New Topologies on the Rings of Continuous Functions

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Abstract. Two new topologies are defined on $C(X)$. These topologies make $C(X)$ to be a zero-dimensional (completely regular) Hausdorff space. $C(X)$ endowed by these topologies is denoted by $C_o(X)$ and $C_{o^{-1}}(X)$. The relations between $X$, $C_o(X)$ and $C_{o^{-1}}(X)$ are studied and closedness of $z$-ideals and maximal ideals are investigated in $C_o(X)$ and $C_{o^{-1}}(X)$.

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1. Introduction

In this paper, $X$ assumed to be completely regular Hausdorff space and $C(X)$ ($C^*(X)$) stands for the ring of all real valued (bounded) continuous functions on $X$. Whenever $C(X) = C^*(X)$, we call $X$ a pseudocompact space. An ideal $I$ in $C(X)$ is said to be a $z$-ideal if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in C(X)$ imply that $g \in I$, where $Z(f) = \{x \in X : f(x) = 0\}$. Equivalently, $I$ is a $z$-ideal if $M_f \subseteq I$ for each $f \in I$, where $M_f$ is intersection of all maximal ideals containing $f$, see [2] and [5,4]. Therefore every maximal ideal is a $z$-ideal. A space $X$ is called a $P$-space if every $G_\delta$-set (every zero set) in $X$ is open and it is called zero-dimensional if it contains a base of closed-open sets.

In this article we define two topologies on $C(X)$ and $C(X)$ endowed by these topologies denote by $C_o(X)$ and $C_{o^{-1}}(X)$. We show that these
spaces are Hausdorff, completely regular and zero-dimensional spaces. We study the relations between topological properties of the space $X$, $C_o(X)$ and $C_{o-1}(X)$. For example we have shown that $X$ is a $P$-space if and only if $C_o(X)$ is discrete, and $X$ is pseudocompact if and only if the set of units of $C(X)$ (those members $u$ of $C(X)$ with $Z(u) = \emptyset$) is a discrete subspace of $C_{o-1}(X)$. Finally we have investigated the closed ideals of $C_o(X)$ and $C_{o-1}(X)$ and we observed that $z$-ideals and maximal ideals are closed in $C_o(X)$ and $z$-ideals are open in $C_{o-1}(X)$ as well. We have also observed that real maximal ideals are closed in $C_{o-1}(X)$ and it turns out that whenever $X$ is pseudocompact then every maximal ideals is closed in $C_{o-1}(X)$ and whenever $X$ is normal, the converse is also true. For the definition of real maximal ideals and undefined terms and notations, the reader is refered to [4].

2. $C'_o(X)$ and $C'_{o-1}(X)$

Several topologies are defined on $C(X)$ and are studied by topologists, such as pointwise convergence which $C(X)$ considered as the subspace of $\mathbb{R}^X$ with product topology [1], compact open topology or uniform topology [6], $m$-topology which is finer than uniform topology [4] and [5] and many other topologies on $C(X)$, for example see [3]. Here we introduce two new topologies on $C(X)$.

For each $f \in C(X)$ and each open subset $G$ in $X$, such that $Z(f) \subseteq G$, we define

$$B(f, G) = \{g \in C(X) : G_f^c \subseteq Z(f - g)\},$$

where $G_f^c = Z(f) \cup G^c$.

It is evident that the collection $\{B(f, G) : G \text{ is open in } X, \text{ and } Z(f) \subseteq G\}$ is the base for the neighborhood system at $f$, for each $f \in C(X)$. In fact $f \in B(f, G)$, for all open set $G$ which $Z(f) \subseteq G$, and $B(f, G \cap H) \subseteq B(f, G) \cap B(f, H)$, for all open sets $G, H$ such that $Z(f) \subseteq G$ and $Z(f) \subseteq H$. Finally for open set $G$ that $Z(f) \subseteq G$, whenever $g \in B(f, G)$, then $B(g, G) \subseteq B(f, G)$. We call the topology generated by this base, open-topology and $C(X)$ endowed with this topology denotes by $C'_o(X)$.

To introduce another topology on $C(X)$, let $r$ be a positive rational
number, and $f \in C(X)$. We consider the set $G_{r,f} = f^{-1}((-r,+r))$, and define
\[
B(f, G_{r,f}) = \{g \in C(X) : G_{r,f}^c \subseteq Z(f-g)\},
\]
where $G_{r,f}^c = Z(f) \cup G_{r,f}^c$.

The collection $\{B(f, G_{r,f}) : r \in \mathbb{Q}^+\}$ is also the base for neighborhood system at $f$, for each $f \in C(X)$. In fact $f \in B(f, G_{r,f})$, for all $r \in \mathbb{Q}^+$, $B(f, G_{r,f}) \cap B(f, G_{s,f}) = B(f, G_{r,f})$, for all $r, s \in \mathbb{Q}^+$ such that $r \leq s$, and finally for $r \in \mathbb{Q}^+$, whenever $g \in B(f, G_{r,f})$, then $B(g, G_{r,g}) \subseteq B(f, G_{r,f})$, for in this case $G_{r,f}^c \subseteq G_{r,g}^c$. We call the topology generated by this base, invers open-topology and $C(X)$ endowed with this topology denotes by $C_{\alpha-1}(X)$.

**Proposition 2.1.** The following statements hold:

(a) $C_{\alpha}(X)$ and $C_{\alpha-1}(X)$ are Hausdorff spaces.

(b) $C_{\alpha}(X)$ and $C_{\alpha-1}(X)$ are zero-dimensional spaces.

(c) $C_{\alpha}(X)$ and $C_{\alpha-1}(X)$ are completely regular spaces.

**Proof.** We prove the properties for $C_{\alpha-1}(X)$, the proof for $C_{\alpha}(X)$ is similar. To prove (a) let $f, g \in C(X)$ and $f \neq g$. There exists $x_0 \in X$, such that $f(x_0) \neq g(x_0)$. Now consider three cases:

Case 1: $x_0 \notin Z(f) \cup Z(g)$. Then there exists $i \in \mathbb{Q}^+$ such that $x_0 \notin f^{-1}((-i,i))$ and $x_0 \notin g^{-1}((-i,i))$. Hence $x_0 \in G_{i,f}^c \cap G_{i,g}^c$, and therefore $B(f, G_{i,f}) \cap B(g, G_{i,g}) = \emptyset$.

Case 2: $x_0 \in Z(g) \setminus Z(f)$. Then there exists $i \in \mathbb{Q}^+$ such that $x_0 \notin f^{-1}((-i,i))$. Hence $x_0 \in G_{i,f}^c \cap G_{i,g}^c$, and therefore $B(f, G_{i,f}) \cap B(g, G_{i,g}) = \emptyset$.

Case 3: $x_0 \in Z(f) \setminus Z(g)$. This is similar to case 2.

To prove (b), it is sufficient to show that $B(f, G_{r,f})$ is closed, for all $f \in C(X)$ and $r \in \mathbb{Q}^+$, $g \notin B(f, G_{r,f})$, then there exists $x_0 \in G_{r,f}^c$ such that $g(x_0) \neq f(x_0)$. Now consider two cases:

Case 1: $x_0 \notin Z(g)$. Then there exists $i \in \mathbb{Q}^+$ such that $x_0 \notin g^{-1}((-i,i))$. Hence $x_0 \in G_{i,f}^c \cap G_{i,g}^c$, and therefore $B(g, G_{i,g}) \subseteq B(f, G_{r,f})^c$.

Case 2: $x_0 \in Z(g)$. This implies that $x_0 \in G_{r,f}^c \cap G_{r,g}^c$ and therefore $B(g, G_{r,g}) \subseteq B(f, G_{r,f})^c$. 
Finally it is clear that part (b) implies part (c). □

One of our goals is to find the relationship between topological structures of the spaces $X$, $C_o(X)$ and $C_{o-1}(X)$. For this purpose, we give the following propositions.

**Proposition 2.2.** $X$ is a $P$-space if and only if $C_o(X)$ is discrete.

**Proof.** Let $X$ be a $P$-space and $f \in C(X)$. Then $Z(f)$ is an open set in $X$. Take $G = Z(f)$, hence $G_f^c = Z(f) \cup Z(f)^c = X$. Therefore $B(f, G) = \{f\}$ and this means that $C_o(X)$ is discrete.

Conversely, suppose that $C_o(X)$ is discrete and $Z(f)$ is a zeroset in $X$. Then there exists open subset $G$ of $X$ such that $Z(f) \subseteq G$ and $B(f, G) = \{f\}$. We claim that $G_f^c = X$. For the otherwise, there exists $t \in G \setminus Z(f)$. Hence $t \not\in G_f^c$. But $G_f^c$ is closed and $X$ is a completely regular space, therefore there exists $g \in C(X)$ such that $g(t) = 0$ and $g(G_f^c) = \{1\}$. Then $G_f^c \subseteq Z(fg - f)$, thus $fg \in B(f, G)$. But $fg \neq f$ and this is a contradiction. □

**Proposition 2.3.** $X$ is connected if and only if every nonzero isolated point in $C_o(X)$ is a unit in $C_o(X)$.

**Proof.** Let $X$ be connected and $f \in C_o(X)$ is a nonzero isolated point. Then $Z(f)$ is open in $C_o(X)$, by the proof of Proposition 2.2. Thus $Z(f)$ is an open-closed subset of $X$, therefore $Z(f) = \emptyset$, i.e., $f$ is a unit.

Conversely, suppose that $f$ is an idempotent of $C(X)$. Then $Z(f)$ is an open subset in $X$, for $Z(f) = f^{-1}((-1, +1))$. Hence $f$ is isolated in $C_o(X)$. Therefore $f = 0$ or $f = 1$, i.e., $X$ is connected, by [4]. □

In the following proposition, $U(X)$ is the set of all units of $C(X)$.

**Proposition 2.4.** The following statements hold:

(a) $X$ is pseudocompact space if and only if $U(X)$ is a discrete subspace of $C_{o-1}(X)$.

(b) $X$ is finite if and only if $C_{o-1}(X)$ is discrete.

**Proof.** Let $X$ be pseudocompact and $f \in U(X)$. Then there exists $i \in \mathbb{Q}^+$ such that $|f(x)| > i$, for all $x \in X$. We take $G_{i,f} = f^{-1}((-i, i))$, hence $G_{i,f}^c = X$. Therefore $B(f, G_{i,f}) = \{f\}$, i.e., $f$ is an isolated point.
Conversely, suppose that $f \in C(X)$ and $g = \frac{1}{|f| + 1}$, so $g$ is unit. Hence there exists $i \in \mathbb{Q}^+$ such that $B(g, G_{i,g}) \cap U(X) = \{g\}$, for $U(X)$ is a discrete subspace of $C_{o-1}(X)$. The function $h$ defined by

$$h(x) = \begin{cases} g(x) & \text{if } i \\ i & \text{otherwise} \end{cases}$$

is continuous and clearly $h \in B(g, G_{i,g}) \cap U(X)$. Hence $h = g$ and this means that $g$ is bounded away from zero. So $f$ is bounded.

To prove (b), we let $X = \{x_1, x_2, ..., x_n\}$ and $f \in C(X)$. If $f = 0$, then it is clearly that $f$ is an isolated point, for $B(0, G_{r,0}) = \{0\}$, for all $r \in \mathbb{Q}^+$. But if $f \neq 0$, then there exists $r \in \mathbb{Q}^+$ such that $r < \min\{|f(x_i)| : x_i \in \text{Coz}(f)\}$. Hence $G_{r,f}^c = X$, so $B(f, G_{r,f}) = \{f\}$.

Conversely, Suppose that $C_{o-1}(X)$ is discrete. Then $C_o(X)$ is also discrete, for $C_o(X)$ is finer than $C_{o-1}(X)$. Hence $X$ is a $P$-space, by Proposition 2.2. On the other hand $X$ is pseudocompact by (a), therefore $X$ must be finite. □

It is not hard to show that whenever $X$ is countably compact, then $C_o(X) = C_{o-1}(X)$ and whenever $C_o(X) = C_{o-1}(X)$, then $X$ is pseudocompact. The next proposition provides necessary and sufficient condition for the concidence of two spaces. At first, we define strongly pseudocompact space.

**Definition 2.5.** A topological space $X$ is strongly pseudocompact if for every closed subset $F \subseteq X$ and for every $f \in C(X)$, whenever $f|_F$ is unit in $C(F)$, then $f|_F$ is bounded away from zero.

Clearly, every countably compact space is a strongly pseudocompact space and every strongly pseudocompact space is a pseudocompact space.

**Proposition 2.6.** A topological space $X$ is strongly pseudocompact if and only if $C_o(X) = C_{o-1}(X)$.

**Proof.** Let $X$ be strongly pseudocompact and $B(f, U)$ be a nhood base at $f \in C_o(X)$, where $U$ is an open subset in $X$ such that $Z(f) \subseteq U$. If $g \in B(f, U)$, then $g|_{U^c}$ is unit in $C(U^c)$, for $Z(g) \subseteq U$. Hence $g|_{U^c}$ is bounded away from zero, i.e., there exists $i \in \mathbb{Q}^+$ such that $|g(x)| > i$, for
for each $x \in U^c$. Take $G_{i,q} = g^{-1}((-i,i))$, then $U^c \subseteq G_{i,q}$ and $Z(f) \subseteq Z(g)$, hence we have $B(g, G_{i,q}) \subseteq B(f, U)$, therefore $B(f, U)$ is open in $C_{o-1}(X)$. This means that $C_0(X) = C_{o-1}(X)$.

Conversely, suppose that $F \subseteq X$ is closed and $f \in C(X)$ such that $f|_F$ is unit element in $C(F)$. We consider the nhhood base $B(f, F^c)$ at $f \in C_0(X)$ (note that $Z(f) \subseteq F^c$). Then there exists $i \in \mathbb{Q}^+$ such that $B(f, G_{i,f}) \subseteq B(f, F^c)$, for $C_0(X) = C_{o-1}(X)$. Now we have $G_{i,f} \subseteq F^c$, for if $x_0 \in G_{i,f} \setminus F^c$, then there exists $h \in C(X)$ such that $h(G_{i,f}^c) = \{1\}$ and $h(x_0) = 0$. Hence $fh \in B(f, G_{i,f})$, but $fh \not\in B(f, F^c)$, a contradiction. Therefore $F \subseteq G_{i,f}^c$ and this means that $|fh(x)| \geq r$, for all $x \in F$, i.e., $f|_F$ is bounded away from zero. □

3. Maximal Ideals in $C_0(X)$ and $C_{o-1}(X)$

We know that maximal ideals are closed in $C(X)$ with $m$–topology ($C_m(X)$), see 2N in [4]. In this section we investigate the closedness maximal ideals in $C_0(X)$ and $C_{o-1}(X)$, and we will observe that the maximal ideals in $C_0(X)$ and the real maximal ideals in $C_{o-1}(X)$ are closed. But at first, in the next proposition we show that maximal ideals are also open.

**Proposition 3.1.** Every $z$–ideal is open in $C_0(X)$ and in $C_{o-1}(X)$.

**Proof.** Let $I$ be a $z$–ideal of $C(X)$ and $f \in I$. We show that $B(f, X) \subseteq I$. In fact if $g \in B(f, X)$, then $Z(f) \subseteq Z(f - g)$ and hence $Z(f) \subseteq Z(g)$, therefore $g \in I$, for $I$ is $z$–ideal. Thus $I$ is an open subset in $C_0(X)$. Similarly, $I$ is open in $C_{o-1}(X)$. □

**Proposition 3.2.** Every maximal ideal is closed in $C_0(X)$.

**Proof.** Let $M$ be a maximal ideal of $C_0(X)$ and $g \in \text{cl}_o M \setminus M$, where $\text{cl}_o$ means the closure with respect to the topology of $C_0(X)$. Then there exists $k \in M$ such that $Z(k) \cap Z(g) = \emptyset$, by Theorem 2.6 in [4], and hence $Z(g) \subseteq \text{Coz}(k) = X \setminus Z(k)$. Now we consider nhhood base $B(g, \text{Coz}(k))$ at $g$. Clearly $B(g, \text{Coz}(k)) \cap M \neq \emptyset$, for $g \in \text{cl}_o M$. So there exists $h \in M$ such that $\text{Coz}(k)_g \subseteq Z(g - h)$, hence $Z(g) \cup Z(k) \subseteq Z(g - h)$ and
therefore \(Z(k) \cap Z(h) = \emptyset\). This is a contradiction, because \(k, h \in M\). □

Proposition 3.2. shows that every maximal ideal in \(C_o(X)\) is closed, however maximal ideals are not necessarily closed in \(C_{o-1}(X)\). But real maximal ideals are closed in \(C_{o-1}(X)\).

**Proposition 3.3.** Every real maximal ideal is closed in \(C_{o-1}(X)\).

**Proof.** Let \(M\) be a real maximal ideal in \(C_{o-1}(X)\) and \(f \in \text{cl}_{o-1}M\), where \(\text{cl}_{o-1}\) means the closure with respect to the topology of \(C_{o-1}(X)\). Consider \(G_1 = \frac{1}{n}f\), for all \(n \in \mathbb{N}\), then there exists \(g_n \in C(X)\) such that \(g_n \in B(f, G_1, f) \cap M\), for all \(n \in \mathbb{N}\). Since \(M\) is a real maximal ideal, \(\bigcap_{n \in \mathbb{N}} Z(g_n) \in Z[M]\) by Theorem 5.14 in [4], hence there exists \(l \in M\) such that \(Z(l) = \bigcap_{n \in \mathbb{N}} Z(g_n)\). Now we claim that \(Z(l) \subseteq Z(f)\). In fact if \(x_0 \in Z(l) - Z(f)\) then \(f(x_0) \neq 0\), hence there exists \(n_0 \in \mathbb{N}\) such that \(|f(x_0)| > \frac{1}{n_0}\) and therefore \(x_0 \in G_{\frac{1}{n_0}} f\). Since \(g_{n_0} \in B(f, G_{\frac{1}{n_0}}, f)\), \(x_0 \in Z(f - g_{n_0})\). So \(x_0 \in Z(f)\), a contradiction. But \(M\) is a \(z\)--ideal, then \(f \in M\) and hence \(M\) is closed. □

**Corollary 3.4.** If \(X\) is pseudocompact, then every maximal ideal in \(C_{o-1}(X)\) is closed.

Now it is natural to ask that “is the converse of the above result true?” The next proposition shows that the answer is positive, whenever \(X\) is normal or a P-space. In the proof of this proposition we have used the notation \(\text{Neg}(f) = \{x \in X : f(x) < 0\}, f \in C(X)\). We could not yetsettled this question in general.

**Proposition 3.5.** The following statements hold:

(a) If \(X\) is normal, then hyper real maximal ideals in \(C_{o-1}(X)\) are not closed.

(b) If \(X\) is P-space, then hyper real maximal ideals in \(C_{o-1}(X)\) are not closed.

**Proof.** Let \(X\) be normal and \(M\) be a hyper real maximal ideal in \(C_{o-1}(X)\). Then there exists \(g \in C(X)\) such that \(Mg\) is infinitely small, see Theorem 5.6 in [4]. This means that \(g \notin M\) and \(|M|g| < \frac{1}{n}\), for all
$n \in \mathbb{N}$. Hence there exists $h_n \in M$ such that $Z(h_n) \subseteq \text{Neg}(|g| - \frac{1}{n})$, for all $n \in \mathbb{N}$, by Theorem 5.4 in [4]. We consider the continuous functions $k_n : X \setminus \text{Neg}(|g| - \frac{1}{n}) \to \mathbb{R}$, defined by $k_n(x) = \frac{1}{h_n(x)}$, for all $n \in \mathbb{N}$. Since $X$ is normal and $X \setminus \text{Neg}(|g| - \frac{1}{n})$ is a closed subset in $X$, there exists $\tilde{k}_n \in C(X)$ such that $\tilde{k}_n|_{X \setminus \text{Neg}(|g| - \frac{1}{n})} = k_n$. Now we see that $gh_n \tilde{k}_n \in M \cap B(g, G_g, \frac{1}{n})$, for all $n \in \mathbb{N}$, hence $g \in \text{cl}_{\sigma-1} M \setminus M$ and therefore $M$ is not closed. Part (b) will be proved by a similar method. □

**Corollary 3.6.** (a) If $X$ is a normal space, then $X$ is pseudocompact if and only if every maximal ideal in $C_{\sigma-1}(X)$ is closed.
(b) If $X$ is P-space, then every maximal ideal in $C_{\sigma-1}(X)$ is closed if and only if $X$ is finite.

**References**


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