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Original Research Paper

A Stabilized P_1-P_0-P_1 Finite Element Method for the Approximation of Linearized Viscoelastic Fluid Flow Problems

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Abstract. In this article, we present a stabilized finite element (FE) method for the linearized viscoelastic fluid flow problems by Discontinuous Galerkin (DG) Method. The FE spaces for the unknown variables are chosen as $P1-P0-P1$, where the fluid velocity and the pressure are discretized by the lowest-order Lagrange elements and the stress tensor is discretized by piecewise $P1$ polynomial. In order to get a stable scheme, we added a stabilization term in the discretized weak formulation. This method has some prominent features: parameter-free, avoiding calculation of higher-order derivatives and its behaviour towards pressure is totally local. We obtained optimal error estimates and presented several numerical experiments to verify the proposed scheme.

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1 Introduction

Viscoelastic fluids are actually a type of non-Newtonian fluids. All those fluids that can not be modeled by a linear constitutive law equation are called non-Newtonian fluids. They are characterized by having difficult and high molecular weight molecules with many degrees of freedom. The polymer solutions and molten polymers are good examples of viscoelastic fluids. Many cases can be found in industrials and practical applications like biological fluids, adhesives, food products, greases, blood in some typical cases, and so on [1, 2].

The importance of viscoelastic fluid flow models has increased in the last few years and motivates researchers due to several reasons, among them the following are common: These fluids have long chain molecules with many internal degrees of freedom, these type of fluids need hyperbolic type constitutive equations with advective non-linear terms, this highly non-linear term may create oscillations both globally and locally in numerical approximation. Indeed it needs some stabilization terms to deal with these difficulties. However, to study viscoelastic properties of the fluids many models have been proposed and applied i.e. Phan-Thien-Tanner model, Maxwell mode, Jeffrey model, Oldroyd-B model [3], and Johnson and Segalman model [4].

Moreover, the literature of the finite element method is increasing day by day to approximate the viscoelastic fluid flow models and a variety of alternative stabilization formulations are continuously developing. Additionally, so far the well-known methods available to approximate the viscoelastic fluid flow in finite element literature are as following; streamline-upwind Petrov-Galerkin (SUPG) method [5], discontinuous Galerkin method (DG), two-level Oseen viscoelastic fluid flow model [6], variational multiscale method [11, 9, 10] and multiscale method based on two local Gauss integrations [12] etc. We are interested to apply the DG method for the FE approximation of stress. As our best knowledge, this method was first introduced by Reed, and Hill in [13]. After that, to deal with the hyperbolic nature of the constitutive equation, Lesaint and Raviart [14] discussed and applied the DG method on the neutron transport equation. later, Baranger and Sandri [16] have proved and analysed the stability and error estimates of the DG method.

However, the viscoelastic fluid flow models are strongly non-linear models, because the non-Newtonian fluid flow obeying the Oldroyd-B model is the combination of conservation of momentum equations and constitutive equations. This model equations are the combination or couples with the Navier-Stokes equations. Its no doubt that the Navier-Stokes equations itself are non-linear. The non-linear convective term of the Navier-Stokes equation can be reduced in a linear system by replacing the unknown velocity with a known velocity field [18]. So in the viscoelastic fluid flow model, the non-linearity occurs only in the constitutive equation [17] which may be also replaced by a known value in the convective term i.e., $\vec{u}(x) = \vec{b}(x)$. By doing this process the non-linear equations now modified into a linear form, what we call here a linearized viscoelastic model.

We study the mixed FE method to approximate linearized viscoelastic fluid flow, which was developed to approximate both pressure and velocity simultaneously. To find the well-posedness of the velocity and pressure, it is important to satisfy the inf-sup conditions. There are very few finite element spaces which are applicable to full fill these conditions see in [21, 22]. But some other spaces are also available with some modifications worked well. So in this contribution, we are interested to choose the lowest finite elements $P1 - P0 - P1$. Undispute, these pairs of elements fail to satisfy the inf-sup condition [23] but can work similarly well with the available spaces i.e., Taylor-Hood and MINI elements. To make lowest-equal order elements applicable, we introduce a technical stabilization term in the discrete variational formulation. This stabilization technique is already discussed in the literature for the Stokes problems and also for the viscoelastic fluid flow problems (for example see [7, 8, 24, 25] and the references cited therein). The Lowest order FE and stabilization methods have also been considered in Stokes-Darcy and Navier-Stokes (see[27, 26]). So far, this method is novel to solve the linearized version of viscoelastic fluid flow model equations with $P1 - P0 - P1$ elements. For further justification, we have given several numerical tests for the applicability of these elements in the second last section.

This method has some important features, it is free of parameters, it does not need any higher-order derivatives, and it can be easily overwritten computational code in the existing framework of the viscoelastic fluid flow easily. We prove the well-posedness of the finite element scheme with the weak coercivity method, and optimal convergence order is obtained. To show the validation of the theoretical analysis we present several numerical examples.

The paper is organized as follows: In the next section, we give an introduction of the governing equations for the linearized viscoelastic fluid flow model and some notations. In section 3, we discuss the variational formulation, and some well-known lemmas. To justify the proposed scheme, the stability and optimal convergence analysis are derived in section 4. The results of the numerical simulations are illustrated in section 5. We end this work with short conclusion. We deal with the following two-dimensional steady-state modeling equations under the influence of applied forces and stresses which are reported in [29, 28].

$$\tau + \lambda((\vec{u} \cdot \nabla\tau) + \widehat{g}(\tau, \nabla\vec{u})) - 2\alpha D(\vec{u}) = 0, \tag{1}$$

where λ is denoted as Weissenberg number [30]. Hence the $\widehat{g}(\tau, \nabla\vec{u})$ is defined as:

$$\widehat{g}(\tau, \nabla\vec{u}) = \frac{1-a}{2}(\tau(\nabla\vec{u}) + (\nabla\vec{u})^T\tau) - \frac{1+a}{2}((\nabla\vec{u})\tau + \tau(\nabla\vec{u})^T). \tag{2}$$

$a \in [-1, 1]$ is considered as a material parameter. The momentum equation under influence of forces $\mathbf{f}$ can be written as:

$$R^e(\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u}) - \nabla \cdot \tau^{tot} = \mathbf{f}.$$

Here R^e is known as Reynolds number, while Oldroyd model is described by $\tau^T = -pI + \tau_N + \tau$ where τ_N and τ are, the Newtonian and viscoelastic part of the extra stress tensor. The Newtonian part is given by $\tau_N = 2(1-\alpha)D(\vec{u})$ and deformation tensor is defined as:

$$D(\vec{u}) = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T).$$

The gradient of $\vec{u}$ is defined in such that $(\nabla\vec{u})_{ij} = \frac{\partial \vec{u}_i}{\partial x_j} = \vec{u}_{j,i}$. The viscoelastic flows are important for understanding several problems in non-Newtonian fluid mechanics especially those related to flow instabilities [29]. The $(1-\alpha)$ denotes the total viscosity; which is considered Newtonian. Hence $\alpha \in (0,1)$ implies the proportion of total viscosity that is supposed to be viscoelastic in nature. We re-write the momentum equation based on the above information as [7, 8].

$$R^e(\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u}) - 2(1-\alpha)\nabla\cdot D(\vec{u}) - \nabla\cdot\tau + \nabla p = \mathbf{f}.$$

1.1 Governing equations

We consider the open bounded and connected domain Ω with homogenous Dirichlet boundary conditions in the viscoelastic case; but there is no inflow boundary, and, thus, no boundary condition is required for stress tensors in this model equation [8, 7]. So the model equations are given by:

$$\begin{aligned} \tau + \lambda(\vec{u}\cdot\nabla)\tau + \lambda\widehat{g}(\tau,\nabla\vec{u}) - 2\alpha D(\vec{u}) &= 0, \quad \text{in } \Omega, \quad (3)\\ \nabla p - 2(1-\alpha)\nabla\cdot D(\vec{u}) - \nabla\cdot\tau &= \mathbf{f}, \quad \text{in } \Omega, \quad (4)\\ \nabla\cdot\vec{u} &= 0, \quad \text{in } \Omega, \quad (5)\\ \vec{u} &= 0, \quad \text{on } \Gamma. \quad (6) \end{aligned}$$

Where $\mathbf{f}$ represents a body force, τ denotes polymeric stress tensor, $\vec{u}$ the velocity vector, p pressure of the fluid and λ is the Weissenberg number. The well-posedness of the (3)-(6) are discussed in [33, 31].

In the main analysis part, we will consider the Oseen system as a linearization of the model equations as
Find the solution $(\tau,\vec{u},p)$ such that:

$$\begin{aligned} \tau + \lambda(\mathbf{b}\cdot\nabla)\tau + \lambda\widehat{g}(\tau,\nabla\vec{b}) - 2\alpha D(\vec{u}) &= 0 \quad \text{in } \Omega, \quad (7)\\ \nabla p - 2(1-\alpha)\nabla\cdot D(\vec{u}) - \nabla\cdot\tau &= \mathbf{f} \quad \text{in } \Omega, \quad (8)\\ \nabla\cdot\vec{u} &= 0 \quad \text{in } \Omega, \quad (9)\\ \vec{u} &= 0 \quad \text{on } \partial\Omega. \quad (10) \end{aligned}$$

It is well-known from [33] as :
$\vec{b} \in H_0^1(\Omega), \quad \nabla\cdot\vec{b} = 0, \quad \|\vec{b}\|_\infty \leq \mathcal{M}, \quad \|\nabla\vec{b}\|_\infty \leq \mathcal{M} < \infty.$

1.2 Variational formulation

We consider some notations used in function spaces are as Sobolev space written as $H^1(\Omega)$ with the norm $||\cdot||_1$ and seminorm $|\cdot|_1$. The $\mathcal{L}^2(\Omega)$ inner product and norm are denoted by $(\cdot,\cdot)$, $||\cdot||=||\cdot||_0$ respectively, the $\mathcal{L}^p(\Omega)$ norm by $||\cdot||_{\mathcal{L}^p}$, with the special cases of $\mathcal{L}^2(\Omega)$ and $\mathcal{L}^\infty(\Omega)$ norms being written as $||\cdot||$ and $||\cdot||_\infty$.

For further discussion, we introduce spaces for the unknown velocity $\vec{u}$, the pressure p, and the stress tensor τ as:

$$
\begin{aligned}
\mathcal{X}: &= H_0^1(\Omega)^2 := \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v} = 0 \text{ on } \partial\Omega\}, \\
\mathcal{Q}: &= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}, \\
\mathcal{S}: &= \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} : \tau_{ij} \in L^2(\Omega); i,j = 1,2\} \\
& \quad \cap \{\tau = (\tau_{ij}); \vec{b} \cdot \nabla\tau \in L^2(\Omega), \forall\, \vec{b} \in X\}.
\end{aligned}
$$

In order to find the corresponding variational formulation, by taking the inner product of stress, velocity, and pressure test functions σ, $\mathbf{v}$, and q respectively.
The given $\mathbf{f} \in H^{-1}$ (Ω), to seek $(\tau,\ \vec{u},\ \text{p}) \in \mathcal{S} \times \mathcal{X} \times \mathcal{Q}$ such that

$$(\tau,\sigma) + \lambda((\vec{b}\cdot\nabla)\tau,\sigma) + \lambda(\widehat{g}(\tau,\nabla\vec{b}),\sigma) - 2\alpha(D(\vec{u}),\sigma) = 0, \quad \forall\sigma \in S, \tag{11}$$

$$-(p,\nabla\cdot\mathbf{v}) + 2(1-\alpha)(D(\vec{u}),D(\mathbf{v})) + (\tau,D(\mathbf{v})) = (\mathbf{f},\mathbf{v}), \quad \forall\mathbf{v} \in X, \tag{12}$$

$$(q,\nabla\cdot\vec{u}) = 0, \quad \forall q \in Q. \tag{13}$$

Note that the velocity and pressure spaces, $\mathcal{X}$ and $\mathcal{Q}$, satisfy the inf-sup (or LBB) condition, we refer the reader to [7]. For further simplification, we have multiplied 2α with equation (12) and equation (13) and adding all equations together. After that, for ease of calculations we introduce two operators $\mathcal{J}((\cdot,\cdot,\cdot),(\cdot,\cdot,\cdot))$ and $\mathcal{B}((\cdot,\cdot,\cdot)$ to make the compact form as:

$$\begin{aligned}\bar{\mathcal{J}}((\tau,\vec{u},p),(\sigma,\mathbf{v},q)) &= (\tau,\sigma)+\lambda(\widehat{g}(\tau,\nabla\vec{b}),\sigma)-2\alpha(D(\vec{u}),\sigma)\\ &\quad +2\alpha(\tau,D(\mathbf{v}))+4\alpha(1-\alpha)(D(\vec{u}),D(\mathbf{v}))\\ &\quad -2\alpha(p,\nabla\cdot\mathbf{v})+2\alpha(q,\nabla\cdot\vec{u}), \qquad (14)\\ \lambda\mathcal{B}(\vec{b},\tau,\sigma) &= \lambda((\vec{b}\cdot\nabla)\tau,\sigma). \qquad (15)\end{aligned}$$

An equivalent formulation to (14)-(15) can be written as:

$$\bar{\mathcal{J}}((\tau,\vec{u},p),(\sigma,\mathbf{v},q))+\lambda\mathcal{B}(\vec{b},\tau,\sigma) = 2\alpha(\mathbf{f},\mathbf{v}). \qquad (16)$$

2 Discontinuous finite element approximation

Mostly in finite element literature, there are two main approaches have been used to approximate the viscoelastic model problems i.e., *DG* method and *SUPG* method [34, 37]. Here we concern, the *DG* method as an upwinding technique. Let $\mathcal{T}^h$ be a triangulation of Ω such that $\widetilde{\Omega}=\{\cup K : K\in T^h\}$. We assume that there exists a positive constants r_1,r_2, hence

$$r_1 h \;\leq\; h_k \;\leq\; r_2 R_k.$$

Where h_k is the diameter of an element (triangle) K, R_k is the diameter of the greatest ball included in K, and $h = max_{K\in\mathcal{T}^h} h_k$.
Let us consider $P_k(K)$ denotes the space of polynomials of degree less or equal to k on $K\in\mathcal{T}^h$. We introduce discrete subspaces for the finite element approximation of the equation (16).

$$\begin{aligned}\mathcal{X}^h &:= \{\mathbf{v}\in X\cap C^0(\widetilde{\Omega})^2; \mathbf{v}_{|K}\in P_1(K)^2, \forall K\in\mathcal{T}^h\},\\ \mathcal{Q}^h &:= \{q\in Q\cap C^0(\widetilde{\Omega}); q_{|K}\in P_0(K); \forall K\in\mathcal{T}^h\},\\ \mathcal{S}^h &:= \{\sigma\in S; \sigma_{|K}\in P_1(K)^{2\times 2}; \forall K\in\mathcal{T}^h\},\end{aligned}$$

We re-introduced some notations for approximating the stress which is analyzed elsewhere [28, 32], as we define $\partial K^{h-}(\vec{b}) = \{x\in\partial K^k; \vec{b}(x)\cdot\vec{n}(x)<0\}$ where ∂K^h is the boundary of $K^h\in\mathcal{T}^h$ and $\vec{n}$ is the outward unit normal to ∂K^h, and

$$\begin{aligned}\partial\Omega^h &= \{\cup\partial K : K\in\mathcal{T}^h\}\setminus\partial\Omega,\\ \tau^{\pm}(\vec{b}(x)) &= \lim_{\varepsilon\to 0}\tau(x\pm\varepsilon\vec{b}(x)).\end{aligned}$$

Also, for any $(\tau, \sigma) \in \prod_{K^h \in \mathcal{T}^h} [H^1(K^h)]^4$, we define

$$\begin{aligned}
(\tau, \sigma)_h &= \sum_{K^h \in \mathcal{T}^h} (\tau, \sigma)_{K^h}, \\
\langle \tau^{\pm}, \sigma^{\pm} \rangle_{h,\vec{b}} &= \sum_{K^h \in \mathcal{T}^h} \int_{\partial K^{h-}(\vec{b})} (\tau^{\pm}(\vec{b}), \sigma^{\pm}(\vec{b})) |\vec{n} \cdot \vec{b}| ds, \\
\langle\langle \tau^{\pm} \rangle\rangle^2_{h,\vec{b}} &= \langle \tau^{\pm}, \tau^{\pm} \rangle_{h,\vec{b}}, \\
\| \tau \|_{0,\partial\Omega^h} &= \Big(\sum_{K^h \in \mathcal{T}^h} | \tau |^2_{0,\partial K^h} \Big)^{1/2}.
\end{aligned}$$

The term $((\vec{b} \cdot \nabla)\tau, \sigma)$ is defined by [29]

$$\begin{aligned}
\mathcal{B}^h(\vec{b}, \tau^h, \sigma^h) &= ((\vec{b} \cdot \nabla)\tau^h, \sigma^h)_h + (1/2)(\nabla \cdot \vec{b}\tau^h, \sigma^h) \qquad (17) \\
&\quad + \langle \tau^{h+} - \tau^{h-}, \sigma^{h+} \rangle_{h,\vec{b}}, \\
&= -((\vec{b} \cdot \nabla)\sigma^h, \tau^h)_h - (1/2)(\nabla \cdot \vec{b}\sigma^h, \tau^h) \qquad (18) \\
&\quad + \langle \tau^{h-}, \sigma^{h-} - \sigma^{h+} \rangle_{h,\vec{b}}, \\
&= ((\vec{b} \cdot \nabla)\tau^h, \sigma^h)_h \qquad (19) \\
&\quad + \langle \tau^{h+} - \tau^{h-}, \sigma^{h+} \rangle_{h,\vec{b}}, \quad if \quad \nabla \cdot \vec{b} = 0.
\end{aligned}$$

Thus,

$$\mathcal{B}^h(\vec{b}, \tau^h, \tau^h) = (1/2)\langle\langle \tau^{h+} - \tau^{h-} \rangle\rangle^2_{h,\vec{b}} \geq 0. \qquad (20)$$

To find stability of discrete problem, the finite element pair $(\mathcal{X}^h, \mathcal{Q}^h)$ must satisfy inf-sup condition:

$$\sup_{\mathbf{v}^h \in \mathcal{X}^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{|| \mathbf{v}^h ||_1} \geq C \, || q^h ||_0, \qquad \forall q^h \in \mathcal{Q}^h, \qquad (21)$$

where C is a positive constant independent of h.
Obviously, it is well-known for the finite element pairs, some space pairs do not satisfy the inf-sup conditions and results a pressure oscillation in numerical formulation. Hence it needs stabilization term to circumvent the inf-sup condition. Here our aim is to add the stabilization term. We define the following operator $\mathcal{G}(p, q) = ((I - \Pi)p, (I - \Pi)q)$ as stabilization term in this contribution, this has been discussed in the literature

[22, 27]. To the best of our knowledge, so far, this method has not been considered in the existing literature for the linearized viscoelastic fluids flows. For conciseness, we restate some important lemmas directly from the [22, 26, 25] to estimate the inequalities.

Lemma 2.1. *Let $\mathcal{Q}^h$ and $\mathcal{X}^h$ be the spaces defined above. Then, there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that*

$$\sup_{\boldsymbol{v}^h \in \mathcal{X}^h} \frac{\int_\Omega p^h \nabla \cdot \boldsymbol{v}^h d\Omega}{\| \boldsymbol{v}^h \|_1} \quad \geq \quad \mathcal{C}_1 \| p^h \|_0 - \mathcal{C}_2 h \| \nabla p^h \|_0, \forall p^h \in \mathcal{Q}^h. \tag{22}$$

Proof. See detail in [22].

Lemma 2.2. *There exists a positive non-zero constant C such that*

$$Ch \| \nabla p^h \|_0 \quad \leq \quad \| p^h - \Pi p^h \|_0 \, . \tag{23}$$

Proof. As given in [22].

The local pressure projection $\Pi : \mathcal{L}^2(\Omega) \to \mathbb{R}^2$ [22], this relation is symmetric and bilinear.

$$\mathcal{G}(p^h, q^h) \quad = \quad (p^h - \Pi p^h, q^h - \Pi q^h).$$

Now, we can write the finite element scheme with the additional term as: Find $(\vec{u}^h, \tau^h, p^h) \in (\mathcal{X}^h \times \mathcal{S}^h \times \mathcal{Q}^h)$ such that

$$\begin{aligned}
(\tau^h, \sigma^h) + \lambda \mathcal{B}^h(\vec{b}, \tau^h, \sigma^h) + \lambda(\widehat{g}(\tau^h, \nabla \vec{b}), \sigma^h) & \\
-2\alpha(D(\vec{u}^h), \sigma^h) &= 0, \forall \sigma^h \in \mathcal{S}^h, & (24)\\
-(p^h, \nabla \cdot \mathbf{v}^h) + 2(1-\alpha)(D(\vec{u}^h), D(\mathbf{v}^h)) & \\
+(\tau, D(\mathbf{v}^h)) &= (\mathbf{f}, \mathbf{v}^h), \forall \mathbf{v}^h \in \mathcal{X}^h, & (25)\\
(q^h, \nabla \cdot \vec{u}^h) + \mathcal{G}(p^h, q^h) &= 0, \forall q^h \in \mathcal{Q}^h. & (26)
\end{aligned}$$

We can rewrite the discrete system of equations (24)-(26) in the operator form, $\mathcal{J}$ as a coupled equations defined on $(\mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h)$ by

$$\begin{aligned}
\mathcal{J}((\tau^h, \vec{u}^h, p^h), (\sigma^h, \mathbf{v}^h, q^h)) = \; & (\tau^h, \sigma^h) + \lambda(g_a(\tau^h, \nabla \vec{b}), \sigma^h) \\
& - 2\alpha(D(\vec{u}^h), \sigma^h) + 2\alpha(\tau^h, D(\mathbf{v}^h)) \\
& + 4\alpha(1-\alpha)(D(\vec{u}^h), D(\mathbf{v}^h)) - 2\alpha(p^h, \nabla \cdot \mathbf{v}^h) \\
& + 2\alpha(q^h, \nabla \cdot \vec{u}^h) + 2\alpha \mathcal{G}(p^h, q^h). \qquad (27)
\end{aligned}$$

Further simplification, we can rewrite the discrete equations with operator $\mathcal{J}$ and $\mathcal{B}$ for the approximation of the unknowns.

Find $(\vec{u}^h, \tau^h, p^h) \in (\mathcal{X}^h \times \mathcal{S}^h \times \mathcal{Q}^h)$ satisfies $\forall(\sigma^h, \mathbf{v}^h, q^h) \in (\mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h)$

$$\mathcal{J}((\tau^h, \vec{u}^h, p^h), (\sigma^h, \mathbf{v}^h, q^h)) + \lambda\mathcal{B}^h((\vec{b}, \tau^h, \sigma^h)) \quad = \quad 2\alpha(\mathbf{f}, \mathbf{v}^h). \tag{28}$$

We make the following assumptions and inequalities:

$$\| \vec{u} - \tilde{\vec{u}}^h \|_1 \leq Ch \| \vec{u} \|_2, \tag{29}$$

$$\| p - \Pi\tilde{p}^h \|_0 \leq Ch \| p \|_1, \tag{30}$$

$$\begin{aligned} \| \tau - \tilde{\tau}^h \|_0 + h \| \nabla(\tau - \tilde{\tau}^h) \|_0 & \\ + h^{1/2} \| \tau - \tilde{\tau}^h \|_{0,\partial\Omega^h} &\leq Ch \| \tau \|_2, \end{aligned} \tag{31}$$

$$\| \Pi p^h \|_0 \leq C \| p \|_0 \,. \tag{32}$$

The inverse inequalities are recalled as [32]:

$$\begin{aligned} \| \nabla\tau^h \|_{0,h} &\leq Ch^{-1} \| \tau^h \|_0, \tau^h \in \mathcal{S}^h. \\ \| \tau^h \|^2_{0,\partial k} &\leq C\frac{1}{h_k} \| \tau^h \|^2_{0,k}, \tau^h \in \mathcal{S}^h. \end{aligned}$$

3 Stability and error estimates

In this section, we illustrate the existence and uniqueness of the finite element scheme. We assume that the problem satisfies the solution under the condition $\mathcal{M} > 0$.

Theorem 3.1. *Given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, if $1 - 2\lambda\mathcal{M}d > 0$. There exists a unique solution $(\tau^h, \vec{u}^h, p^h) \in (\mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h)$ satisfying (28).*

Proof.

$$\begin{aligned} \mathcal{J}((\tau^h, \vec{u}^h, p^h), (\sigma^h, \mathbf{v}^h, q^h)) + \lambda\mathcal{B}^h(\vec{b}, \tau^h, \sigma^h) \quad &= \quad F(\sigma^h, \mathbf{v}^h, q^h) \\ \forall(\sigma^h, \mathbf{v}^h, q^h) \in \mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h), & \end{aligned} \tag{33}$$

the right hand side of equation (33) can be explained as $\mathcal{F}(\cdot):\mathcal{S}^h\times\mathcal{X}^h\to\mathbb{R}$ is a functional so,

$$\mathcal{F}(\sigma^h,\mathbf{v}^h,q^h) \;=\; 2\alpha(\mathbf{f},\mathbf{v}^h).$$

We can easily get the continuity of the right hand side as:

$$\begin{aligned}|\,\mathcal{F}(\sigma^h,\mathbf{v}^h,q^h)\,| &\le 2\alpha \parallel \mathbf{f}\parallel_{-1}\parallel \mathbf{v}^h\parallel_1 \\ &\le 2\alpha \parallel \mathbf{f}\parallel_{-1}|||\;(\sigma^h,\mathbf{v}^h,q^h)\;|||_{(\mathcal{S}^h\times\mathcal{X}^h\times\mathcal{Q}^h)},\end{aligned} \tag{34}$$

where $|||\;(\sigma^h,\mathbf{v}^h,q^h)\;|||_{(\mathcal{S}^h\times\mathcal{X}^h\times\mathcal{Q}^h)}=(\parallel\sigma^h\parallel_0^2+\parallel\mathbf{v}^h\parallel_1^2+\parallel q^h\parallel_0^2)^{\frac{1}{2}}$. The operators $\mathcal{J}$ and $\mathcal{B}$ used in (33) is continuous and coercive in $(\mathcal{S}^h\times\mathcal{X}^h\times\mathcal{Q}^h)$, if $1-2\lambda\mathcal{M}d>0$. By using inverse inequalities we can get,

$$\begin{aligned}\mathcal{B}^h(\vec{b},\tau^h,\sigma^h) &= ((\vec{b}\cdot\nabla)\tau^h,\sigma^h)_h+\langle\tau^{h+}-\tau^{h-},\sigma^{h+}\rangle_{h,\vec{b}} \\ &\le C\big[\parallel\vec{b}\parallel_\infty\parallel\nabla\tau^h\parallel_{0,h}\parallel\sigma^h\parallel_0 \\ &\quad+\parallel\vec{b}\parallel_\infty\parallel\tau^h\parallel_{0,\partial\Omega^h}\parallel\sigma^h\parallel_{0,\partial\Omega^h}\big] \\ &\le C\big[\mathcal{M}\parallel\nabla\tau^h\parallel_0\parallel\sigma^h\parallel_0+\parallel\vec{b}\parallel_\infty(h^{-1/2}\parallel\tau^h\parallel_0) \\ &\quad\times(h^{-1/2}\parallel\sigma^h\parallel_0)\big] \\ &\le C\big[\mathcal{M}(h^{-1}\parallel\tau^h\parallel_0)\parallel\sigma^h\parallel_0 \\ &\quad+\mathcal{M}h^{-1}\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0\big] \\ &\le C\mathcal{M}h^{-1}\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0\,.\end{aligned} \tag{35}$$

Note that $\mathcal{J}$ is continuous and coercive, if $1-2\lambda\mathcal{M}d>0$:

$$\begin{aligned}\lambda(g_a(\tau^h,\nabla\vec{b}),\sigma^h) &\le 2d\lambda\parallel\nabla\vec{b}\parallel_\infty\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0 \\ &\le 2d\mathcal{M}\lambda\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0,\end{aligned} \tag{36}$$

Now (33) can be summarized as:

$$\begin{aligned}&\mathcal{J}((\tau^h,\vec{u}^h,p^h),(\sigma^h,\mathbf{v}^h,q^h))+\lambda\mathcal{B}^h(\vec{b},\tau^h,\sigma^h) \\ \le\;&\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0+2\mathcal{M}d\lambda\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0 \\ &+2\alpha\parallel D(\vec{u}^h)\parallel_0\parallel\sigma^h\parallel_0+2\alpha\parallel\tau^h\parallel_0\parallel D(\mathbf{v}^h)\parallel_0 \\ &+4\alpha(1-\alpha)\parallel D(\vec{u}^h)\parallel_0\parallel D(\mathbf{v}^h)\parallel_0 \\ &+2\alpha d\parallel q^h\parallel_0\parallel\nabla\vec{u}^h\parallel_0+2\alpha d\parallel p^h\parallel_0\parallel\nabla\mathbf{v}^h\parallel_0 \\ &+2\alpha\parallel p^h\parallel_0\parallel q^h\parallel_0+C\mathcal{M}h^{-1}\parallel\tau^h\parallel_0\parallel\sigma^h\parallel_0 \\ \le\;&C\;|||\;(\tau^h,\vec{u}^h,p^h)\;|||_{(\mathcal{S}^h\times\mathcal{X}^h\times\mathcal{Q}^h)}|||\;(\sigma^h,\mathbf{v}^h,q^h)\;|||_{(\mathcal{S}^h\times\mathcal{X}^h\times\mathcal{Q}^h)},\end{aligned} \tag{37}$$

where C is a constant independent on h.
To find weakly coercivity, for all $p^h \in \mathcal{Q}^h \subset \mathcal{Q}$, there exists a positive constant Υ used hereinafter represents a constant and $\vec{w} \in \mathcal{X}$ such that: $(\nabla \cdot \vec{w}, p^h) =\| p^h \|_0^2$, $\| \vec{w} \|_1 \leq \Upsilon \| p^h \|_0$. Assigning the value in FE approximation for normalization of $\vec{w}^h \in X^h$ of $\vec{w}$([22, 26, 35, 25]), assume

$$\| \vec{w}^h \|_1 \quad \leq \quad \Upsilon_0 \| p^h \|_0 . \tag{38}$$

As from the reference paper [22], we have

$$\int_\Omega p^h \nabla \cdot \vec{w}^h d\Omega \quad \geq \quad C_1 \| p^h \|_0^2 - C_2 \| (I - \Pi)p^h \|_0 \| p^h \|_0 . \tag{39}$$

For weak coercivity we substitute $(\mathbf{v}^h = \vec{u}^h - \xi \vec{w}^h,\ q^h = p^h,\ \sigma^h = \tau^h)$ in the bilinear form of $\mathcal{J}$, where $\xi \in \mathbb{R}$,

$$\begin{aligned} & \mathcal{J}((\tau^h, \vec{u}^h, p^h), (\tau^h, \vec{u}^h - \xi \vec{w}^h, p^h)) + \lambda \mathcal{B}^h(\vec{b}, \tau^h, \tau^h) \qquad (40) \\ = \quad & \mathcal{J}((\tau^h, \vec{u}^h, p^h), (\tau^h, \vec{u}^h, p^h)) + \mathcal{J}((\tau^h, \vec{u}^h, p^h), (0, -\xi \vec{w}^h, 0)) \\ & + \lambda \mathcal{B}^h(\vec{b}, \tau^h, \tau^h). \end{aligned}$$

The right hand side of the above equation (40) appears in three new terms, each term can be bounded as,
First term of (40).
From (27) and (36), we have

$$\begin{aligned} & \mathcal{J}((\tau^h, \vec{u}^h, p^h), (\tau^h, \vec{u}^h, p^h)) \\ \geq \quad & \| \tau^h \|_0^2 + \lambda (g_a(\tau^h, \nabla \vec{b}), \tau^h) \\ & + 4\alpha(1-\alpha) \| D(\vec{u}^h) \|_0^2 + 2\alpha \| (I - \Pi)p^h \|_0^2 \\ \geq \quad & \| \tau^h \|_0^2 - 2\lambda \mathcal{M} d \| \tau^h \|_0^2 \\ & + 4\alpha(1-\alpha) \| D(\vec{u}^h) \|_0^2 + 2\alpha \| (I - \Pi)p^h \|_0^2 \\ \geq \quad & (1 - 2\lambda \mathcal{M} d) \| \tau^h \|_0^2 \\ & + 4\alpha(1-\alpha) \| D(\vec{u}^h) \|_0^2 + 2\alpha \| (I - \Pi)p^h \|_0^2 . \end{aligned}$$

The second term of (40):

$$\begin{aligned} & \mathcal{J}((\tau^h, \vec{u}^h, p^h), (0, -\xi \vec{w}^h, 0)) \\ = \quad & -4\alpha(1-\alpha)\xi(D(\vec{u}^h), D(\vec{w}^h)) - 2\alpha\xi(\tau^h, D(\vec{w}^h)) \\ & + 2\alpha\xi(p^h, \nabla \cdot \vec{w}^h). \end{aligned} \tag{41}$$

To estimate right hand side of equation (41), by using (29),(38), (39), and Young's inequality. To further majorisation of the right-hand side in (41), we deduce that,

$$\begin{aligned}
\mid 4\alpha(1-\alpha)\xi(D(\vec{u}^h), D(\vec{w}^h)) \mid &\leq 4\alpha(1-\alpha)\xi \parallel D(\vec{u}^h) \parallel_0 \parallel D(\vec{w}^h) \parallel_0 \\
&\leq 4\alpha(1-\alpha)\xi \parallel D(\vec{u}^h) \parallel_0 \Upsilon_0 \parallel p^h \parallel_0 \\
&\leq \frac{12\alpha(1-\alpha)^2\xi\Upsilon_0^2}{C_1} \parallel D(\vec{u}^h) \parallel_0^2 \\
&\quad +\alpha\xi\frac{C_1}{3} \parallel p^h \parallel_0^2,
\end{aligned}$$

$$\begin{aligned}
\mid 2\alpha\xi(\tau^h, D(\vec{w}^h)) \mid &\leq 2\alpha\xi \parallel \tau^h \parallel_0 \parallel D(\vec{w}^h) \parallel_0 \\
&\leq 2\alpha\xi\Upsilon_0 \parallel \tau^h \parallel_0 \parallel p^h \parallel_0 \\
&\leq 3\alpha\xi\Upsilon_0^2\frac{1}{C_1} \parallel \tau^h \parallel_0^2 + \frac{C_1\alpha\xi}{3} \parallel p^h \parallel_0^2,
\end{aligned}$$

$$\begin{aligned}
\mid 2\alpha\xi(p^h, \nabla\cdot w^h) \mid &\geq 2\alpha\xi(C_1 \parallel p^h \parallel_0^2 - C_2 \parallel (I-\Pi)p^h \parallel_0 \parallel p^h \parallel_0) \\
&\geq 2\alpha\xi C_1 \parallel p^h \parallel_0^2 - 2\alpha\xi C_2 \parallel (I-\Pi)p^h \parallel_0 \parallel p^h \parallel_0 \\
&\geq 2\alpha\xi C_1 \parallel p^h \parallel_0^2 - 3\alpha\xi\frac{C_2^2}{C_1} \parallel (I-\Pi)p^h \parallel_0^2 \\
&\quad -\alpha\xi\frac{C_1}{3} \parallel p^h \parallel_0^2,
\end{aligned}$$

by substituting all the bounds above in the equation (41), we obtain:

$$\begin{aligned}
\mathcal{J}\big((\tau^h, \vec{u}^h, p^h), (0, -\xi\vec{w}^h, 0)\big) &\geq -3\alpha\xi\Upsilon_0^2\frac{1}{C_1} \parallel \tau^h \parallel_0^2 \\
&\quad -\frac{12\alpha(1-\alpha)^2\xi\Upsilon_0^2}{C_1} \parallel D(\vec{u}^h) \parallel_0^2 \\
&\quad +\alpha\xi C_1 \parallel p^h \parallel_0^2 - 3\alpha\xi\frac{C_2^2}{C_1} \parallel (I-\Pi)p^h \parallel_0^2 .
\end{aligned}$$

The third term of (40):

$$\lambda\mathcal{B}^h(\vec{b}, \tau^h, \tau^h) = (\lambda/2)\langle\langle\tau^{h+} - \tau^{h-}\rangle\rangle_{h,\vec{b}}^2 \geq 0. \tag{42}$$

By using equation (40) and substitute the three bounded terms, we get

$$\begin{aligned}
&\mathcal{J}((\tau^h, \vec{u}^h, p^h), (\tau^h, \vec{u} - \xi\vec{w}^h, p^h)) \\
\geq\ & (1 - 2\lambda\mathcal{M}d - \epsilon_1\delta\lambda\mathcal{M}d - \frac{12\alpha(1-\alpha)\xi}{C_1}) \parallel \tau^h \parallel_0^2 \\
&+(4\alpha(1-\alpha) - \frac{\alpha^2\delta^2}{\epsilon_2} - \frac{48\alpha(1-\alpha)(1-\alpha)^2\xi}{C_1}) \parallel D(\vec{u}^h) \parallel_0^2 \\
&+\alpha\xi C_1 \parallel p^h \parallel_0^2 + (2\alpha - 3\alpha\xi\frac{C_2^2}{C_1}) \parallel (I - \Pi)p^h \parallel_0^2 \\
\geq\ & C_3 \parallel \tau^h \parallel_0^2 + C_4 \parallel D(\vec{u}^h) \parallel_0^2 + C_5 \parallel p^h \parallel_0^2 \\
\geq\ & C^* \mid\mid\mid (\tau^h, \vec{u}^h, p^h) \mid\mid\mid^2_{\mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h} . \qquad (43)
\end{aligned}$$

□

The error estimate provides an approximate solution with the same order as the chosen pair of the FE solution for the linearized viscoelastic fluid flow equations.

Theorem 3.2. *For* $(\tau^h, \vec{u}^h, p^h) \in (\mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h)$ *satisfies equation* (28) *and* $(\tau, \vec{u}, p) \in (S \times \mathcal{X} \times \mathcal{Q})$ *satisfies* (16)*, then for* $\mathcal{M}$ *satisfying* $1 - 2\lambda\mathcal{M}d > 0$*, as follows:*

$$\parallel \tau - \tau^h \parallel_0 + \parallel (\vec{u} - \vec{u}^h) \parallel_1 + \parallel p - p^h \parallel_0 \quad \leq \quad Ch. \qquad (44)$$

Proof. Subtracting equation (28) from equation (16) yields;

$$\begin{aligned}
&\mathcal{J}((\tau - \tau^h, \vec{u} - \vec{u}^h, p - p^h), (\sigma^h, \mathbf{v}^h, q^h)) \\
&+\lambda\mathcal{B}^h(\vec{b}, \tau - \tau^h, \sigma^h) \ = \ 0 \ \forall(\sigma^h, \mathbf{v}^h, q^h) \in (\mathcal{S}^h \times \mathcal{X}^h \times \mathcal{Q}^h), \qquad (45)
\end{aligned}$$

by adding and subtracting the projection terms $(\tilde{\tau}^h, \tilde{\vec{u}}^h, \tilde{p}^h)$ and using the orthogonality gives,

$$\begin{aligned}
\mathcal{J}((\tilde{\tau}^h - \tau^h, &\tilde{\vec{u}}^h - \vec{u}^h, \tilde{p}^h - p^h), (\sigma^h, \mathbf{v}^h, q^h)) + \lambda B^h(\vec{b}, \tilde{\tau}^h - \tau^h, \sigma^h) \\
&= \mathcal{J}((\tilde{\tau}^h - \tau, \tilde{\vec{u}}^h - \vec{u}, \tilde{p}^h - p), (\sigma^h, \mathbf{v}^h, q^h)) \\
&+ \lambda B^h(\vec{b}, \tilde{\tau}^h - \tau, \sigma^h) + 2\alpha\mathcal{G}(p, q^h). \qquad (46)
\end{aligned}$$

To get error bounds, we choose left hand side of the equation (46), by setting $\sigma^h = (\tilde{\tau}^h - \tau^h)$, $\mathbf{v}^h = (\tilde{\vec{u}}^h - \vec{u}^h)$ and $q^h = (\tilde{p}^h - p^h)$, we have

$$\begin{aligned}
&\mathcal{J}((\tilde{\tau}^h - \tau^h, \tilde{\vec{u}}^h - \vec{u}^h, \tilde{p}^h - p^h), (\tilde{\tau}^h - \tau^h, \tilde{\vec{u}}^h - \vec{u}^h, \tilde{p}^h - p^h)) \\
&\quad + \lambda B^h(\vec{b}, \tilde{\tau}^h - \tau^h, \tilde{\tau}^h - \tau^h) \\
&\quad \geq (1 - 2\lambda M d) \parallel \tilde{\tau}^h - \tau^h \parallel_0^2 + 4\alpha(1-\alpha) \parallel (\tilde{\vec{u}}^h - \vec{u}^h) \parallel_1^2 \\
&\quad + 2\alpha \parallel (I - \Pi)(\tilde{p}^h - p^h) \parallel_0^2 + \frac{\lambda}{2} \langle\langle (\tilde{\tau}^h - \tau^h)^+ - (\tilde{\tau}^h - \tau^h)^- \rangle\rangle_{h,\vec{b}}^2 \\
&\quad \geq \beta \, ||| \, (\tilde{\tau}^h - \tau^h, D(\tilde{\vec{u}}^h - \vec{u}^h), \tilde{p}^h - p^h) \, |||^2 \, . \qquad (47)
\end{aligned}$$

Right hand side of equation (46) gives with the setting of $\sigma^h = (\tilde{\tau}^h - \tau^h)$, $\mathbf{v}^h = (\tilde{\vec{u}}^h - \vec{u}^h)$ and $q^h = (\tilde{p}^h - p^h)$

$$\begin{aligned}
&\mathcal{J}((\tilde{\tau}^h - \tau, \tilde{\vec{u}}^h - \vec{u}, \tilde{p}^h - p), (\tilde{\tau}^h - \tau^h, \tilde{\vec{u}}^h - \vec{u}^h, \tilde{p}^h - p^h)) \\
&\qquad + \lambda B^h(\vec{b}, \tilde{\tau}^h - \tau, \tilde{\tau}^h - \tau^h) \\
&\qquad \leq \widehat{C} \, \big(h^{-1} \parallel \tilde{\tau}^h - \tau \parallel_0 + \parallel (\tilde{\vec{u}}^h - \vec{u}) \parallel_1 + \parallel \tilde{p}^h - p \parallel_0 \\
&\qquad + 2\alpha \parallel (I - \Pi) p \parallel \big) \, ||| \, (\tilde{\tau}^h - \tau^h, (\tilde{\vec{u}}^h - \vec{u}^h), \tilde{p}^h - p^h) \, ||| \, . \qquad (48)
\end{aligned}$$

By using (47), (48) and triangle inequality such that:

$$\begin{aligned}
&|||(\tilde{\tau}^h - \tau^h, (\tilde{\vec{u}}^h - \vec{u}^h), \tilde{p}^h - p^h) \, |||^2 \\
&\leq \frac{\widehat{C}}{\beta} \, \big(h^{-1} \parallel \tilde{\tau}^h - \tau \parallel_0 + \parallel (\tilde{\vec{u}}^h - \vec{u}) \parallel_1 + \parallel \tilde{p}^h - p \parallel_0 \\
&+ \parallel (I - \Pi) p \parallel \big) \, ||| \, (\tilde{\tau}^h - \tau^h, (\tilde{\vec{u}}^h - \vec{u}^h), \tilde{p}^h - p^h) \, ||| \, . \qquad (49)
\end{aligned}$$

Hence the desired optimal error estimate is proved as;

$$\parallel \tau - \tau^h \parallel_0 + \parallel (\vec{u} - \vec{u}^h) \parallel_1 + \parallel p - p^h \parallel_0 \quad \leq \quad Ch.$$

The proof is completed. □

4 Numerical tests

This section illustrates the numerical result which is analyzed theoretically in Theorem 4.2. Our motive is to confirm the theoretical results for the proposed stabilized lowest equal order FE pair for the Oseen viscoelastic fluid flow model. For numerical evaluation, we design and examine three types of different experiments: a non-physical example with exact solution, a viscoelastic cavity flow problem and a benchmark 4-to-1 contraction channel flow [21]. In the analytical solution test, we demonstrate the optimal convergence order for lowest equal order. The second experiment elucidates the viscoelastic cavity flow to show the characteristics of the pressure contour and its behavior. The flow speed, behavior of the contours, streamlines patterns, and the pressure oscillation, are examined by the 4-to-1 contraction channel flow. In order to show the distinguishing features of the new stabilized model, we compared newly formulated method for the lowest-equal-order FE ($P1-P0-P1$) with the standard elements Taylor-Hood ($P2-P1-P1$).

4.1 Analytical solution test

The theoretical convergence rates are verified by considering fluid flow across a unit square with a known solution. To test the numerical stability of the new stabilized method, we considered the lowest order FE $P1-P0-P1$ pair for velocity, pressure, and stress. Different authors used this experimental pattern for Stokes and Navier-Stokes equation [11, 32, 36], where the function $\vec{b}(x)$ was chosen to be the exact solution of velocity $\vec{u}$. Moreover, the true solution of the problem for velocity $\vec{u}=(u_1,u_2)$, pressure p and polymeric stress τ is given:

$$\left\{\begin{array}{lcl} \vec{u} & = & \left(\begin{array}{c} -10(x^4-2x^3+x^2)(2y^3-3y^2+y) \\ 10(2x^3-3x^2+x)(y^4-2y^3+y^2)) \end{array}\right), \\ p & = & -10.0(2x-1)(2y-1), \\ \tau & = & 2\alpha D(\vec{u}). \end{array}\right.$$

The right-hand sides, initial and boundary conditions are derived by model equations with $Re=1$, $a=0$, $\lambda=10$ and $\alpha=1$. Moreover, we have formulated the following numerical results for Rate of convergence order R

$$E_1 = Ch_1^R$$

$$E_2 = Ch_2^R$$

$$\frac{E_1}{E_2} = C\left(\frac{h_1}{h_2}\right)^R$$

$$\log\left(\frac{E_1}{E_2}\right) = R\log\left(\frac{h_1}{h_2}\right)$$

$$R = \frac{\log\left(\frac{E_1}{E_2}\right)}{\log\left(\frac{h_1}{h_2}\right)}.$$

The numerical results are presented in different tables; In Tables 1-3, we

Table 1: The error estimate for linear viscoelastic fluid flow with Taylor-Hood elements.

h	$\|\|\tau-\tau^h\|\|_0$	$\|\|u-u^h\|\|_0$	$\|\|u-u^h\|\|_1$	$\|\|p-p^h\|\|_0$
1/4	0.03923	0.0018893	0.04920	0.16628
1/8	0.00974	0.0002271	0.01349	0.04050
1/16	0.00245	0.0000265	0.00348	0.01009
1/32	0.00062	0.0000339	0.00088	0.00252
1/64	0.00016	0.0000925	0.00022	0.00063
Rate	1.0	1.1	1.8	1.2

Table 2: The error estimate for linear viscoelastic fluid flow result obtain before addition of stabilization term with $P1-P0-P1$ pairs.

h	$\|\|\tau-\tau^h\|\|_0$	$\|\|u-u^h\|\|_0$	$\|\|u-u^h\|\|_1$	$\|\|p-p^h\|\|_0$
1/4	0.18798	0.02798	0.23593	1.77618
1/8	0.05412	0.00869	0.12337	1.24821
1/16	0.01649	0.00226	0.06111	0.53504
1/32	0.00510	0.00057	0.03032	1.45620
1/64	0.00166	0.00014	0.01509	0.24536
Rate	1.1	1.0	0.9	0.5

Table 3: The error estimate for linear viscoelastic fluid flow result obtain after addition of a stabilization term with $P1-P0-P1$ pairs.

h	$\|\|\tau-\tau^h\|\|_0$	$\|\|u-u^h\|\|_0$	$\|\|u-u^h\|\|_1$	$\|\|p-p^h\|\|_0$
1/4	0.28020	0.05204	0.44811	1.23944
1/8	0.10445	0.01708	0.18149	0.40238
1/16	0.03951	0.00460	0.07026	0.12951
1/32	0.01484	0.00118	0.02956	0.04116
1/64	0.00565	0.00029	0.01348	0.01318
Rate	1.0	1.1	1.3	1.6

illustrate the distinguishing feature of the finite element method for the linearized viscoelastic fluid flow model by comparing the results with the standard Taylor-Hood elements. Table 1: represents the computations of the errors for the standard finite elements with $P2-P1-P1$ pair. We have given the error H^1-norm for velocity, L^2-norm for velocity, L^2-norm for pressure, and L^2-norm for stress, respectively with the varying spacing $h=1/8,\ 1/16,\ 1/32,\ 1/64$. Table 2: provides the result for the approximation of the linearized viscoelastic fluid flow model by $P1-P0-P1$ pair without stabilization. Table 3, shows the error obtained from the approximate values with the stabilization term by using $P1-P0-P1$. From the previous tables, we can observe that the velocity H^1-norm and stress L^2-norm obtain optimal convergence order while the pressure L^2-norm is affected without the stabilization term. The accuracy of the convergence order of the pressure is ensured by adding a stabilization term, which is illustrated in Table 3. This experimental test illustrates that the scheme we have designed can be applied successfully for the linearized viscoelastic fluid flow model to stabilize the pressure.

4.2 4-to-1 contraction channel flow

The second example is the well-known benchmark problem for viscoelastic flow "4-to-1 contraction channel flow problem" which has a huge application in polymeric liquid industries and studied by many authors [37]. The geometry of the 4-to-1 contraction commonly occurs in the forming of 'die' for the viscoelastic fluid. Owing to the sudden reduction in width, in the corner region, a vortex appears. Moreover, 4-to-1 in literature have been widely used to show the convergence, stability, behavior of the streamlines of the contraction channel and the behavior of pressure [6]. The domain is constructed in such a way that the channel lengths are sufficiently long for a fully developed Poiseuille flow at both the inflow and outflow boundaries. The shape of typical geometry for physical representation we presented domain very related to the paper discussed [7]. To discretize this computational domain we have used finite element meshes that are supposed to be isotropic in the region surrounding the corner and non-isotropic and unstructured further way. The linearized viscoelastic fluid flow problem is based on the linear form of viscoelastic fluids. To apply the known function given in theoretical part $\vec{b}(x) = (b1, b2)$ in numerical simulation, we perform following steps in the computational code. For the brief discussion we refer reader to [8, 7].

- We first execute output data of the approximate solution from the non-linear velocity for true solution $\vec{u} = (u1, u2)$.
- We use the executed solution of (u1) and (u2) as a known solution (u1=b1 and u2=b2). Moreover, now the solution for the approximation considers for the linear one and it will be known for the velocity field $\vec{b} = (b1, b2)$ respectively.

Note: These two-steps makes the system non-linear to linear. It is indeed important to formulate for the known values for initial data.

5 Conclusion and future work

In this contribution, a new stabilized method for finite element $P1-P0-P1$ pairs for the linearized viscoelastic fluid flow is presented. Some of the finite element pairs does not fully overcome the requirement of the essential condition i.e., inf-sup (or LBB). We overcome this difficulty with the addition of the stabilization terms in the discrete variational formulation. We proved the well-posedness of the scheme. The desired error estimates are obtained. This method is easy to modify with the existing computational code and also convenient to prove the theoretical analysis. Moreover, in future aspects, this method may apply for the approximation solution of the non-linear viscoelastic fluid flow model.

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