the following inequality for generalized fractional integral holds:*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+\ {}_{b-}I_{\varphi}f(a)\right]\right| \\ \leqslant\ & \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2\Lambda(1)}\left[\int_0^{\frac{1}{2}}\Lambda(t)h(t)dt+\int_0^{\frac{1}{2}}\Lambda(t)h(1-t)dt\right. \\ &\left.+\int_{\frac{1}{2}}^1\Delta(t)h(t)dt+\int_{\frac{1}{2}}^1\Delta(t)h(1-t)dt\right]. \end{aligned}$$

Proof. From Lemma 2.5 and since $f \in SX(h, I)$, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)} \left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)} \left[\int_0^{\frac{1}{2}} |\Lambda(t)| \left| f^{'}(tb + (1-t)a) \right| dt + \int_0^{\frac{1}{2}} |\Lambda(t)| \left| f^{'}(ta + (1-t)b) \right| dt \right. \\
& \left. + \int_{\frac{1}{2}}^1 |\Delta(t)| \left| f^{'}(tb + (1-t)a) \right| dt + \int_{\frac{1}{2}}^1 |\Delta(t)| \left| f^{'}(ta + (1-t)b) \right| dt \right] \\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)} \left[\int_0^{\frac{1}{2}} |\Lambda(t)| \left[h(t) \left| f^{'}(b) \right| + h(1-t) \left| f^{'}(a) \right| \right] dt \right. \\
& + \int_0^{\frac{1}{2}} |\Lambda(t)| \left[h(t) \left| f^{'}(a) \right| + h(1-t) \left| f^{'}(b) \right| \right] dt \\
& + \int_{\frac{1}{2}}^1 |\Delta(t)| \left[h(t) \left| f^{'}(b) \right| + h(1-t) \left| f^{'}(a) \right| \right] dt \\
& \left. + \int_{\frac{1}{2}}^1 |\Delta(t)| \left[h(t) \left| f^{'}(a) \right| + h(1-t) \left| f^{'}(b) \right| \right] dt \right] \\
=\ & \frac{(b-a)\left(\left| f^{'}(a) \right| + \left| f^{'}(b) \right| \right)}{2\Lambda(1)} \left[\int_0^{\frac{1}{2}} \Lambda(t)h(t)dt + \int_0^{\frac{1}{2}} \Lambda(t)h(1-t)dt \right. \\
& \left. + \int_{\frac{1}{2}}^1 \Delta(t)h(t)dt + \int_{\frac{1}{2}}^1 \Delta(t)h(1-t)dt \right].
\end{aligned}$$

Hence the proof is completed. $\square$

Remark 5.2. *If we choose $h(t) = t$ in Theorem 5.1, then we obtain the following inequality*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)} \left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)}{\Lambda(1)} \left[\frac{|f'(a)| + |f'(b)|}{2} \right] \left(\int_0^{\frac{1}{2}} |\Lambda(t)| \, dt + \int_{\frac{1}{2}}^1 |\Delta(t)| \, dt \right),
\end{aligned}$$

which was proved by Sarikaya and Ertuğral in [29].

Corollary 5.3. *Under the assumptions of Theorem 5.1 with $\varphi(t) = t$, then we have the following inequality for h-convex functions via classical integrals:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leqslant\ & (b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)\left[\int_0^{\frac{1}{2}} th(t)dt + \int_{\frac{1}{2}}^1 (1-t)h(t)dt\right]. \end{aligned}$$

Remark 5.4. *In Corollary 5.3,*

i) *if we choose $h(t) = t$, then we have the following inequality for the classical convex functions:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leqslant \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{8},$$

which was given by Kirmaci in [19].

ii) *if we take $\varphi(t) = t^s$, then we have following inequality for s-convex functions in second sense:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leqslant (b-a)\frac{\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{2}\left[\frac{1}{(s+1)(s+2)}\left(1 - \frac{1}{2^{s+1}}\right)\right]. \end{aligned}$$

Corollary 5.5. *Under the assumptions of Theorem 5.1 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we have the following inequality for h-convex functions via Riemann-Liouville fractional integrals:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] \right| \\ \leqslant\ & (b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)\left[\int_0^{\frac{1}{2}} t^\alpha h(t)dt + \int_{\frac{1}{2}}^1 (1-t)^\alpha h(t)dt\right] \end{aligned}$$

Remark 5.6. *In Corollary 5.5,*

i) *if we suppose $h(t) = t$, then we have the following inequality for the classical convex functions:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a)\right] \right| \leqslant \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{2^{\alpha+1}(\alpha+1)}$$

which was obtained by Iqbal et al. in [12].

ii) *If we suppose $h(t) = t^s$, then we have the following inequality for the s-convex functions in second sense:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{a+}^{\alpha} f(b) + I_{b-}^{\alpha} f(a) \right] \right| \\ \leqslant & \frac{(b-a)\left(\left| f'(a) \right| + \left| f'(b) \right| \right)}{2} \\ &\times \left[\frac{1}{\alpha+s+1} + \beta\left(\frac{1}{2}; s+1, \alpha+1\right) + \beta\left(\frac{1}{2}; \alpha+1, s+1\right) \right]. \end{aligned}$$

Corollary 5.7. *Under the assumptions of Theorem 5.1 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequality for h-convex functions via k-Riemann-Liouville fractional integrals:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+,\ k}^{\alpha} f(b) + I_{b-,\ k}^{\alpha} f(a) \right] \right| \\ \leqslant & (b-a)\left(\left| f'(a) \right| + \left| f'(b) \right| \right) \left[\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} h(t)dt + \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} h(t)dt \right]. \end{aligned}$$

Corollary 5.8. *In Corollary 5.7,*

i) *if we suppose $h(t) = t$, then we have the following inequality for the classical convex functions:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+,\ k}^{\alpha} f(b) + I_{b-,\ k}^{\alpha} f(a) \right] \right| \\ &\leqslant \frac{(b-a)\left(\left| f'(a) \right| + \left| f'(b) \right| \right)}{2^{\frac{\alpha}{k}+1}(\frac{\alpha}{k}+1)}. \end{aligned}$$

ii) *if we suppose $h(t) = t^s$, then we have the following inequality for the s-convex functions in second sense:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+,\ k}^{\alpha} f(b) + I_{b-,\ k}^{\alpha} f(a) \right] \right| \\ \leqslant & \frac{(b-a)\left(\left| f'(a) \right| + \left| f'(b) \right| \right)}{2} \\ &\times \left[\frac{1}{\frac{\alpha}{k}+s+1} + \beta\left(\frac{1}{2}; s+1, \frac{\alpha}{k}+1\right) + \beta\left(\frac{1}{2}; \frac{\alpha}{k}+1, s+1\right) \right]. \end{aligned}$$

Theorem 5.9. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be positive function with $0 \leqslant a < b$ and $h^q \in L_1[0,1]$, $f \in L_1[a,b]$. If $\left|f^{'}\right|$ is an h-convex function on $[a,b]$, then the following inequality for generalized fractional integral holds:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right]\right| \tag{33}$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{2\Lambda(1)}\left[\left(\int_0^{\frac{1}{2}} (\Lambda(t))^p dt\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^1 (\Delta(t))^p dt\right)^{\frac{1}{p}}\right]$$
$$\times\left[\left(\int_0^{\frac{1}{2}} (h(t))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}\right],$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.5 and well known Hölder inequality and Minkowski inequality, we can easily prove the inequality (33). $\square$

Corollary 5.10. *Under the assumptions of Theorem 5.9 with $\varphi(t) = t$, we have the following inequality for h-convex functions via classical integrals:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{\left(2^{p+1}(p+1)\right)^{\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}} (h(t))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}\right].$$

Corollary 5.11. *In Corollary 5.10*

i) *if we suppose $h(t) = t$, then we have the following inequality for the classical convex functions:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right)}{\left(2^{p+1}(p+1)\right)^{\frac{1}{p}}}\left[\left(\frac{1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}} + \left(\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}\right].$$

ii) *if we suppose $h(t) = t^s$, then we have the following inequality for the classical* *convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)}{\left(2^{p+1}(p+1)\right)^{\frac{1}{p}}}\left[\left(\frac{1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}\right].$$

Corollary 5.12. *Under the assumptions of Theorem 5.9 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, we have the following inequality for h-convex functions via Riemann-Liouville fractional integrals:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]\right|$$
$$\leqslant \quad \frac{(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)}{\left(2^{\alpha p+1}(\alpha p+1)\right)^{\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(h(t))^q dt\right)^{\frac{1}{q}}\right].$$

Corollary 5.13. *In Corollary 5.12,*

i) *if we take $h(t)=t$, then we obtain the following inequality for classical convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]\right|$$
$$\leqslant \quad \frac{(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)}{\left(2^{\alpha p+1}(\alpha p+1)\right)^{\frac{1}{p}}}\left[\left(\frac{1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}\right].$$

ii) *if we suppose $h(t)=t^s$, then we have the following inequality for s-convex functions in second sense:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]\right|$$
$$\leqslant \quad \frac{(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)}{\left(2^{\alpha p+1}(\alpha p+1)\right)^{\frac{1}{p}}}\left[\left(\frac{1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}\right].$$

Corollary 5.14. *Under the assumptions of Theorem 5.9 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequality for h-convex functions via k-Riemann-Liouville* *fractional integrals:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{\kappa}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{\left(2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)\right)^{\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(h(t))^q dt\right)^{\frac{1}{q}}\right].$$

Corollary 5.15. *In Corollary 5.14,*

i) *if we take $h(t)=t$, then we obtain the following inequality for classical convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{\kappa}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{\left(2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)\right)^{\frac{1}{p}}}\left[\left(\frac{1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right)^{\frac{1}{q}}\right].$$

ii) *if we suppose $h(t)=t^s$, then we have the following inequality for s-convex functions in second sense:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{\kappa}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{\left(2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)\right)^{\frac{1}{p}}}\left[\left(\frac{1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}+\left(\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}\right].$$

Theorem 5.16. *Let $h: J\subset\mathbb{R}\to\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ be positive function with $0\leqslant a<b$ and $h^q\in L_1[0,1]$, $f\in L_1[a,b]$. If $\left|f^{'}\right|$ is an h-convex function on $[a,b]$, then the following inequality for generalized fractional integral holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \tag{34}$$
$$\leqslant \frac{(b-a)\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)}{2\Lambda(1)}\left[B_1-B_2\right],$$

where

$$\begin{aligned} B_1 &= \left(\int_0^{\frac{1}{2}} \Lambda(t)dt\right)^{1-\frac{1}{q}} \\ &\quad\times\left[\left(\int_0^{\frac{1}{2}} \Lambda(t)[h(t)]^q dt\right)^{\frac{1}{q}} + \left(\int_0^{\frac{1}{2}} \Lambda(t)[h(1-t)]^q dt\right)^{\frac{1}{q}}\right], \\ B_2 &= \left(\int_0^{\frac{1}{2}} \Delta(t)dt\right)^{1-\frac{1}{q}} \\ &\quad\times\left[\left(\int_{\frac{1}{2}}^1 \Delta(t)[h(t)]^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 \Delta(1-t)[h(1-t)]^q dt\right)^{\frac{1}{q}}\right], \end{aligned}$$

and $q > 1$.

Proof. From Lemma 2.5 and well known power mean inequality, we can easily prove our desired inequality (34). □

Theorem 5.17. *Let* $h : J \subset \mathbb{R} \to \mathbb{R}$ *and* $f : [a,b] \to \mathbb{R}$ *be positive function with* $0 \leqslant a < b$ *and* $h^q \in L_1[0,1]$, $f \in L_1[a,b]$. *If* $\left|f'\right|^q$ *is an h-convex function on* $[a,b]$, *then the following inequality for generalized fractional integral holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)\right]\right| \qquad (35) \\ \leqslant\ & \frac{(b-a)}{\Lambda(1)}\left(\int_0^{\frac{1}{2}} (|\Lambda(t)|)^p dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} h(t)\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ &+\frac{(b-a)}{\Lambda(1)}\left(\int_{\frac{1}{2}}^1 (|\Delta(t)|)^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 h(t)\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. From Lemma 2.5 and well known Hölder inequality, we obtain the following inequality

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \qquad (36)\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\int_0^{\frac{1}{2}}|\Lambda(t)|\left|f^{'}(tb+(1-t)a)\right|dt+\int_0^{\frac{1}{2}}|\Lambda(t)|\left|f^{'}(ta+(1-t)b)\right|dt\\
&+\int_{\frac{1}{2}}^1|\Delta(t)|\left|f^{'}(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1|\Delta(t)|\left|f^{'}(ta+(1-t)b)\right|dt\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left\{\left(\int_0^{\frac{1}{2}}(|\Lambda(t)|)^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}\left|f^{'}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{2}}\left|f^{'}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{2}}^1(|\Delta(t)|)^p\,dt\right)^{\frac{1}{p}}\\
&\left.\times\left[\left(\int_{\frac{1}{2}}^1\left|f^{'}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|f^{'}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

Since $\left|f^{'}\right|^q\in SX(h,I)$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left\{\left(\int_0^{\frac{1}{2}}(|\Lambda(t)|)^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}\left[h(t)\left|f^{'}(b)\right|^q+h(1-t)\left|f^{'}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{2}}\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{2}}^1(|\Delta(t)|)^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1\left[h(t)\left|f^{'}(b)\right|^q+h(1-t)\left|f^{'}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^1\left[h(t)\left|f^{'}(a)\right|^q+h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]\right\}\\
=\ & \frac{(b-a)}{\Lambda(1)}\left(\int_0^{\frac{1}{2}}(|\Lambda(t)|)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}h(t)\left[\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{(b-a)}{\Lambda(1)}\left(\int_{\frac{1}{2}}^1(|\Delta(t)|)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1h(t)\left[\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}.
\end{aligned}$$

and thus the proof is completed. $\square$

Corollary 5.18. *Under the assumptions of Theorem 5.17 with $\varphi(t) = t$, then we obtain the following inequality for h-convex functions via classical integrals:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-1}\int_a^b f(x)dx \right| \\
&\leqslant \frac{(b-a)}{\left(2^{p+1}(p+1)\right)^{\frac{1}{p}}} \left[\left(\int_0^{\frac{1}{2}} h(t) \left[\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q \right] dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 h(t) \left[\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q \right] dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 5.19. *In Corollary 5.18,*

i) *if we take $h(t) = t$, then we have the following inequality for classical convex functions:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-1}\int_a^b f(x)dx \right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{p+1}(p+1)\right)^{\frac{1}{p}}} \left[\left(\frac{\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3\left(\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q\right)}{8} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

ii) *if we take $h(t) = t^s$, then we have the following inequality for s-convex functions in second sense:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-1}\int_a^b f(x)dx \right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{p+1}(p+1)\right)^{\frac{1}{p}}} \\
\times & \left[\left(\frac{\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{2^{s+1}(s+1)} \right)^{\frac{1}{q}} + \left(\frac{(2^{s+1}-1)\left(\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q\right)}{2^{s+1}(s+1)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 5.20. *Under the assumptions of Theorem 5.17 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we obtain the following inequality for h-convex functions via Riemann-*

Liouville fractional integrals:

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{a+}^{\alpha} f(b) + I_{b-}^{\alpha} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{\alpha p+1}(\alpha p+1)\right)^{\frac{1}{p}}} \left[\left(\int_0^{\frac{1}{2}} h(t) \left[\left| f^{'}(a) \right|^q + \left| f^{'}(b) \right|^q \right] dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\int_{\frac{1}{2}}^{1} h(t) \left[\left| f^{'}(a) \right|^q + \left| f^{'}(b) \right|^q \right] dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 5.21. *In Corollary 5.20,*

i) *if we suppose $h(t) = t$, Then we obtain the following inequality for the classical convex functions:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{a+}^{\alpha} f(b) + I_{b-}^{\alpha} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{\alpha p+1}(\alpha p+1)\right)^{\frac{1}{p}}} \\
&\times \left[\left(\frac{3\left| f^{'}(a) \right|^q + \left| f^{'}(b) \right|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3\left(\left| f^{'}(a) \right|^q + \left| f^{'}(b) \right|^q \right)}{8} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

ii) *if we choose $h(t) = t^s$, then we obtain the following inequality for s-convex functions in second sense:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{a+}^{\alpha} f(b) + I_{b-}^{\alpha} f(a) \right] \right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{\alpha p+1}(\alpha p+1)\right)^{\frac{1}{p}}} \\
&\times \left[\left(\frac{\left| f^{'}(a) \right|^q + \left| f^{'}(b) \right|^q}{2^{s+1}(s+1)} \right)^{\frac{1}{q}} + \left(\frac{(2^{s+1}-1)\left(\left| f^{'}(a) \right|^q + \left| f^{'}(b) \right|^q \right)}{2^{s+1}(s+1)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 5.22. *Under the assumptions of Theorem 5.17 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we obtain the following inequality for h-convex functions via k-Riemann-*

Liouville fractional integrals:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{\left(2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)\right)^{\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}}h(t)\left[\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}h(t)\left[\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right].$$

Corollary 5.23. *In Corollary (5.22),*

i) *If we suppose $h(t)=t$, then we obtain the following inequality for the classical convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{\left(2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)\right)^{\frac{1}{p}}}$$
$$\times\left[\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left(\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right)}{8}\right)^{\frac{1}{q}}\right].$$

ii) *if we choose $h(t)=t^s$, then we obtain the following inequality for s-convex functions in second sense:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{\left(2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)\right)^{\frac{1}{p}}}$$
$$\times\left[\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)\left(\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q\right)}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right].$$

Theorem 5.24. *Let $h:J\subset\mathbb{R}\to\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ be positive function with $0\leqslant a<b$ and $h^q\in L_1[0,1]$, $f\in L_1[a,b]$. If $\left|f^{'}\right|^q$, $q\geqslant 0$ is an h-convex function on $[a,b]$, then the following inequality for generalized fractional integral holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right]\right| \qquad (37)\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left[\left(\int_0^{\frac{1}{2}} |\Lambda(t)|\, dt\right)^{1-\frac{1}{q}}\right.\\
&\times\left\{\left(\int_0^{\frac{1}{2}} \Lambda(t)\left[h(t)\left|f^{'}(b)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right] dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^{\frac{1}{2}} \Lambda(t)\left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right] dt\right)^{\frac{1}{q}}\right\}\\
&+\frac{(b-a)}{2\Lambda(1)}\left(\int_{\frac{1}{2}}^1 |\Delta(t)|\, dt\right)^{1-\frac{1}{q}}\\
&\times\left\{\left(\int_{\frac{1}{2}}^1 \Delta(t)\left[h(t)\left|f^{'}(b)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right] dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^1 \Delta(t)\left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right] dt\right)^{\frac{1}{q}}\right\}\right],
\end{aligned}$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. From Lemma 2.6 and power mean inequality, we have the following inequality

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right]\right| \qquad (38)\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left[\int_0^{\frac{1}{2}} |\Lambda(t)|\left|f^{'}(tb+(1-t)a)\right| dt + \int_0^{\frac{1}{2}} |\Lambda(t)|\left|f^{'}(ta+(1-t)b)\right| dt\right.\\
&\left.+\int_{\frac{1}{2}}^1 |\Delta(t)|\left|f^{'}(tb+(1-t)a)\right| dt + \int_{\frac{1}{2}}^1 |\Delta(t)|\left|f^{'}(ta+(1-t)b)\right| dt\right]\\
\leqslant\ & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^{\frac{1}{2}} |\Lambda(t)|\, dt\right)^{1-\frac{1}{q}}\left\{\left(\int_0^{\frac{1}{2}} |\Lambda(t)|\left|f^{'}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^{\frac{1}{2}} |\Lambda(t)|\left|f^{'}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&+\frac{(b-a)}{2\Lambda(1)}\left(\int_{\frac{1}{2}}^1 |\Delta(t)|\, dt\right)^{1-\frac{1}{q}}\left\{\left(\int_{\frac{1}{2}}^1 |\Delta(t)|\left|f^{'}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1 |\Delta(t)|\left|f^{'}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\left|f'\right|^q \in SX(h,I)$, then (38) becomes

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\
\leqslant\; & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^{\frac{1}{2}}|\Lambda(t)|\right)^{1-\frac{1}{q}} \\
& \times\left\{\left(\int_0^{\frac{1}{2}}\Lambda(t)\left[h(t)\left|f'(b)\right|^q+h(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_0^{\frac{1}{2}}\Lambda(t)\left[h(t)\left|f'(a)\right|^q+h(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
& +\frac{(b-a)}{2\Lambda(1)}\left(\int_{\frac{1}{2}}^1|\Delta(t)|\right)^{1-\frac{1}{q}} \\
& \times\left\{\left(\int_{\frac{1}{2}}^1\Delta(t)\left[h(t)\left|f'(b)\right|^q+h(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1\Delta(t)\left[h(t)\left|f'(a)\right|^q+h(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

Corollary 5.25. *Under the assumptions of Theorem 5.24 with $\varphi(t)=t$, we have the following inequality for h-convex functions via classical integrals:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leqslant\; & \frac{(b-a)}{2(8)^{1-\frac{1}{q}}}\left[\left(\int_0^{\frac{1}{2}}t\left[h(t)\left|f'(b)\right|^q+h(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
& +\left(\int_0^{\frac{1}{2}}t\left[h(t)\left|f'(a)\right|^q+h(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{2}}^1(1-t)\left[h(t)\left|f'(b)\right|^q+h(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\
& \left.+\left(\int_{\frac{1}{2}}^1(1-t)\left[h(t)\left|f'(a)\right|^q+h(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 5.26. *In Corollary 5.25,*

i) *if we take $h(t) = t$, then we have the following new inequality for the classical convex functions:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leqslant \frac{(b-a)}{8(3)^{\frac{1}{q}}}\left[\left(\left|f^{'}(b)\right|^q + 2\left|f^{'}(a)\right|^q\right)^{\frac{1}{q}} + \left(2\left|f^{'}(b)\right|^q + \left|f^{'}(a)\right|^q\right)^{\frac{1}{q}}\right].$$

ii) *if we take $h(t) = t^s$, then we have the following new inequality for the s-convex functions in second sense:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leqslant \frac{(b-a)}{(8)^{1-\frac{1}{q}}}\left[\left(\frac{\left|f^{'}(b)\right|^q}{2^{s+2}(s+2)} + \frac{(2^{s+2}-s-3)\left|f^{'}(a)\right|^q}{2^{s+2}(s+1)(s+2)}\right)\right.$$
$$\left. + \left(\frac{\left|f^{'}(a)\right|^q}{2^{s+2}(s+2)} + \frac{(2^{s+2}-s-3)\left|f^{'}(b)\right|^q}{2^{s+2}(s+1)(s+2)}\right)\right].$$

Corollary 5.27. *Under the assumptions of Theorem 5.24 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we have the following inequality for h-convex functions via Riemann-Liouville fractional integrals:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[I_{a+}^{\alpha} f(b) + I_{b-,}^{\alpha} f(a)\right] \right|$$
$$\leqslant \frac{(b-a)}{2(2^{\alpha+1}(\alpha+1))^{1-\frac{1}{q}}}\left[\left(\int_0^{\frac{1}{2}} t^\alpha \left[h(t)\left|f^{'}(b)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$+ \left(\int_0^{\frac{1}{2}} t^\alpha \left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+ \left(\int_{\frac{1}{2}}^1 (1-t)^\alpha \left[h(t)\left|f^{'}(b)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^\alpha \left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right].$$

Corollary 5.28. *In Corollary 5.27*
i) *if we suppose $h(t) = t$, we obtain the following inequality for classical convex functions:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I^{\alpha}_{a+,}f(b) + I^{\alpha}_{b-}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{\alpha+1}(\alpha+1)\right)^{1-\frac{1}{q}}}\left[\left(\frac{\left|f^{'}(b)\right|^{q}}{2^{\alpha+2}(\alpha+2)} + \frac{(\alpha+3)\left|f^{'}(a)\right|^{q}}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{\left|f^{'}(a)\right|^{q}}{2^{\alpha+2}(\alpha+2)} + \frac{(\alpha+3)\left|f^{'}(b)\right|^{q}}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

ii) *if we suppose $h(t) = t^{s}$, then we obtain the following inequality for the s-convex functions in second sense:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I^{\alpha}_{a+,}f(b) + I^{\alpha}_{b-}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)}{\left(2^{\alpha+1}(\alpha+1)\right)^{1-\frac{1}{q}}} \\
&\times\left[\left(\frac{\left|f^{'}(b)\right|^{q}}{2^{s+\alpha+1}(s+\alpha+1)} + \left|f^{'}(a)\right|^{q}\beta\left(\frac{1}{2}; s+1, \alpha+1\right)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{\left|f^{'}(a)\right|^{q}}{2^{s+\alpha+1}(s+\alpha+1)} + \left|f^{'}(b)\right|^{q}\beta\left(\frac{1}{2}; s+1, \alpha+1\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 5.29. *Under the assumptions of Theorem 5.24 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequality for h-convex functions via k-Riemann-Liouville fractional integrals:*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b) + I^{\alpha}_{b-,\ k}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)}{2(2^{\frac{\alpha}{k}+1}(\frac{\alpha}{k}+1))^{1-\frac{1}{q}}} \\
&\times\left[\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}\left[h(t)\left|f^{'}(b)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}\left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}}\left[h(t)\left|f^{'}(b)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}}\left[h(t)\left|f^{'}(a)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 5.30. *In Corollary 5.27,*

i) *if we suppose $h(t) = t$, we obtain the following inequality for classical convex functions:*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b) + I^{\alpha}_{b-\ k}f(a)\right]\right| \\
\leqslant\ & \frac{(b-a)}{(2^{\frac{\alpha}{k}+1}(\frac{\alpha}{k}+1))^{1-\frac{1}{q}}} \\
&\times\left[\left(\frac{\left|f^{'}(b)\right|^q}{2^{\frac{\alpha}{k}+2}(\frac{\alpha}{k}+2)} + \frac{(\frac{\alpha}{k}+3)\left|f^{'}(a)\right|^q}{2^{\frac{\alpha}{k}+2}(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{\left|f^{'}(a)\right|^q}{2^{\frac{\alpha}{k}+2}(\frac{\alpha}{k}+2)} + \frac{(\frac{\alpha}{k}+3)\left|f^{'}(b)\right|^q}{2^{\frac{\alpha}{k}+2}(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

ii) *if we suppose $h(t) = t^s$, then we obtain the following inequality for the s-convex functions in second sense:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{(2^{\frac{\alpha}{k}+1}(\frac{\alpha}{k}+1))^{1-\frac{1}{q}}}\left[\left(\frac{\left|f^{'}(b)\right|^q}{2^{s+\frac{\alpha}{k}+1}(s+\frac{\alpha}{k}+1)}+\left|f^{'}(a)\right|^q\beta\left(\frac{1}{2};s+1,\frac{\alpha}{k}+1\right)\right)^{\frac{1}{q}}\right.$$
$$\left.\left(\frac{\left|f^{'}(a)\right|^q}{2^{s+\frac{\alpha}{k}+1}(s+\frac{\alpha}{k}+1)}+\left|f^{'}(b)\right|^q\beta\left(\frac{1}{2};s+1,\frac{\alpha}{k}+1\right)\right)^{\frac{1}{q}}\right].$$

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