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Original Research Paper

Recent Thought of α_*-Geraghty F-Contraction with Application

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Abstract. In this paper, we use the concept of a generalized α-Geraghty contraction type mapping to introduce the new notion of α_*-Geraghty type F-contraction multivalued mapping and prove some new common fixed point results for such contraction in b-metric-like spaces. Also, we give some examples to illustrate our main results, we also discuss existence a solution for a system of non-linear integral equation.

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1 Introduction and Preliminaries

Fixed point theorem is known as Banach's contraction principle [21] which is the base of functional analysis and plays a main role in several branches of mathematics and applied sciences whereas it shows the existence and uniqueness of solutions of many problems. Later Banach contraction principle has been generalized by using different forms of contractive conditions in diverse spaces. Many authors obtained interesting generalized results of the classical Banach contraction principle in several spaces see [1, 5, 14, 17, 20, 24, 27, 28, 30, 33, 36, 42, 45]. On the other hand, in 1973, Geraghty [30] generalized the Banach contraction principle. In 2013, Cho et al. [25] introduced the notion of α-Geraghty contractive type mappings and deduced the unique fixed point theorems for such mappings in a complete metric space. The notion of alpha in fixed point theory is important since it combines three types results: fixed point in the setting of cyclic contraction, fixed point in the metric space endowed with a partial order and standard results, where Samet et al. [45] and Karapinar et al. [40] opened a new wide field in fixed point theory by introducing the concepts of α-admissible and triangular α-admissible mappings and established various fixed point results for such type of mappings in the context of complete metric spaces. In 2014, Popescu [44] defined the concepts of α-orbital and triangular α-orbital admissible mappings and verified the unique fixed point theorems for the said mappings which are generalized α-Geraghty contraction type mappings. Ameer et al. [14] produced the notion of α_*-orbital and triangular α_*-orbital admissible mappings with proving some fixed point theorems in b-metric spaces. After that many articles have been dedicated to generalize the Geraghty contraction mappings in different spaces see [16, 22, 25, 26, 34, 38, 39, 42, 43, 44]. In 2012, Wardowski [47] introduced definition of F-contraction and proved fixed point results as a generalization of the Banach contraction principle in complete metric spaces. Furthermore some studies are dedicated to apply this definition on several contraction mappings see [2, 35, 37, 46].

Definition 1.1 [33]. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a metric-like on X, if for all $x, y, z \in X$;*
(1) $d(x, y) = 0$ *then* $x = y$;

(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \leq d(x,z) + d(z,y)$. *Then the pair* (X,d) *is said to be metric-like (or dislocated metric) space.*

Definition 1.2 [5]. *Let* X *be a non-empty set and* $d : X \times X \to [0,\infty)$ *be a function, called a b-metric-like if there exists a real number* $s \geq 1$ *such that the following conditions hold for every* $x, y, z \in X$,
(1) $d(x,y) = 0$ *then* $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \leq s[d(x,z) + d(z,y)]$. *Then the pair* (X,d) *is said to be b-metric-like space.*

Definition 1.3 [3]. *Let* X *be a nonempty set. Then* $(X, d, \preceq)$ *is called partially ordered b-metric space if and only if* d *is a b-metric on a partially ordered set* $(X, \preceq)$.

Definition 1.4 [5]. *Let* (X,d) *be a b-metric-like space,* $\{x_n\}$ *be a sequence in* X*, and* $x \in X$. *The sequence* $\{x_n\}$ *converges to* x *if and only if* $\lim\limits_{n\to\infty} d(x_n, x) = d(x,x)$.

Remark 1.5 [5]. In a b-metric-like space, the limit for a convergent sequence is not unique in general.

Example 1.6. Let $X = [0,\infty)$ and $d : X \times X \longrightarrow [0,\infty)$ defined by $d(x,y) = (\max\{x,y\})^2$. Then (X,d) is a b-metric-like space, with a coefficient $s = 2$. Assume that

$$\{x_n\} = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 1 & \text{when } n \text{ is even} \end{cases}.$$

For $x \geq 1$, $\lim\limits_{n\to\infty} d(x_n, x) = \lim\limits_{n\to\infty} (\max\{x_n, x\})^2 = x^2 = d(x,x)$.
Therefore, it is a convergent sequence and $x_n \to x$ for all $x \geq 1$. That is, limit of the sequence is not unique.

Definition 1.7 [5]. *Let* (X,d) *be a b-metric-like space and* $\{x_n\}$ *be a sequence in* X. *We say that* $\{x_n\}$ *is Cauchy if and only if* $\lim\limits_{n,m\to\infty} d(x_n, x_m)$ *exists and is finite.*

Definition 1.8 [5]. *Let (X,d) be a b-metric-like space. We say that (X,d) is complete if and only if each Cauchy sequence in X converges to $x \in X$ so that*

$$\lim_{n\to\infty} d(x_n, x) = d(x,x) = \lim_{m,n\to\infty} d(x_m, x_n).$$

Definition 1.9 [28]. *Let X be a metric space and $T : X \to X$ be a self-mapping. For $A \subseteq X$, let $\delta(A) = \sup\{d(a,b) : a,b \in A\}$ and for each $x \in X$, let $O(x,n) = \left\{x, Tx, T^2x, ..., T^nx\right\}$, $n = 0,1,2,...$, $O(x,\infty) = \left\{x, Tx, T^2x, ...\right\}$. The set $O(x,\infty)$ is called the orbit of T and the metric space X is said to be T-orbitally complete, if every Cauchy sequence in $O(x,\infty)$ is convergent in X.*

Definition 1.10 [44]. *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-orbital admissible if $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.*

Definition 1.11 [44]. *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1$.*

Lemma 1.12 [44]. *Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

Definition 1.13 [14]. *Let $S,T : X \to CB(X)$ be two multivalued mappings and $\alpha : X \times X \to [0,+\infty)$ be a function. Then the pair (S,T) is said to be α_*-orbital admissible if the following condition hold:*

$$\alpha_*(x,Sx) \geq 1,\ \alpha_*(x,Tx) \geq 1 \text{ implies } \alpha_*(Sx,T^2x) \geq 1,\ \alpha_*(Tx,S^2x) \geq 1.$$

Definition 1.14 [14]. *Let $S, T : X \to CB(X)$ be two multivalued mappings and $\alpha : X \times X \to [0, +\infty)$ be a function. Then the pair (S, T) is said to be triangular α_*-orbital admissible, if the following conditions hold:*
(i) (S, T) is α_*-orbital admissible.
(ii) $\alpha(x, y) \geq 1$, $\alpha_*(y, Sy) \geq 1$ and $\alpha_*(y, Ty) \geq 1$ imply $\alpha_*(x, Sy) \geq 1$ and $\alpha_*(x, Ty) \geq 1$.

Lemma 1.15 [14]. *Let $S, T : X \to CB(X)$ be two multivalued mappings such that the pair (S, T) is triangular α_*-orbital admissible. Assume that there exists an $x_0 \in X$ such that $\alpha_*(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\} \in X$ by $x_{2i+1} \in Sx_{2i}$ and $x_{2i+2} \in Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then for $n, m \in N \cup \{0\}$ with $m > n$, we have $\alpha(x_n, x_m) \geq 1$.*

Definition 1.16 [29]. *Let (X, d) be a b-metric-like space and $CB(X)$ be the family of all nonempty, closed and bounded subsets of X. For $A, B \in CB(X)$ and $x \in X$,*

$$\begin{aligned} D(x, A) &= \inf\{d(x, a) : a \in A\}, \\ \delta(A, B) &= \sup\{D(a, B) : a \in A\}, \\ \delta(B, A) &= \sup\{D(b, A) : b \in B\}. \end{aligned}$$

Also, define a mapping $H : CB(X) \times CB(X) \to [0, \infty)$ by

$$H(A, B) = \max\{\delta(A, B),\ \delta(B, A)\}.$$

Then H is called a Pompeiu-Hausdorff b-metric-like. For A and B two nonempty subsets of a b-metric-like space (X, d), define $d(A, B) =$ $\inf\{d(a, b) : a \in A,\ b \in B\}$.

Lemma 1.17 [29]. *Let (X, d) be a b-metric-like space. For $x \in X$ and $A, B, C \in CB(X)$, we have*
(1) $H(A, A) = \delta(A, A) = \sup\{D(a, A) : a \in A\}$;
(2) $H(A, B) = H(B, A)$;
(3) $H(A, B) = 0$ *implies* $A = B$;
(4) $H(A, B) \leq s[H(A, C) + H(C, B)]$;
(5) $D(x, A) \leq s[d(x, y) + D(y, A)]$.

Theorem 1.18. [30]. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that $\forall\ x,y \in X$,*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y),$$

where $\beta : [0,\infty) \to [0,1)$ is a function satisfying $\beta(t_n) \to 1 \implies t_n \to 0$ as $n \to \infty$.

Then T has a uniqeu fixed point $x^* \in X$.

Theorem 1.19. [39]. *Let (X,d) be a complete metric-like space and $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y), \text{ for all } x,y \in X,$$

where $\beta \in \xi$ and ξ is the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then T has a unique fixed point $x^* \in X$ with $d(x^*,x^*) = 0$.

Cho et al. [25]. proved the following interesting result.

Definition 1.20. [25]. *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called a generalized α-Geraghty contraction type mapping if $\exists\ \beta \in \xi$ such that $\forall\ x,y \in X$,*

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(M(x,y)\right)M(x,y),$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Theorem 1.21. [25]. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a mapping. Suppose that the following conditions are satisfied:*

(i) T is a generalized α-Geraghty contraction type map;

(ii) T is triangular α-admissible;

(iii) $\exists\ x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(iv) T is continuous.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Definition 1.22. [44]. *Let (X, d) be a metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called a generalized α-Geraghty contraction type mapping if $\exists\ \beta \in \xi$ such that $\forall\ x, y \in X$,*

$$\alpha(x,y)d(Tx,Ty) \le \beta\left(M(x,y)\right)M(x,y),$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

Theorem 1.23. [44]. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a mapping. Suppose that the following conditions are satisfied:*
(i) T is a generalized α-Geraghty contraction type mapping;
(ii) T is a triangular α-orbital admissible mapping;
(iii) $\exists\ x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
(iv) T is continuous.
Then T has a fixed point $x^* \in X$ and $\{T^n x\}$ converges to x^*.

Definition 1.24 [31]. *Let (X, d) be a b-metric-like space, and let T be a self-mapping on X. T is called orbitally continuous whenever for each $x, z \in X$*

$$\lim_{n\to\infty} d\left(T^n x, z\right) = d\left(z, z\right) \Longrightarrow \lim_{n\to\infty} d\left(TT^n x, Tz\right) = d\left(Tz, Tz\right).$$

Definition 1.25 [47]. *Let Δ_F be the family of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions*
$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta \Rightarrow\ F(\alpha) < F(\beta)$;
$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F\left(\alpha_n\right) = -\infty$;
$(F3)$ There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F\left(\alpha\right) = 0$.

2 Main Results

In this section, we improve the notion of Geraghty contraction type mappings and estaplish some new common fixed point theorem for pair of generalized α_*-Geraghty F-contraction for multivalued mapping in a b-metric-like space.

Definition 2.1. *Let (X,d) be a b-metric-like space, $\alpha : X\times X \to [0,\infty)$ be a function. Two multivalued mappings $S,T : X \to CB(X)$ is called a pair of generalized α_*-Geraghty F-contraction mapping if there exist $\beta \in \xi$ and $F \in \Delta_F$ such that for all $x, y \in X$, $s \geq 1$ and $\tau \in R_+$ with $H(Sx,Ty) > 0$,*

$$\tau + F\left(\alpha(x,y)\, s^3 H(Sx,Ty)\right) \leq F\left(\beta\left(M(x,y)\right).M(x,y)\right), \tag{1}$$

where

$$M(x,y) = \max\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{4s}\}.$$

Theorem 2.2. *Let (X,d) be a complete b-metric-like space, $\alpha : X \times X \to [0,\infty)$ be a function, and $S,T : X \to CB(X)$ two multivalued mappings. Suppose that the following conditions hold:*
(i) A pair (S,T) is generalized α_*-Geraghty F-contraction;
(ii) A pair (S,T) is triangular α_*-orbital admissible;
(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Sx_0) \geq 1$;
(iv) S and T are continuous.
Then (S,T) has a common fixed point $x^* \in X$.

Proof. Due to (iii), we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by letting $x_1 \in Sx_0$ such that $\alpha(x_0,x_1) \geq 1$ and $x_2 \in Tx_{1,}$ $x_3 \in Sx_2, ..., x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$. As (S,T) is triangular α_*-orbital admissible, From Lemma 1.15, we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. Since (S,T) is generalized α_*-Geraghty F-contraction for multivalued mappings, then from (1) we have

$$\begin{aligned} F\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) &\leq F\left(s^3 H\left(Sx_{2n}, Tx_{2n+1}\right)\right) 2 \qquad (1) \\ &\leq F\left(\alpha\left(x_{2n}, x_{2n+1}\right) s^3 H\left(Sx_{2n}, Tx_{2n+1}\right)\right) \\ &\leq F\left(\beta\left(M(x_{2n}, x_{2n+1})\right) . M(x_{2n}, x_{2n+1})\right) - \tau, \end{aligned}$$

we evaluate

$$\begin{aligned} M\left(x_{2n}, x_{2n+1}\right) &= \max\left\{ \begin{array}{c} d\left(x_{2n}, x_{2n+1}\right), D\left(x_{2n}, Sx_{2n}\right), \\ D\left(x_{2n+1}, Tx_{2n+1}\right), \\ \frac{D(x_{2n}, Tx_{2n+1}) + D(x_{2n+1}, Sx_{2n})}{4s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d\left(x_{2n}, x_{2n+1}\right), d\left(x_{2n}, x_{2n+1}\right), \\ d\left(x_{2n+1}, x_{2n+2}\right), \\ \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{4s} \end{array} \right\}. \end{aligned}$$

Since

$$\begin{aligned} &\frac{d\left(x_{2n}, x_{2n+2}\right) + d\left(x_{2n+1}, x_{2n+1}\right)}{4s} \\ \leq\; &\max\left\{d\left(x_{2n}, x_{2n+1}\right), d\left(x_{2n+1}, x_{2n+2}\right)\right\}. \end{aligned}$$

We conclude that

$$M\left(x_{2n}, x_{2n+1}\right) = \max\left\{d\left(x_{2n}, x_{2n+1}\right), d\left(x_{2n+1}, x_{2n+2}\right)\right\}.$$

Now if

$$\max\left\{d\left(x_{2n}, x_{2n+1}\right), d\left(x_{2n+1}, x_{2n+2}\right)\right\} = d\left(x_{2n+1}, x_{2n+2}\right) \text{ for } n \geq 1,$$

then from (2), we get

$$F\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) \leq F\left(\beta\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) . d\left(x_{2n+1}, x_{2n+2}\right)\right) - \tau.$$

Since $\beta \in \xi$ and $\tau > 0$, we have

$$F\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) \leq F\left(d\left(x_{2n+1}, x_{2n+2}\right)\right),$$

which is a contradiction as $d\left(x_{2n+1}, x_{2n+2}\right) \geq 0$. Therefore

$$\max\left\{d\left(x_{2n}, x_{2n+1}\right), d\left(x_{2n+1}, x_{2n+2}\right)\right\} = d\left(x_{2n}, x_{2n+1}\right),$$

by (2), we have

$$\begin{aligned} F\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) &\leq F\left(\beta\left(d\left(x_{2n}, x_{2n+1}\right)\right).d\left(x_{2n}, x_{2n+1}\right)\right) - \tau \\ &< F\left(\beta\left(d\left(x_{2n}, x_{2n+1}\right)\right).d\left(x_{2n}, x_{2n+1}\right)\right). \end{aligned}$$

Since $F \in \Delta_F$, we conclude

$$d\left(x_{2n+1}, x_{2n+2}\right) < \beta\left(d\left(x_{2n}, x_{2n+1}\right)\right).d\left(x_{2n}, x_{2n+1}\right).$$

As $\beta \in \xi$, then

$$d\left(x_{2n+1}, x_{2n+2}\right) < d\left(x_{2n}, x_{2n+1}\right).$$

Hence, $\{d\left(x_{2n+1}, x_{2n+2}\right)\}$ is a decreasing sequence of positive real numbers. Again from (2) for all $n \geq 1$, we get

$$\begin{aligned} F\left(d\left(x_{2n+1}, x_{2n+2}\right)\right) &\leq F\left(s^3 H\left(Sx_{2n}, Tx_{2n+1}\right)\right) \\ &\leq F\left(\alpha\left(x_{2n}, x_{2n+1}\right) s^3 H\left(Sx_{2n}, Tx_{2n+1}\right)\right) \\ &\leq F\left(\beta\left(d\left(x_{2n}, x_{2n+1}\right)\right).d\left(x_{2n}, x_{2n+1}\right)\right) - \tau \\ &\leq F\left(d\left(x_{2n}, x_{2n+1}\right)\right) - \tau \\ &\leq F(\alpha\left(x_{2n-1}, x_{2n}\right) s^3 H(Tx_{2n-1}, Sx_{2n})) - \tau \\ &\leq F(d(x_{2n-1}, x_{2n})) - 2\tau \\ &\leq F(\alpha\left(x_{2n-2}, x_{2n-1}\right) s^3 H(Sx_{2n-2}, Tx_{2n-1})) - 2\tau \\ &\leq F(d(x_{2n-2}, x_{2n-1})) - 3\tau, \end{aligned}$$

by continuing this manner, we obtain

$$F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_0, x_1)) - (2n+1)\tau. \tag{3}$$

Taking limit as $n \to \infty$ in (3), we get

$$\lim_{n\to\infty} F(d(x_{2n+1}, x_{2n+2})) = -\infty.$$

By (F2), we have

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0. \tag{4}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X. Suppose on the contrary that $\{x_n\}$ is not Cauchy sequence. Then there exists $\epsilon > 0$ and the subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \epsilon, \tag{5}$$

we choose n_k to be the smallest number such that (5) holds, then we have

$$d(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{6}$$

By triangular inequality and $(5), (6)$, we obtain

$$\begin{aligned} \epsilon &\leq d(x_{n_k}, x_{m_k})\, 7 \\ &\leq s[d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k})] \\ &< s[d(x_{n_k}, x_{n_k-1}) + \epsilon]. \end{aligned} \tag{2}$$

Taking the limit as $k \to \infty$ in (7) and using (4), we get

$$\epsilon \leq \liminf_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) < s\epsilon. \tag{8}$$

Again by triangular inequality, taking the limit as $k \to \infty$ and using (4) and (8), we conclude

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{n_k+1}, x_{m_k}) \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k}) < s^2\epsilon. \tag{9}$$

Similarly

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+2}) < s^2\epsilon. \tag{10}$$

Also by triangular inequality, taking the limit as $k \to \infty$ and utilizing (4) and (9), we conclude

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) \leq s^3\epsilon. \tag{11}$$

Now from (2), we write

$$\begin{aligned} F(d(x_{n_k+1}, x_{m_k+1})) &\leq F\left(\alpha(x_{n_k}, x_{m_k})\, s^3 d(Sx_{n_k}, Tx_{m_k})\right) \\ &\leq F(\beta(M(x_{n_k}, x_{m_k}))\, M(x_{n_k}, x_{m_k})) - \tau \\ &< F(\beta(M(x_{n_k}, x_{m_k}))\, M(x_{n_k}, x_{m_k})), \end{aligned}$$

this implies that

$$\begin{aligned} d\left(x_{n_k+1}, x_{m_k+1}\right) &< \beta\left(M\left(x_{n_k}, x_{m_k}\right)\right).M\left(x_{n_k}, x_{m_k}\right) \\ &< M\left(x_{n_k}, x_{m_k}\right), \end{aligned}$$

where

$$M\left(x_{n_k}, x_{m_k}\right) = \max\left\{ \begin{array}{c} d\left(x_{n_k}, x_{m_k}\right), d\left(x_{n_k}, x_{n_k+1}\right), d\left(x_{m_k}, x_{m_k+1}\right), \\ \frac{d\left(x_{n_k}, x_{m_k+1}\right) + d\left(x_{m_k}, x_{n_k+1}\right)}{4s} \end{array} \right\}.$$

Taking upper limit as $k \to \infty$ and using (4), (8) and (10), we get

$$\begin{aligned} \epsilon &= \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{4s}\right\} \leq \lim_{k\to\infty} \sup M\left(x_{n_k}, x_{m_k}\right) \\ &\leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2}\right\} = s\epsilon. \end{aligned}$$

Similarly

$$\begin{aligned} \epsilon &= \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{4s}\right\} \leq \lim_{k\to\infty} \inf M\left(x_{n_k}, x_{m_k}\right) \\ &\leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2}\right\} = s\epsilon. \end{aligned}$$

Hence, from (11), it follows

$$\begin{aligned} F\left(s\epsilon\right) &= F\left(s^3 \frac{\epsilon}{s^2}\right) \leq F\left(s^3 \lim_{k\to\infty} \sup d\left(x_{n_k+1}, x_{m_k+1}\right)\right) \\ &\leq F\left(\alpha\left(x_{n_k}, x_{m_k}\right) s^3 \lim_{k\to\infty} \sup d\left(Sx_{n_k}, Tx_{m_k}\right)\right) \\ &\leq F\left(\alpha\left(x_{n_k}, x_{m_k}\right) s^3 \lim_{k\to\infty} \sup H\left(Sx_{n_k}, Tx_{m_k}\right)\right) \\ &\leq F\left(\beta\left(\lim_{k\to\infty} M\left(x_{n_k}, x_{m_k}\right)\right)\left(\lim_{k\to\infty} \sup M\left(x_{n_k}, x_{m_k}\right)\right)\right) - \tau \\ &\leq F\left(\beta\left(s\epsilon\right)\left(s\epsilon\right)\right) - \tau \\ &< F\left(\beta\left(s\epsilon\right)\left(s\epsilon\right)\right), \end{aligned}$$

as $F \in \Delta_F$ and $\beta \in \xi$, we get $s\epsilon < s\epsilon$. Which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists

$x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, implies $x_{2i+1} \rightarrow x^*$ and $x_{2i+2} \rightarrow x^*$ as $i \rightarrow \infty$. As T is continuous. Then we have

$$\begin{aligned} D\left(x_*, Tx^*\right) &= \lim_{i\rightarrow\infty} D\left(x_{2i+2}, Tx^*\right) 12 \\ &\leq \lim_{i\rightarrow\infty} H\left(Tx_{2i+1}, Tx^*\right) = H\left(Tx^*, Tx^*\right). \end{aligned} \tag{3}$$

Utilizing the triangular inequality, we get

$$D\left(x^*, Tx^*\right) \leq s[d\left(x^*, x_{2i+2}\right) + D\left(x_{2i+2}, , Tx^*\right)].$$

Letting $i \rightarrow \infty$ and using (12), we have

$$\begin{aligned} D\left(x^*, Tx^*\right) &\leq \lim_{i\rightarrow\infty} sd\left(x^*, x_{2i+2}\right) + \lim_{i\rightarrow\infty} sD\left(x_{2i+2}, , Tx^*\right)] \\ &\leq sH\left(Tx^*, Tx^*\right). \end{aligned}$$

Thus, we have

$$D\left(x^*, Tx^*\right) \leq sH\left(Tx^*, Tx^*\right). \tag{13}$$

Now we show that $x^* \in Tx^*$. suppose that $x^* \notin Tx^*$. from (13), we find that $D\left(x^*, Tx^*\right) \neq 0$, moreover

$$\begin{aligned} F\left(D\left(x^*, Tx^*\right)\right) &\leq F\left(sH\left(Tx^*, Tx^*\right)\right) \leq F\left(s^3 H\left(Tx^*, Tx^*\right)\right) 14\,(4) \\ &\leq F\left(\alpha\left(x^*, x^*\right) s^3 H\left(Tx^*, Tx^*\right)\right) \\ &\leq F\left(\beta\left(M\left(x^*, x^*\right)\right).M\left(x^*, x^*\right)\right) - \tau \\ &< F\left(\beta\left(M\left(x^*, x^*\right)\right).M\left(x^*, x^*\right)\right) \\ &< F\left(M\left(x^*, x^*\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} M\left(x^*, x^*\right) &= \max\left\{ \begin{array}{c} d\left(x^*, x^*\right), D\left(x^*, Tx^*\right), D\left(x^*, Tx^*\right), \\ \frac{D(x^*, Tx^*) + D(x^*, Tx^*)}{4s} \end{array} \right\} \\ &= D\left(x^*, Tx^*\right). \end{aligned}$$

From (14), we get

$$F\left(D\left(x^*, Tx^*\right)\right) < F\left(D\left(x^*, Tx^*\right)\right),$$

since $F \in \Delta_F$, we obtain

$$D\left(x^*, Tx^*\right) < D\left(x^*, Tx^*\right),$$

which is a contradiction. Therefore $x^* \in Tx^*$, similarly, $x^* \in Sx^*$.

Hence the pair (S,T) has a common fixed point $x^* \in X$.

Theorem 2.3. *Let $(X,d,\preceq)$ be a partially ordered complete b-metric-like space, $\alpha : X \times X \to [0,\infty)$ be a function. Two non-decreasing multivalued mappings $S,T : X \to CB(X)$ is called a pair of generalized α_*-Geraghty F-contraction mapping if there exist $\beta \in \xi$ and $F \in \Delta_F$ such that for all $x,y \in X$, $s \geq 1$ and $\tau \in R_+$,*

$$\tau + F\left(\alpha(x,y)s^3 H(Sx,Ty)\right) \leq F\left(\beta\left(M(x,y)\right).M(x,y)\right),$$

where

$$M(x,y) = \max\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{4s}\},$$

satisfying the following conditions:
(i) (S,T) is a triangular α_*-orbital admissible;
(ii) X is (S,T)-orbitally complete for each $x,y \in O(x)$ with $x \preceq y$;
(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0,Sx_0) \geq 1$ with $x_0 \preceq Sx_0$;
(iv) S or T is orbitally continuous at $x^* \in X$.

Then (S,T) has a common fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ such that $x_0 \preceq Sx_0$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by letting $x_1 \in Sx_0$ such that $x_0 \preceq x_1$ and $x_2 \in Tx_1$ then $x_1 \preceq x_2$. As S,T are non-decreasing, we have $x_3 \in Sx_2$ such that $x_2 \preceq x_3$. Continuing the same procedures, we obtain a sequence $\{x_n\} \subseteq O(x)$ such that $\forall$ $n = 0,1,2,...$

$$x_{2n+1} \in Sx_{2n} \text{ and } x_{2n+2} \in Tx_{2n+1}, \ \Rightarrow x_{2n} \preceq x_{2n+1} \text{ and } x_{2n+1} \preceq x_{2n+2},$$

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ...$$

As (S,T) is triangular α_*-orbital admissible, From Lemma 1.15, we have $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
since (S,T) is generalized α_*-Geraghty F-contraction for multivalued mappings, and by the analogous proof as in Theorem 2.2, we conclude that $\{x_n\} \subseteq O(x)$ is a Cauchy sequence in X. Since X is orbitally complete, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, implies

$x_{2i+1} \to x^*$ and $x_{2i+2} \to x^*$ as $i \to \infty$. As T is orbitally continuous at $x^* \in X$, then for each $x,\ x^* \in X$

$$\lim_{n\to\infty} D\left(T^n x, x^*\right) = d\left(x^*, x^*\right) \Longrightarrow \lim_{n\to\infty} H\left(TT^n x, Tx^*\right) = H\left(Tx^*, Tx^*\right).$$

Then we have

$$\begin{aligned} D\left(x^*, Tx^*\right) &= \lim_{i\to\infty} D\left(x_{2i+2}, Tx^*\right) \\ &\leq \lim_{i\to\infty} H\left(Tx_{2i+1}, Tx^*\right) = H\left(Tx^*, Tx^*\right). \end{aligned}$$

Utilizing the triangular inequality, we get

$$D\left(x^*, Tx^*\right) \leq s[d\left(x^*, x_{2i+2}\right) + D\left(x_{2i+2}, , Tx^*\right)]$$

Letting $i \to \infty$, we have

$$\begin{aligned} D\left(x^*, Tx^*\right) &\leq \lim_{i\to\infty} sd\left(x^*, x_{2i+2}\right) + \lim_{i\to\infty} sD\left(x_{2i+2}, , Tx^*\right)] \\ &\leq sH\left(Tx^*, Tx^*\right). \end{aligned}$$

Thus, we have

$$D\left(x^*, Tx^*\right) \leq sH\left(Tx^*, Tx^*\right).$$

Now we show that $x^* \in Tx^*$. suppose that $x^* \notin Tx^*$. from above inequality, we find that $D\left(x^*, Tx^*\right) \neq 0$, moreover

$$\begin{aligned} F\left(D\left(x^*, Tx^*\right)\right) &\leq F\left(sH\left(Tx^*, Tx^*\right)\right) \leq F\left(s^3 H\left(Tx^*, Tx^*\right)\right) \\ &\leq F\left(\alpha\left(x^*, x^*\right) s^3 H\left(Tx^*, Tx^*\right)\right) \\ &\leq F\left(\beta\left(M\left(x^*, x^*\right)\right) .M\left(x^*, x^*\right)\right) - \tau \\ &< F\left(\beta\left(M\left(x^*, x^*\right)\right) .M\left(x^*, x^*\right)\right) \\ &< F\left(M\left(x^*, x^*\right)\right). \end{aligned}$$

Whereas

$$\begin{aligned} M\left(x^*, x^*\right) &= \max\left\{ \begin{array}{c} d\left(x^*, x^*\right), D\left(x^*, Tx^*\right), D\left(x^*, Tx^*\right), \\ \frac{D(x^*, Tx^*) + D(x^*, Tx^*)}{4s} \end{array} \right\} \\ &= D\left(x^*, Tx^*\right). \end{aligned}$$

Now we have

$$F\left(D\left(x^{*},Tx^{*}\right)\right)<F\left(D\left(x^{*},Tx^{*}\right)\right)\text{ implies }D\left(x^{*},Tx^{*}\right)<D\left(x^{*},Tx^{*}\right),$$

which is a contradiction. Therefore $x^{*}\in Tx^{*}$, similarly, $x^{*}\in Sx^{*}$. Hence the pair (S,T) has a common fixed point $x^{*}\in X$.

From Theorem 2.2, if $s=1$, we deduce the following theorem.

Theorem 2.4. *Let (X,d) be a complete metric-like space, $\alpha:X\times X\to[0,\infty)$ be a function. Two multivalued mappings $S,T:X\to CB(X)$ is called a pair of generalized α_{*}-Geraghty F-contraction mapping if there exist $\beta\in\xi$ and $F\in\Delta_{F}$ such that for all $x,y\in X$ and $\tau\in R_{+}$,*

$$\tau+F\left(\alpha(x,y)H(Sx,Ty)\right)\leq F\left(\beta\left(M(x,y)\right).M(x,y)\right),$$

where

$$M(x,y)=\max\{d(x,y),D(x,Sx),D(y,Ty),\frac{D(x,Ty)+D(y,Sx)}{4}\},$$

satisfying the following conditions:
(i) (S,T) is a triangular α_{*}-orbital admissible;
(ii) there exists an $x_{0}\in X$ such that $\alpha_{*}(x_{0},Sx_{0})\geq1$;
(iii) S and T are continuous.
Then (S,T) has a common fixed point $x^{*}\in X$.

The following Theorem comes directly from Theorem 2.2, when we consider $M\left(x,y\right)=d\left(x,y\right).$

Theorem 2.5. *Let (X,d) be a complete b-metric-like space, $\alpha:X\times X\to[0,\infty)$ be a function. Two multivalued mappings $S,T:X\to CB(X)$ is called a pair of generalized α_{*}-Geraghty F-contraction mapping if there exist $\beta\in\xi$ and $F\in\Delta_{F}$ such that for all $x,y\in X$, $s\geq1$, $\tau\in R_{+}$ and $H(Sx,Ty)>0$,*

$$\tau+F\left(\alpha(x,y)s^{3}H(Sx,Ty)\right)\leq F\left(\beta\left(d(x,y)\right).d(x,y)\right),$$

satisfying the following conditions:
(i) (S,T) is a triangular α_{*}-orbital admissible;

(ii) there exists an $x_0 \in X$ *such that* $\alpha_*(x_0, Sx_0) \geq 1$*;*
(iii) S *and* T *are continuous.*

Then (S,T) *has a common fixed point* $x^* \in X$*.*

Proof. By congruous way of proof of theorem 2.2 with considering that $M(x,y) = d(x,y)$, we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, implies $x_{2i+1} \to x^*$ and $x_{2i+2} \to x^*$ as $i \to \infty$. As T is continuous. Then we have

$$\begin{aligned} D(x_*, Tx^*) &= \lim_{i\to\infty} D(x_{2i+2}, Tx^*) \\ &\leq \lim_{i\to\infty} H(Tx_{2i+1}, Tx^*) = H(Tx^*, Tx^*). \end{aligned}$$

Utilizing the triangular inequality, we get

$$D(x^*, Tx^*) \leq s[d(x^*, x_{2i+2}) + D(x_{2i+2},, Tx^*)].$$

Letting $i \to \infty$ and using above inequality, we have

$$\begin{aligned} D(x^*, Tx^*) &\leq \lim_{i\to\infty} sd(x^*, x_{2i+2}) + \lim_{i\to\infty} sD(x_{2i+2},, Tx^*)] \\ &\leq sH(Tx^*, Tx^*). \end{aligned}$$

Thus, we have

$$D(x^*, Tx^*) \leq sH(Tx^*, Tx^*).$$

Again by the triangular inequality and Letting $i \to \infty$, we get

$$d(x^*, x^*) \leq sD(x^*, Tx^*)$$

Now we show that $x^* \in Tx^*$. suppose that $x^* \notin Tx^*$. from above inequalities, we find that $D(x^*, Tx^*) \neq 0$, moreover

$$\begin{aligned} F(d(x^*, x^*)) &\leq F(sD(x^*, Tx^*)) \leq F\left(s^3 H(Tx^*, Tx^*)\right) \\ &\leq F\left(\alpha(x^*, x^*)\, s^3 H(Tx^*, Tx^*)\right) \\ &\leq F(\beta(d(x^*, x^*)) .d(x^*, x^*)) - \tau \\ &< F(\beta(d(x^*, x^*)) .d(x^*, x^*)) \\ &< F(d(x^*, x^*)). \end{aligned}$$

From (2.16) since $F \in \triangle_F$, we get $d(x^*, x^*) < d(x^*, x^*)$, which is a contradiction. Therefore $x^* \in Tx^*$, similarly, $x^* \in Sx^*$. Hence the pair (S, T) has a common fixed point $x^* \in X$.

We can extract the following Corollary by taking $S = T$ in Theorem 2.2.

Corollary 2.6. *Let (X, d) be a complete b-metric-like space, $\alpha : X \times X \to [0, \infty)$ be a function. A multivalued mapping $T : X \to CB(X)$ is called generalized α_*-Geraghty F-contraction mapping if there exist $\beta \in \xi$ and $F \in \Delta_F$ such that for all $x, y \in X$, $s \geq 1$ and $\tau \in R_+$,*

$$\tau + F\left(\alpha(x,y)s^3 H(Tx, Ty)\right) \leq F\left(\beta\left(M(x,y)\right).M(x,y)\right),$$

where

$$M(x,y) = max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{4s}\},$$

satisfying the following conditions:
(i) T is a triangular α_*-orbital admissible;
(ii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$;
(iii) T is a continuous.
Then T has a fixed point $x^* \in X$.

We can study Theorem 2.2 for single valued mappings to conclude the following Corollary.

Corollary 2.7. *Let (X, d) be a complete b-metric-like space and $\alpha : X \times X \to [0, \infty)$ be a function. Two mappings $S, T : X \to X$ are called a pair of generalized α-Geraghty F-contraction type mappings if there exists $\beta \in \xi$ and $F \in \Delta_F$ such that for all $x, y \in X$, $s \geq 1$ and $\tau > 1$,*

$$\tau + F\left(\alpha(x,y)s^3 d(Sx, Ty)\right) \leq F\left(\beta\left(M(x,y)\right).M(x,y)\right),$$

where

$$M(x,y) = max\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty) + d(y,Sx)}{4s}\},$$

satisfying the following conditions:

(i) (S,T) *is a generalized* α*-Geraghty* F*-contraction type mappings;*
(ii) (S,T) *is a triangular* α*-orbital admissible;*
(iii) there exists an $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$*;*
(iv) S *and* T *are continuous.*

Then S *and* T *have a common fixed point* $x^* \in X$.

Example 2.8. Let $X = \mathbb{R}$ and $d : X \times X \longrightarrow [0,\infty)$ defined by $d(x,y) = (|x| + |y| + a)^p$. Then (X,d) is a b-metric-like space, where $p > 1$, $a \geq 0$ and $s = 2^{p-1}$. Define $S, T : X \to CB(X)$ by

$$Sx = \begin{cases} \left\{\frac{x}{64}\right\}, & \text{if } x \in [0,1] \\ \{0\} & \text{otherwise} \end{cases} \text{ and } Tx = \begin{cases} \left\{\frac{x}{64}\right\} & \text{if } x \in [0,1] \\ \left\{\frac{1}{2}, \frac{1}{4}\right\} & \text{otherwise} \end{cases}.$$

Also, we define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1] \\ \frac{1}{16} & \text{otherwise} \end{cases}$$

and $F : R^+ \to R$ is defined by $F(x) = \ln(x)$, and $\beta : X \times X \to [0,1)$ by $\beta(x,y) = \frac{4}{5}$. Now for $x, y \in [0,1]$

$$\begin{aligned} F(\alpha(x,y)\, s^3 H(Sx,Ty)) &= F(2^{3p-3} H(\left\{\frac{x}{64}\right\}, \left\{\frac{y}{64}\right\})) \\ &= F\left(2^{3p-3} d\left(\frac{x}{64}, \frac{y}{64}\right)\right) \\ &= F\left(2^{3p-3}\left(\left|\frac{x}{64}\right| + \left|\frac{y}{64}\right| + a\right)^p\right) \\ &= F\left(\frac{2^{3p-3}}{64^p}\left(|x| + |y| + 64a\right)^2\right) \\ &= F\left(\frac{2^{3p-3}}{2^{6p}}\left(|x| + |y| + a_1\right)^p\right) \\ &= F\left(\frac{1}{2^{3p+3}} d(x,y)\right) \\ &\leq F\left(\frac{4}{5} d(x,y)\right) \\ &\leq F\left(\beta\left(M(x,y)\right).M(x,y)\right) - \tau \end{aligned}$$

Otherwise, we have

$$F(\alpha\,(x,y)\,s^3H(Sx,Ty)) \leq F\,(\beta\,(M(x,y))\,.M(x,y)) - \tau$$

Hence for $\tau \in (0, 0.2)$ and $a \in [0, \frac{1}{64})$,

$$\tau + F(\alpha\,(x,y)\,s^3H(Sx,Ty)) \leq F\,(\beta\,(M(x,y))\,.M(x,y))\,.$$

Similarly for each $x, y \in X$ we can find some $\tau > 0$ that satisfy the inequality. Now we show that (S,T) is a pair of triangular α_*-orbital admissible mapping. For $x, y \in [0,1]$, then $\alpha\,(x,y) \geq 1$, $Sx \leq 1$, $Sy \leq 1$, $Tx \leq 1$, and $Ty \leq 1$, also $S^2x = S(Sx) \leq 1$ and $T^2x = T(Tx) \leq 1$, so it follows that $\alpha_*(x,Sx) \geq 1$ and $\alpha_*(x,Tx) \geq 1$ imply $\alpha_*(Sx,T^2x) \geq 1$ and $\alpha_*(Tx,S^2x) \geq 1$. Thus (S,T) is α_*-orbital admissible. Let $x, y \in [0,1]$ be such that $\alpha(x,y) \geq 1$, $\alpha_*(y,Sy) \geq 1$ and $\alpha_*(y,Ty) \geq 1$ imply $\alpha_*(x,Sy) \geq 1$ and $\alpha_*(x,Ty) \geq 1$. Therefore (S,T) is triangular α_*-orbital admissible.

Furthermore, if $\{x_n\}$ is a sequence in X such that $\alpha\,(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $x_n \subseteq [0,1]$ and hence $x \in [0,1]$. This implies that $\alpha_*\,(x,Sx) \geq 1$. Let $x_0 = \frac{1}{2}$. Then

$$\alpha_*\left(\frac{1}{2}, S\left(\frac{1}{2}\right)\right) = \alpha\left(\frac{1}{2}, \frac{1}{128}\right) \geq 1.$$

Hence, all conditions of Theorem 2.2 are satisfied and $x^* = 0$ is the common fixed point of S and T.

3 Application for Existence a Solution to the System of Non-Linear Integral Equations

In this section, we study an existence a common solution to the system of non-linear integral equations by using Corollary 2.7. Consider the system of nonlinear quadratic integral equations:

$$\begin{cases} x\,(t) = \int_0^1 F\,(t,r)\,\Phi_1(r, x\,(r))dr; \\ y\,(t) = \int_0^1 F\,(t,r)\,\Phi_2(r, y\,(r))dr, \end{cases} \tag{15}$$

where $F : [0,1] \times [0,1] \to [0,\infty)$ is continuous at $t \in [0,1]$ for every $r \in [0,1]$ and measurable at $r \in [0,1]$ for every $t \in [0,1]$, and also $\Phi_1, \Phi_2 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions.

Define the operators $S, T : X \to X$ by:

$$\begin{cases} Sx\,(t) = \int_0^1 F\,(t,r)\,\Phi_1(r, x\,(r))dr; \\ Ty\,(t) = \int_0^1 F\,(t,r)\,\Phi_2(r, y\,(r))dr, \end{cases} \tag{16}$$

where $X = C([0,1])$ (the space of continuous functions defined on $[0,1]$ with b-metric-like defined by $d\,(x,y) = \max\limits_{t \in [0,1]} \left(|x(t)| + |y(t)|\right)^p$, $\forall\ x, y \in X$. Obviously, (X,d) is a complete b-metric-like space with the constant $s = 2^{p-1}$ and $p \geq 1$.

Theorem 3.1. *Define two mappings as in* (16). *Suppose that the following conditions hold:*
(i) $F : [0,1] \times [0,1] \to [0,\infty)$ *is continuous at* $t \in [0,1]$ *for every* $r \in [0,1]$ *and measurable at* $r \in [0,1]$ *for every* $t \in [0,1]$ *such that* $\int_0^1 F\,(t,r)\,dr \leq \sqrt[p]{\frac{e^{-\tau}}{s^3 \alpha(x,y)}}$;
(ii) $\Phi_1, \Phi_2 : [0,1] \times \mathbb{R} \to [0,\infty)$ *are continuous functions, such that there exists a constant* $0 \leq \lambda < 1$ *and* $\forall\ x, y \in X$

$$|\Phi_1(r, x(r))| + |\Phi_2(r, y(r))| \leq \lambda(|x(r)| + |y(r)|);$$

(iii) $\lambda \leq \frac{1}{2}$;
(iv) Define two functions $\beta : [0,\infty) \to [0,1)$ *by* $\beta\,(t) = \frac{1}{2^p}$, *and* $\alpha : X \times X \to [0,\infty)$ *by* $\alpha\,(x,y) \geq 1$. $\forall x, y \in X$.
Then the system of integral equations (15) *has a common solution in* X.

Proof. Define the operators $S,T:X\to X$ as in (16), we get

$$
\begin{aligned}
\left(|Sx(r)|+|Ty(r)|\right)^p &= \left(\begin{array}{c}\left|\int_0^1 F(t,r)\Phi_1(r,x(r))dr\right|+\\ \left|\int_0^1 F(t,r)\Phi_2(r,y(r))dr\right|\end{array}\right)^p\\
&\leq \left(\begin{array}{c}\int_0^1 |F(t,r)\Phi_1(r,x(r))|\,dr+\\ \int_0^1 |F(t,r)\Phi_2(r,y(r))|\,dr\end{array}\right)^p\\
&\leq \left(\int_0^1\left(\begin{array}{c}|F(t,r)\Phi_1(r,x(r))|+\\ |F(t,r)\Phi_2(r,y(r))|\end{array}\right)dr\right)^p\\
&\leq \left(\int_0^1 F(t,r)\left(|\Phi_1(r,x(r))|+|\Phi_2(r,y(r))|\right)dr\right)^p\\
&\leq \left(\int_0^1 F(t,r)\left(\lambda(|x(r)|+|y(r)|)\right)dr\right)^p\\
&= \left(\int_0^1 F(t,r)\lambda^p\left((|x(r)|+|y(r)|)^p\right)^{\frac{1}{p}}dr\right)^p\\
&\leq d\left(x,y\right)\lambda^p\left(\sqrt[p]{\frac{e^{-\tau}}{s^3\alpha\left(x,y\right)}}\right)^p\\
&= \frac{e^{-\tau}}{s^3\alpha\left(x,y\right)}\lambda^p d\left(x,y\right).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\alpha\left(x,y\right)s^3 d\left(Sx,Ty\right) &\leq e^{-\tau}\lambda^p d\left(x,y\right)\\
&\leq e^{-\tau}\frac{1}{2^p}d\left(x,y\right)\\
&\leq e^{-\tau}\beta\left(M\left(x,y\right)\right)M\left(x,y\right).
\end{aligned}
$$

Applying natural logarithm on above inequality and after some simplification, we get

$$
\tau+\ln\left(\alpha\left(x,y\right)s^3 d\left(Sx,Ty\right)\right)\leq\ln\left(\beta\left(M\left(x,y\right)\right)M\left(x,y\right)\right).
$$

Thus (S,T) is a pair of generalized α-Geraghty F-contraction type mappings with $F\left(x\right)=\ln x$. All other conditions of Corllary 2.7 immediately follows by the hypothesis. Therefore, the operators S,T have a common fixed point, that is, the system of nonlinear quadratic integral equations (15) has a common solution.

Competing interests
The authors declare that they have no competing interests.

Authors' Contributions
The second author made the first draft of this paper. All authors read and approved the final manuscript.

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