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Fixed Point Methods in the Stability of the Cauchy Functional Equations

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Abstract. By using the fixed point methods, we prove some generalized Hyers-Ulam stability of homomorphisms for Cauchy and Cauchy-Jensen functional equations on the product algebras and on the triple systems.

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1. Introduction and Preliminaries

In 1940, S. M. Ulam states a question concerning the stability of group homomorphisms. In fact, for a group G_1 and a metric group G_2 with metric d and for any given $\varepsilon > 0$, if there exists a $\delta > 0$ such that for any function $h : G_1 \longrightarrow G_2$ that satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta, \qquad x, y \in G_1,$$

there exists a homomorphisms $H : G_1 \longrightarrow G_2$ such that $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

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Following this question, in [2], D. H. Hyers gave the first affirmative answer to the Ulam's question for linear mappings on Banach spaces. Then T. M. Rassias [14] and P. Găvruta [1] and some other researchers, generalized the Hyers's Theorem and gave some approaches of the stability of Ulam-Hyers-Rassias problem. See for instance [6], [12] and [13].
In 2003, Radu in [11] used the following fixed point Theorem for the proof of the stability of additive functional equation of Rassias [14]:

Theorem 1.1. *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a contraction map with a Lipschitz constant $0 \leqslant L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1. *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geqslant n_0$,*
2. *the sequence $\{J^n x\}$ converges to a fixed point x^* of J,*
3. *x^* is the unique fixed point of J in the set $Y := \{y \in X \mid d(J^{n_0} x, y) < \infty\}$,*
4. *$d(y, x^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Following the Radu's paper, some authors interested the same method in the stability problems. For example, Park and Rassias in [10] and [8] used this method for solving the Cauchy and Cauchy-Jensen functional equations. Here, by using the Radu's method of fixed point, we first prove some extensions of the stability of Cauchy functional equations of [10] and then the stability of functional equations in triple systems.

2. Stability of the Cauchy Functional Equations

In [10], the authors proved that for Banach algebras A and B, if $f : A \longrightarrow B$ is a mapping for which there exists a function $\varphi : A \times A \longrightarrow [0, \infty)$ that satisfy the following conditions:

$$\begin{aligned}\lim_{j\longrightarrow\infty} 2^{-j}\varphi(2^j x, 2^j y) &= 0,\\ \|\mu f(x+y) - f(\mu x) - f(\mu y)\| &\leqslant \varphi(x,y),\\ \|f(xy) - f(x)f(y)\| &\leqslant \varphi(x,y),\\ \varphi(2x,2x) &\leqslant 2L\varphi(x,x),\end{aligned}$$

for some $0 \leqslant L < 1$ and for all scalar μ with absolute value 1 and all $x, y \in A$; then there exists a unique homomorphism $H : A \longrightarrow B$ such that for all $x \in A$, $\|f(x)-H(x)\| \leqslant \frac{1}{2-2L}\varphi(x,x)$. That is, H is a solution of the Cauchy functional equation

$$\mu f(x+y) - f(\mu x) - f(\mu y) = 0,$$

that satisfies the homomorphism equation $f(xy) - f(x)f(y) = 0$. Here, we will obtain some refinement of it on the product algebras.

Theorem 2.1. *Let G be an additive group and F be a Banach space. If $f : G \times G \longrightarrow F$ and $\varphi : G \times G \longrightarrow [0,\infty)$ are mappings such that for all $a, b, c \in G$ the following conditions hold:*

$$\|f((a,b)+(c,d)) - f(a,b) - f(c,d)\| \leqslant \varphi(a+c, b+d), \tag{1}$$

$$\|f(a,b) - f(b,a)\| \leqslant \varphi(a,b),$$

$$\varphi(2a,2b) \leqslant 2L\varphi(a,b), \tag{2}$$

for some $0 \leqslant L < 1$, then the map

$$T : G\times G \longrightarrow F \quad , \quad T(a,b) = \lim_{n\longrightarrow\infty} 2^{-n} f(2^n a, 2^n b) \quad , \quad a, b \in G,$$

is the unique additive map such that for all $a, b \in G$,

$$\|f(a,b) - T(a,b)\| \leqslant \frac{L}{1-L}\varphi(a,b),$$

$$T(a,b) = T(b,a).$$

Proof. Consider X as the set of all functions $g : G \times G \longrightarrow F$ and define a generalized metric d on X by

$$d(g,h) := \inf\{c \in [0,\infty] : \|g(a,b) - h(a,b)\| \leqslant c\,\varphi(a,b),\ for\, all\, a, b \in G\}.$$

Then, in fact, d is a complete generalized metric on X. Now define $J : X \longrightarrow X$ by $Jg(a,b) := \frac{1}{2}g(2a,2b)$. since

$$\begin{aligned} \|\frac{1}{2}g(2a,2b) - \frac{1}{2}h(2a,2b)\| &\leqslant \frac{1}{2}d(g,h)\varphi(2a,2b) \\ &\leqslant Ld(g,h)\varphi(a,b), \end{aligned}$$

and by (1),

$$\|f(2a,2b) - 2f(a,b)\| \leqslant \varphi(2a,2b) \leqslant 2L\varphi(a,b),$$

for all $g, h \in X$ and all $a, b \in G$; J is a contraction with constant at most L, such that $d(f, Jf) \leqslant L$ and so by Theorem 1.1, J has a unique fixed point function T in $Y = \{g \in X : d(f,g) < \infty\}$. Furthermore,

$$\begin{aligned} d(f,T) &\leqslant \frac{1}{1-L}d(f,Jf) \leqslant \frac{L}{1-L}, \\ T(a,b) &= \lim_{n\longrightarrow\infty} 2^{-n}f(2^n a, 2^n b), \end{aligned}$$

for all $a, b \in G$. So for all $a, b \in G$,

$$\begin{aligned} \|T(a,b) - T(b,a)\| &= \lim_{n\longrightarrow\infty} 2^{-n}\|f(2^n a, 2^n b) - f(2^n b, 2^n a)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^n a, 2^n b) = 0, \end{aligned}$$

where the last equality holds by (2). This shows that $T(a,b) = T(b,a)$, for all $a, b \in G$.

For the proof of additivity of T, it is sufficient to note that

$$\begin{aligned} \|T((a,b) &+ (c,d)) - T(a,b) - T(c,d)\| \\ &= \lim_{n\longrightarrow\infty} 2^{-n}\|f((2^n a, 2^n b) + (2^n c, 2^n d)) \\ &- f(2^n a, 2^n b) - f(2^n c, 2^n d)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^n(a+c), 2^n(b+d)) = 0. \end{aligned}$$

Finally, we will prove that T is unique. If H is an additive function on $G \times G$, such that

$$\|f(a,b) - H(a,b)\| \leqslant \frac{L}{1-L}\varphi(a,b),$$

for all $a, b \in G$, then $d(f,H) \leqslant \frac{L}{1-L} < \infty$ and so $H \in Y$. On the other hands,

$$JH(a,b) = \frac{1}{2}H(2a,2b) = H(a,b).$$

This shows that H is a fixed point of J in Y and so $H = T$, thanks to the uniqueness of fixed point of J in Y. □

As a corollary, we have the following refinement of Theorem 2.1 of [10].

Theorem 2.2. *Suppose that A and B are two algebras such that B is also a Banach space. If $f : A \times A \longrightarrow B$ and $\varphi : A \times A \longrightarrow [0,\infty)$ are two mappings that satisfy the following conditions:*

$$\|\mu f((a,b)+(c,d)) - f(\mu a,\mu b) - f(\mu c,\mu d)\| \leqslant \varphi(a+c,b+d), \tag{3}$$

$$\|f(a,b) - f(b,a)\| \leqslant \varphi(a,b),$$

$$\|f((ac,bd)) - f(a,b)f(c,d)\| \leqslant \varphi(ac,bd), \tag{4}$$

$$\varphi(2a,2b) \leqslant 2L\varphi(a,b). \tag{5}$$

for some $0 \leqslant L < 1$ and all $a,b,c,d \in A$ and all scalar μ with absolute value 1,then there exists a unique algebraic homomorphism $H : A \times A \longrightarrow B$ such that for all $a,b \in A$,

$$\|f(a,b) - H(a,b)\| \quad \leqslant \quad \frac{L}{1-L}\varphi(a,b), \tag{6}$$

$$H(a,b) = H(b,a).$$

Proof. By Theorem 2.1, the additive function $H : A \times A \longrightarrow B$ defined by

$$H(a,b) = \lim_{n \longrightarrow \infty} 2^{-n} f(2^n a, 2^n b), \quad a,b \in A,$$

is the unique additive map satisfying the above conditions (6). We only need to prove that it is an algebraic homomorphism. By hypothesis (3), one has

$$\|\mu f(2^{n+1}a, 2^{n+1}b) - 2f(2^n\mu a, 2^n\mu b)\| \leqslant \varphi(2^{n+1}a, 2^{n+1}b),$$

and

$$\|f(2^{n+1}\mu a, 2^{n+1}\mu b) - 2f(2^n\mu a, 2^n\mu b)\| \leqslant \varphi(2^{n+1}\mu a, 2^{n+1}\mu b).$$

Thus

$$\begin{aligned}\|\mu H(2a, 2b) \; &- \; H(2\mu a, 2\mu b)\| \\ &= \lim_{n\longrightarrow\infty} 2^{-n}\|\mu f(2^{n+1}a, 2^{n+1}b) - f(2^{n+1}\mu a, 2^{n+1}\mu b)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 2^{-n}(\|\mu f(2^{n+1}a, 2^{n+1}b) - 2f(2^n\mu a, 2^n\mu b)\| \\ &+ \|f(2^{n+1}\mu a, 2^{n+1}\mu b) - 2f(2^n\mu a, 2^n\mu b)\|) \\ &\leqslant \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^{n+1}a, 2^{n+1}b) \\ &+ \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^{n+1}\mu a, 2^{n+1}\mu b) = 0,\end{aligned}$$

where the last equality holds by (5). So $H(2\mu a, 2\mu b) = \mu H(2a, 2b)$. Since $H(2a, 2b) = 2H(a, b)$, for all $a, b \in A$, we have

$$H(\mu a, \mu b) = \mu H(a, b),$$

for all $a, b \in A$ and all scalar μ with absolute value 1.
Now, for the $\mathbb{C}$-linearity of H, assume that $\lambda \in \mathbb{C}$ is an arbitrary non zero scalar and M is an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$ and so by Theorem 1 of [3], there are three scalars μ_1, μ_2, μ_3 with absolute value 1 such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$.
Also, for all $x \in A \times A$, by additivity of H, we have

$$H(x) = H(3\frac{1}{3}x) = 3H(\frac{1}{3}x).$$

So, $H(\frac{1}{3}x) = \frac{1}{3}H(x)$, for all $x \in A \times A$, and then

$$\begin{aligned} H(\lambda x) &= H(\frac{M}{3}3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \lambda H(x). \end{aligned}$$

Finally, the following assertion shows that H is an algebraic homomorphism. In fact, for each (a, b) and (c, d) in $A \times A$ the relation (4) guaranties that

$$\begin{aligned} &\|H((a,b)(c,d)) - H(a,b)H(c,d)\| \\ &= \\ &\|\lim_{n\longrightarrow\infty} 2^{-n} f(2^n ac, 2^n bd) - \lim_{n\longrightarrow\infty} 4^{-n} f(2^n a, 2^n b) f(2^n c, 2^n d)\| \\ &= \lim_{n\longrightarrow\infty} 4^{-n} \|f(4^n ac, 4^n bd) - f(2^n a, 2^n b) f(2^n c, 2^n d)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 4^{-n} \varphi(4^n ac, 4^n bd) = 0. \quad \Box \end{aligned}$$

The final Theorem of this section solves the additive functional equation, for groups.

Theorem 2.3. *Let G be an additive group and E be a Banach space. If $f : G \longrightarrow E$ and $\varphi : G \times G \longrightarrow [0, \infty)$ are mappings that satisfy the conditions*

$$\lim_{n\longrightarrow\infty} 2^{-n} \varphi(2^n x, 2^n y) = 0 \quad , \quad \sup_{x\in G} \varphi(x, x) < \infty,$$

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varphi(x, y),$$

then there exists an unique additive function $T : G \longrightarrow E$ such that

$$\sup_{x\in G} \|f(x) - T(x)\| \leqslant \sup_{x\in G} \varphi(x, x).$$

Proof. Suppose that X is the set of all functions $g : G \longrightarrow E$ and define a complete generalized metric d on X by

$$d(g, h) = \sup_{x\in G} \|g(x) - h(x)\|.$$

If we define the mapping $J : X \longrightarrow X$ via $Jg(x) = \frac{1}{2}g(2x)$, then a straightforward computation shows that J is a contraction with Lipschitz constant L at most $\frac{1}{2}$ such that

$$d(f, Jf) \leqslant \frac{1}{2} \sup_{x \in G} \varphi(x, x) < \infty.$$

So by generalized Banach's contraction Theorem 1.1, J has a fixed point map $T : G \longrightarrow E$ that satisfies the conditions 1-4 of that theorem. Since by hypothesis,

$$\begin{aligned} \|T(x+y) - T(x) - T(y)\| &= \lim_{n \longrightarrow \infty} 2^{-n} \|f(2^n x, 2^n y) - f(2^n x) - f(2^n y)\| \\ &\leqslant \lim_{n \longrightarrow \infty} 2^{-n} \varphi(2^n x, 2^n y) \\ &= 0, \end{aligned}$$

the function T is additive. Also by Theorem 1.1,

$$\begin{aligned} d(f, T) &\leqslant \frac{1}{1-L} d(f, Jf) \\ &\leqslant 2d(f, Jf) \\ &\leqslant \sup_{x \in G} \varphi(x, x). \qquad (7) \end{aligned}$$

The uniqueness of the additive function T satisfying the above condition (7), is completely similar to the uniqueness part of Theorem 2.1. □

3. Cauchy Functional Equations in Triple Systems

A triple system is a vector space V together with a trilinear mapping $V \times V \times V \longrightarrow V$, called a triple product, and usually denoted by $\{.,.,.\}$. A triple system V is called continuous if its triple product $\{.,.,.\}$ is separately continuous, and is called $*$-triple system, if V admits an involution $*$. The most important examples of triple systems are Lie triple systems and Jordan triple systems (see for instance, [4] and [7]).

Also every JB^*-triple is an $*$-triple system [7]. We remember that a complex Banach space J with a continuous triple product $\{.,.,.\}$ on J is a JB^*-triple, if it is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfy the following conditions:

(1) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(a,b)y,z\} + \{x,y,L(a,b)z\}$, for all $a,b,x,y,z \in J$; where the operator $L(a,b) : J \longrightarrow J$ is defined by $L(a,b)(x) = \{a,b,x\}$,

(2) The operator $L(a,b) : J \longrightarrow J$ is an hermitian operator with non-negative spectrum,

(3) $\|\{x,x,x\}\| = \|a\|^3$ for all $a \in J$.

In [7], the author proved the following Theorem 3.1, which solved the Cauchy functional equation in JB^*-triples. Here, by using the familiar generalized Banach's contraction Theorem, we improve Theorem 1 of [3] and we give another proof for it.
We note that, as usual, a $\mathbb{C}$-linear map $H : A \longrightarrow B$ between two triple systems A and B is called triple homomorphism if it satisfies the fallowing condition

$$H\{x,y,z\} = \{H(x),H(y),H(z)\},$$

for all $x,y,z \in A$. We need this definition in the rest of this article.

Theorem 3.1. *Suppose that A is a normed space and B is a Banach space and $f : A \longrightarrow B$ and $\varphi : A \times A \times A \longrightarrow [0,\infty)$ are mappings such that $f(0) = 0$ and*

$$\lim_{n\longrightarrow\infty} 3^{-n}\varphi(3^n x, 3^n y, 3^n z) \quad = \quad 0,$$

$$\varphi(3x,3x,3x) \leqslant 3L\varphi(x,x,x),$$

$$\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\| \quad \leqslant \quad \varphi(x,y,z), \qquad (8)$$

for some $0 \leqslant L < 1$ *and all* $x, y, z \in A$ *and for scalar* μ *with absolute value* 1*. Then there exists an unique* $\mathbb{C}$*-linear map* $H : A \longrightarrow B$ *such that*

$$\|f(x) - H(x)\| \leqslant \frac{1}{3-3L}\varphi(x, x, x),$$

for all $x \in A$*. If furthermore,* A *and* B *are* $*$*-triple systems such that* B *is continuous and the following conditions hold:*

$$\|f(x^*) - f(x)^*\| \leqslant \varphi(x, x, x), \tag{9}$$

$$\|f\{x, y^*, z\} - \{f(x), f(y^*), f(z)\}\| \leqslant \varphi(x, y, z), \tag{10}$$

for all $x, y, z \in A$*, then the* $\mathbb{C}$*-linear map* H *is an* $*$*-homomorphism and triple homomorphism.*

Proof. Define the complete generalized metric

$$d(g, h) = \inf\{c \in [0, \infty] : \|g(x) - h(x)\| \leqslant c\varphi(x, x, x),\ for\, all\, x \in A\}$$

on the space X consisting of all functions $g : A \longrightarrow B$. Since

$$\|\frac{1}{3}g(3x) - \frac{1}{3}h(3x)\| \leqslant \frac{1}{3}d(g, h)\varphi(3x, 3x, 3x)$$

$$\leqslant Ld(g, h)\varphi(x, x, x),$$

for all $x \in A$; the mapping

$$J : X \longrightarrow X \quad , \quad Jg(x) = \frac{1}{3}g(3x),$$

for $g \in X$ and $x \in A$, is a contraction with Lipschitz constant at most L and so has an unique fixed point map $H : A \longrightarrow B$ such that

$$H(x) = \lim_{n \longrightarrow \infty} 3^{-n} f(3^n x),$$

$$d(f, H) \leqslant \frac{1}{1-L} d(f, Jf).$$

Since $f(0)=0$, by substituting $z=0$, $\mu=1$ and replacing $3^n x$ and $3^n y$ instead of x and y respectively, in (8), one has

$$\begin{aligned}\|H(x+y)-H(x)-H(y)\| &= \lim_{n\longrightarrow\infty} 3^{-n}\|f(3^n(x+y)) \\ &- f(3^n x)-f(3^n y)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 3^{-n}\varphi(3^n x, 3^n y, 0) \\ &= 0.\end{aligned}$$

So H is additive. On the other hands, similar to the proof of Theorem 2.2, one can obtains that $H(\mu x)=\mu H(x)$, for all scalar μ with absolute value 1 and all $x\in A$. This is a critical point for the proof of $\mathbb{C}$-linearity of H, as shown in the proof of Theorem 2.2. Also, by (8),

$$\|\frac{1}{3}f(3x)-f(x)\|\leqslant\frac{1}{3}\varphi(x,x,x),$$

and so

$$d(f,Jf)\leqslant\frac{1}{3}.$$

Hence

$$\begin{aligned}\|f(x)-H(x)\| &\leqslant \frac{1}{1-L}d(f,Jf)\varphi(x,x,x) \\ &\leqslant \frac{1}{3-3L}\varphi(x,x,x).\end{aligned}$$

This finishes the proof of the first part of the theorem. If furthermore, A and B are $*$-triple systems that satisfy the conditions (9) and (10), then for each $x\in A$,

$$\begin{aligned}H(x^*) &= \lim_{n\longrightarrow\infty}3^{-n}f(3^n x^*)=\lim_{n\longrightarrow\infty}3^{-n}f(3^n x)^* \\ &= (\lim_{n\longrightarrow\infty}3^{-n}f(3^n x))^*=H(x)^*,\end{aligned}$$

where the second equality holds by (9). Also

$$\begin{aligned}\|H\{x,y^*,z\} &- \{H(x),H(y^*),H(z)\}\| \\ &= \|\lim_{n\longrightarrow\infty} 3^{-3n} f\{3^n x, 3^n y^*, 3^n z\} \\ &- \{\lim_{n\longrightarrow\infty} 3^{-n} f(3^n x), \lim_{n\longrightarrow\infty} 3^{-n} f(3^n y^*), \lim_{n\longrightarrow\infty} 3^{-n} f(3^n z)\}\| \\ &= \lim_{n\longrightarrow\infty} 3^{-3n} \|f\{3^n x, 3^n y^*, 3^n z\} - \{f(3^n x), f(3^n y^*), f(3^n z)\}\| \\ &\leqslant \lim_{n\longrightarrow\infty} 3^{-3n} \varphi(3^n x, 3^n y, 3^n z) \\ &= 0,\end{aligned}$$

where the second equality holds by continuity of B and the third equality holds by the condition (10). This finishes the proof of the second part of the theorem. □

Theorem 3.2. *Suppose that A and B are triple systems such that B is continuous and $f : A \longrightarrow B$ and $\varphi : A \times A \times A \longrightarrow [0, \infty)$ are mappings such that $f(0) = 0$ and*

$$\lim_{n\longrightarrow\infty} 8^n \varphi(2^{-n}x, 2^{-n}y, 2^{-n}z) = 0,$$

$$\|f(\frac{\mu x + \mu y}{2} + \mu z) + f(\frac{\mu x + \mu z}{2} + \mu y) + f(\frac{\mu y + \mu z}{2} + \mu x)$$

$$-2\mu(f(x) + f(y) + f(z))\| \leqslant \varphi(x, y, z), \tag{11}$$

$$\|f\{x, y, z\} - \{f(x), f(y), f(z)\}\| \leqslant \varphi(x, y, z), \tag{12}$$

$$\varphi(\frac{x}{2}, 0, 0) \leqslant \frac{L}{2}\varphi(x, 0, 0),$$

for some $0 \leqslant L < 1$ and all x, y, z in A. Then there exists an unique triple homomorphism $H : A \longrightarrow B$ such that for all $x \in A$,

$$\|f(x) - H(x)\| \leqslant \frac{1}{1-L}\varphi(x, 0, 0).$$

Proof. Define the complete generalized metric d on X consisting of all functions $g : A \longrightarrow B$, by

$$d(g,h) = \inf\{c \in [0,\infty] : \|g(x) - h(x)\| \leqslant c\varphi(x,0,0), for\, all\, x \in A\},$$

and consider the mapping $J : X \longrightarrow X$ via $Jg(x) = 2g(\frac{x}{2})$. Letting $\mu = 1$ and $y = z = 0$ in (11), we get

$$\|2f(\frac{x}{2}) - f(x)\| \leqslant \varphi(x,0,0),$$

for all $x \in A$. Hence $d(f, Jf) \leqslant 1$. Since J is a contraction map with Lipschitz constant at most L and $d(f, Jf) \leqslant 1$, by Theorem 1.1 there exists a mapping $H : A \longrightarrow B$ such that for all $x \in A$,

$$\begin{aligned} H(x) &= \lim_{n\longrightarrow\infty} 2^n f(\frac{x}{2^n}), \\ \|f(x) - H(x)\| &\leqslant \frac{1}{1-L}\varphi(x,0,0). \end{aligned}$$

It is enough to prove that H is $\mathbb{C}$-linear and triple homomorphism. Since, by (11),

$$\begin{aligned} \|H(\frac{\mu x + \mu y}{2} + \mu z) &+ H(\frac{\mu x + \mu z}{2} + \mu y) + H(\frac{\mu y + \mu z}{2} + \mu x) \\ &- 2\mu(H(x) + H(y) + H(z))\| \\ &\leqslant \lim_{n\longrightarrow\infty} 2^n \varphi(2^{-n}x, 2^{-n}y, 2^{-n}z) = 0, \qquad (13) \end{aligned}$$

by substituting $\mu = 1$ and $y = z = 0$ in (13) we have

$$H(\frac{x}{2}) = \frac{1}{2}H(x),$$

for all $x \in A$. So, if in (13), we set $\mu = 1$, $y = -x$ and $z = 0$, we obtain that $H(-x) = -H(x)$, for all $x \in A$.
Now, if in (13) we set $\mu = 1$, $z = -y$ and then we replace x and y by $x + y$ and $x - y$ respectively, we obtain that

$$H(x + y) = H(x) + H(y),$$

i.e., H is additive. This allows that one can repeat the technique of Theorem 2.2 and conclude that H is $\mathbb{C}$-linear. Finally,

$$\begin{aligned}\| H\{x,y,z\} &- \{H(x),H(y),H(z)\}\| = \| \lim_{n\longrightarrow\infty} 8^n f\{2^{-n}x, y2^{-n}, 2^{-n}z\} \\ &- \{\lim_{n\longrightarrow\infty} 2^n f(2^{-n}x), \lim_{n\longrightarrow\infty} 2^n f(2^{-n}y), \lim_{n\longrightarrow\infty} 2^n f(2^{-n}z)\}\| \\ &= \lim_{n\longrightarrow\infty} 8^n \|f\{2^{-n}x, 2^{-n}y, 2^{-n}z\} \\ &- \{f(2^{-n}x), f(2^{-n}y), f(2^{-n}z)\}\| \\ &\leqslant \lim_{n\longrightarrow\infty} 8^n \varphi(2^{-n}x, 2^{-n}y, 2^{-n}z) \\ &= 0,\end{aligned}$$

where the second equality holds by continuity of B and the last inequality holds by (12). □

By the same method one can prove the following corollary:

Corollary 3.3. *Suppose that A and B are triple systems such that B is continuous and $f : A \longrightarrow B$ and $\varphi : A \times A \times A \longrightarrow [0,\infty)$ are mappings satisfying* (11) *and* (12) *such that*

$$\lim_{n\longrightarrow\infty} 8^{-n}\varphi(2^n x, 2^n y, 2^n z) = 0,$$

$$\varphi(2x,0,0) \leqslant 2L\varphi(x,0,0),$$

for some $0 \leqslant L < 1$ and all x,y,z in A. Then there exists an unique triple homomorphism $H : A \longrightarrow B$ such that for all $x \in A$,

$$\|f(x) - H(x)\| \leqslant \frac{L}{1-L}\varphi(x,0,0).$$

Proof. It is sufficient in the proof of the previous theorem, one defines $Jg(x) = \frac{1}{2}g(2x)$ □.

Theorem 3.4. *Suppose that A and B are triple systems such that B is continuous. If $f : A \longrightarrow B$ and $\varphi : A \times A \longrightarrow [0,\infty)$ are two mappings that satisfy the following conditions:*

$$\lim_{j\longrightarrow\infty} 2^{-j}\varphi(2^j x, 2^j y) = 0,$$

$$\|\mu f(x+y) - f(\mu x) - f(\mu y)\| \leqslant \varphi(x,y), \tag{14}$$
$$f(2^n\{x,y,z\}) = \{f(2^n x), f(y), f(z)\}, \tag{15}$$

$$\varphi(x,x) \leqslant 2L\varphi(\tfrac{x}{2},\tfrac{x}{2}),$$

for some $0 \leqslant L < 1$, *all* μ *with absolute value 1 and all* $x, y \in A$, *then there exists an unique triple homomorphism* $H : A \to B$ *such that for all* $x \in A$,

$$\|f(x) - H(x)\| \leqslant \frac{1}{2-2L}\varphi(x,x). \tag{16}$$

Proof. The proof of the first part of this theorem is similar to the proof of Theorem 2.1 of [10]. In fact, the generalized (complete) metric

$$d(g,h) = \inf\{c \in [0,\infty] : \|g(x) - h(x)\| \leqslant c\,\varphi(x,x),\ for\, all\, x \in A\},$$

on the space X consisting of all functions $g : A \longrightarrow B$, allow us to define the contraction map $J : X \longrightarrow X$ via $Jg(x) = \frac{1}{2}g(2x)$, for all $x \in A$. By Theorem 1.1, J has a fixed point function $H \in X$ that satisfies the conditions 1-4 of that theorem. Furthermore

$$H(x) = \lim_{n\longrightarrow\infty} 2^{-n} f(2^n x),$$

for all $x \in A$. So,

$$\begin{aligned}\|H(x+y) - H(x) - H(y)\| &= \lim_{n\longrightarrow\infty} 2^{-n}\|f(2^n(x+y)) \\ &- f(2^n x) - f(2^n y)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^n x, 2^n y) \\ &= 0,\end{aligned}$$

for all $x, y \in A$; that is, H is additive. In particular,

$$H(2x) = 2H(x), \tag{17}$$

for all $x \in A$. Now, by (14),

$$\begin{aligned}\|\mu H(2x) - H(2\mu x)\| &= \lim_{n\longrightarrow\infty} 2^{-n}\|\mu f(2^{n+1}x) - f(2^{n+1}\mu x)\| \\ &\leqslant \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^n x, 2^n x) \\ &= 0.\end{aligned}$$

Hence $\mu H(2x) = H(2\mu x)$ and so by (17), $\mu H(x) = H(\mu x)$, for all $x \in A$ and all μ with absolute value 1. This implies that H is $\mathbb{C}$-linear. For the proof of (16), letting $\mu = 1$ and $y = x$ in (14), we get

$$\|f(2x) - 2f(x)\| \leqslant \varphi(x, x),$$

and so

$$\|f(x) - \frac{1}{2}f(2x)\| \leqslant \frac{1}{2}\varphi(x, x),$$

for all $x \in A$. Hence $d(f, Jf) \leqslant \frac{1}{2}$ and so

$$d(f, H) \leqslant \frac{1}{1-L}d(f, Jf) \leqslant \frac{1}{2-2L}.$$

This proves that the inequality (16) is satisfied. By a similar method to the proof of Theorem 2.1, one can see the uniqueness of H. Also, it follows from (15) that

$$\begin{aligned}H\{x, y, z\} &= \lim_{n\longrightarrow\infty} 2^{-n} f(2^n\{x, y, z\}) \\ &= \lim_{n\longrightarrow\infty} 2^{-n}\{f(2^n x), f(y), f(z)\} \\ &= \lim_{n\longrightarrow\infty}\{2^{-n} f(2^n x), f(y), f(z)\} \\ &= \{H(x), f(y), f(z)\},\end{aligned}$$

for all x, y, z in A and so by linearity of H and trilinearity of the product on the triple systems A and B we have:

$$\begin{aligned}H\{x, y, z\} &= 4^{-n}H\{x, 2^n y, 2^n z\} \\ &= 4^{-n}\{H(x), f(2^n y), f(2^n z)\} \\ &= \{H(x), 2^{-n}f(2^n y), 2^{-n}f(2^n z)\},\end{aligned}$$

for all positive integer n and all $x, y, z \in A$. This implies the equality

$$\begin{aligned} H\{x, y, z\} &= \{H(x), \lim_{n \longrightarrow \infty} 2^{-n} f(2^n y), \lim_{n \longrightarrow \infty} 2^{-n} f(2^n z)\} \\ &= \{H(x), H(y), H(z)\}, \end{aligned}$$

for all $x, y, z \in A$. Thus, $H : A \longrightarrow B$ is a triple homomorphism satisfying (16), as desired. $\square$

Theorem 3.5. *Assume that A and B are triple systems such that B is continuous. If $f : A \longrightarrow B$ and $\varphi : A \times A \times A \longrightarrow [0, \infty)$ are two mappings such that satisfy the following conditions:*

$$\begin{aligned} \lim_{j \longrightarrow \infty} 2^{-n} \varphi(2^n x, 2^n y, 2^n z) &= 0, \\ \|\mu f(x + y) - f(\mu x) - f(\mu y)\| &\leqslant \varphi(x, y, 0), \end{aligned}$$

$$\|f\{x, y, z\} - \{f(x), f(y), f(z)\}\| \leqslant \varphi(x, y, z), \tag{18}$$

$$\varphi(x, x, x) \leqslant 2L\varphi(\tfrac{x}{2}, \tfrac{x}{2}, \tfrac{x}{2}),$$

for some $0 \leqslant L < 1$ and for all scalar μ with absolute value 1 and all $x, y, z \in A$; then there exists an unique triple homomorphism $H : A \longrightarrow B$ such that for all $x \in A$,

$$\|f(x) - H(x)\| \leqslant \frac{1}{2 - 2L} \varphi(x, x, 0).$$

Proof. By a similar method of the proof of Theorem 3.4, one can show that there exists an unique $\mathbb{C}$-linear mapping $H : A \longrightarrow B$ via $H(x) = \lim_{n \longrightarrow \infty} 2^{-n} f(2^n x)$ such that

$$\|f(x) - H(x)\| \leqslant \frac{1}{2 - 2L} \varphi(x, x, 0).$$

for all $x \in A$. It follows from (18) that

$$\begin{aligned}
&\|H\{x,y,z\} - \{H(x),H(y),H(z)\}\| \\
&\qquad = \lim_{n\longrightarrow\infty} 8^{-n}\|f\{2^n x, 2^n y, 2^n z\} - \{f(2^n x), f(2^n y), f(2^n z)\}\| \\
&\qquad \leqslant \lim_{n\longrightarrow\infty} 8^{-n}\varphi(2^n x, 2^n y, 2^n z) \\
&\qquad \leqslant \lim_{n\longrightarrow\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n z) = 0,
\end{aligned}$$

for all $x, y, z \in A$ and so $H : A \longrightarrow B$ is a triple homomorphism. □

References

[1] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.,* 184 (1994), 431-436.

[2] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.,* 27 (1941), 222-224.

[3] R. V. Kadison and G. Pedersen, Means and convex combinations of unitary operators, *Math. Scand.,* 57 (1985), 249-266.

[4] H. Munthe-Kaas, G. R. W. Quispel, and A. Zanna, *Application of symmetric spaces and Lie triple systems in numerical analysis,* Tech. Report 217, Department of Informatics, University of Bergen, Norway, 2001.

[5] A. Najati, Stability of homomorphisms on JB^* -triples associated to a Cauchy-Jensen functional equation, *J. Math. Ineq.,* 1 (2007), 83-103.

[6] Z. Páles, Generalized stability of the Cauchy functional equation, *Aequations Math.,* 56(3) (1998), 222-232.

[7] C. Park, Approximate homomorphisms on JB^* -triples *J. Math. Anal. Appl.,* 306 (2005), 375-381.

[8] C. Park, Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras, *Fixed Point Theory and Appl.,* 2007 (2007), 1-15.

[9] C. Park, On an approximate automorphisms on a C^*-algebra, *Proc. Amer. Math. Soc.,* 132 (2004), 1739-1745.

[10] C. Park and T. M. Rassias, Fixed points and stability of the Cauchy functional equations, *Aust. J. Math. Anal. Appl.,* 6 (2009), 1-9.

[11] V. Radu, The fixed point alternative and the stability of functional equations, *Fixed Point Theory,* 4 (2003), 91-96.

[12] T. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.,* 62 (2000), 23-130.

[13] T. M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.,* 251 (2000), 264-284.

[14] T. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.,* 72 (1978), 297-300.

[15] S. M. Ulam, A collection of the mathematical problems, *Interscience Publ. New York,* 1960.

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