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On Two Extensions of Hilbert's Integral Inequality

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Abstract. The norm of a Hilbert's type linear operator $T : L^2(0,\infty) \to L^2(0,\infty)$ is given. By introducing some parameters, we give the norms of two extensions of Hilbert's integral operator. Also, we obtain two new extended Hilbert's type inequalities with constant factors and the equivalent forms are obtained.

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1. Introduction

Let H be a real separable Hilbert space, and $T : H \to H$ be a bounded self-adjoint semi-positive definite operator. Then (see[6]),

$$(a, Tb)^2 \leqslant \frac{\|T\|^2}{2}\left(\|a\|^2\|b\|^2 + (a,b)^2\right) \quad , \quad (a, b \in H); \tag{1}$$

where (a, b) is the inner product of a and b, and $\|a\| = \sqrt{(a,a)}$ is the norm of a.
In particular, set $H = L^2(0,\infty)$ and define $T : L^2(0,\infty) \to L^2(0,\infty)$ by

$$(Tf)(y) = \int_0^\infty \frac{1}{x+y} f(x)dx \quad , \quad y \in (0,\infty), f \in L^2(0,\infty). \tag{2}$$

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Then by (1), one has the sharper form of Hilbert's integral inequality as [6, p. 292],

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leqslant$$

$$\frac{\pi}{\sqrt{2}} \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx + \left(\int_0^\infty f(x)g(x)dx \right)^2 \right\}^{\frac{1}{2}}. \tag{3}$$

Also, by using Cauchy's inequality in the term $(\int_0^\infty f(x)g(x)dx)^2$ of (3), the Hilbert's integral inequality is obtained as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leqslant \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}.$$

Another inequality of the same type is given by Li, Wang and He:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\} + B\max\{x,y\}} dxdy <$$

$$D(A,B) \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}; \tag{4}$$

where the constant factor $D(A,B)$ [3, Theorem 2.2] is the best possible constant and $\|T\| = D(A,B)$. By introducing a parameter $\lambda > 0$, Jokar and Behboodian gave a general case of inequality (4):

$$\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f(x)g(y)dxdy <$$

$$\omega_\lambda(x) \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}; \tag{5}$$

where the constant factor $\omega_\lambda(x)$ is the best possible and $\|T\| = \omega_\lambda(x)$ [1, Theorem 3.1].

In this paper, by introducing some parameters, we study the norm of Hilbert's type linear operators with the kernels $\frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}$ and $\frac{\left(min\{(\frac{x}{y}),(\frac{y}{x})\}\right)^{\frac{\lambda}{2}}}{A\min\{x,y\}+B\max\{x,y\}}$ for the continuous forms. Also, we obtain two new extended Hilbert's type inequalities with constant factors and the equivalent forms.

2. Main Results and Applications

Lemma 2.1. *Suppose that, $x > 0$, and $0 \leqslant \varepsilon < 1$,*
i) *For any $\lambda > 0$, $\lambda \neq 1, \frac{1}{2}$, and $A > B \geqslant 0$,define the weight function:*

$$\varpi_\lambda(\varepsilon, x) \;:=\; \int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy; \quad (6)$$

and put $\varpi_\lambda(x) = \varpi_\lambda(0,x)$,then $0 < \varpi_\lambda(x) = E_\lambda(A,B) < \infty$ is a constant.
ii) *For any $\lambda \geqslant 0$, and $A > 0, B \geqslant 0$, define the weight function:*

$$\varpi_\lambda(\varepsilon, x) \;:=\; \int_0^\infty \frac{\Big(min\{(\frac{x}{y}),(\frac{y}{x})\}\Big)^{\frac{\lambda}{2}}}{A\min\{x,y\} + B\max\{x,y\}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy; \quad (7)$$

Then $0 < \varpi_\lambda(x) = F_\lambda(A,B) < \infty$ is a constant.

Proof. For a fixed x, letting $v = \frac{B}{A}(\frac{y}{x})$, $\quad u = \frac{A}{B}(\frac{y}{x})$, we get
i) if $\lambda > 1$ then

$$\begin{aligned}
\varpi_\lambda(\varepsilon, x) &= \int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy \\
&= \frac{B^{\frac{1-\varepsilon-2\lambda}{2}}}{A^{\frac{1-\varepsilon}{2}}} \left\{ \int_0^{\frac{A}{B}} \frac{u^{-(\frac{1+\varepsilon}{2})}}{(1+u)^\lambda} du - (\frac{B}{A})^{\lambda-1} \int_0^{\frac{A}{B}} \frac{u^{(\frac{-3-\varepsilon+2\lambda}{2})}}{(1+u)^\lambda} du \right\} \\
&\quad + \frac{A^{\frac{1-\varepsilon-2\lambda}{2}}}{B^{\frac{1-\varepsilon}{2}}} \left\{ (\frac{A}{B})^{\lambda-1} \int_{\frac{B}{A}}^\infty \frac{v^{(\frac{-3-\varepsilon+2\lambda}{2})}}{(1+v)^\lambda} dv - \int_{\frac{B}{A}}^\infty \frac{v^{-(\frac{1+\varepsilon}{2})}}{(1+v)^\lambda} dv \right\} \\
&\leqslant \frac{B^{1-\varepsilon-\lambda} - A^{1-\varepsilon-\lambda}}{(AB)^{\frac{1-\varepsilon}{2}}} \int_0^\infty \frac{u^{-(\frac{1+\varepsilon}{2})}}{(1+u)^\lambda} du \\
&\quad + \frac{A^{-1-\varepsilon+\lambda} - B^{-1-\varepsilon+\lambda}}{(AB)^{\frac{-1-\varepsilon+2\lambda}{2}}} \int_0^\infty \frac{u^{-(\frac{-3-\varepsilon+2\lambda}{2})}}{(1+u)^\lambda} du.
\end{aligned}$$

By Beta function [4], one has

$$0 < \varpi_\lambda(x) \leqslant \frac{2(A^{\lambda-1} - B^{\lambda-1})}{(AB)^{\lambda-\frac{1}{2}}} \beta(\frac{1}{2}, \lambda - \frac{1}{2}) < \infty.$$

Furthermore if $0 < \lambda \leqslant 1$, and $\lambda \neq \frac{1}{2}$ then in the same way

$$\varpi_\lambda(x) \leqslant \frac{2(B^{\lambda-1} - A^{\lambda-1})}{(AB)^{\lambda-\frac{1}{2}}}\beta(\frac{1}{2}, \lambda - \frac{1}{2}) < \infty.$$

That is, for any $\lambda > 0$, $\lambda \neq \frac{1}{2}, 1$, and $A > B \geqslant 0$,

$$0 < \varpi_\lambda(x) \leqslant |\frac{2(A^{\lambda-1} - B^{\lambda-1})}{(AB)^{\lambda-\frac{1}{2}}}|\beta(\frac{1}{2}, \lambda - \frac{1}{2}) < \infty.$$

Hence $0 < \varpi_\lambda(x) = \varpi_\lambda(y) = E_\lambda(A, B) < \infty$.
ii) for any $\lambda \geqslant 0$,

$$\begin{aligned}
\varpi_\lambda(\varepsilon, x) &= \int_0^\infty \frac{\left(min\{(\frac{x}{y}), (\frac{y}{x})\}\right)^{\frac{\lambda}{2}}}{A\min\{x,y\} + B\max\{x,y\}} \left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{2}} dy \\
&= \frac{B^{\frac{\lambda-1-\varepsilon}{2}}}{A^{\frac{\lambda-\varepsilon+1}{2}}} \int_0^{\frac{A}{B}} \frac{u^{\frac{-1-\varepsilon+\lambda}{2}}}{1+u} du + \frac{A^{\frac{-\lambda-1-\varepsilon}{2}}}{B^{\frac{-\lambda-\varepsilon+1}{2}}} \int_{\frac{B}{A}}^\infty \frac{v^{\frac{-\lambda-1-\varepsilon}{2}}}{1+v} dv \\
&\leqslant \frac{B^{\frac{\lambda-1-\varepsilon}{2}}}{A^{\frac{\lambda-\varepsilon+1}{2}}} \int_0^\infty \frac{u^{\frac{-1-\varepsilon+\lambda}{2}}}{1+u} du + \frac{A^{-(\frac{\lambda+1+\varepsilon}{2})}}{B^{\frac{-\lambda-\varepsilon+1}{2}}} \int_0^\infty \frac{v^{-(\frac{\lambda+1+\varepsilon}{2})}}{1+v} dv.
\end{aligned}$$

By Beta function, one has

$$0 < \varpi_\lambda(x) \leqslant (\frac{2}{A^{\frac{\lambda+1}{2}} B^{\frac{1-\lambda}{2}}})\beta(\frac{1-\lambda}{2}, \frac{1+\lambda}{2}) < \infty.$$

Hence $0 < \varpi_\lambda(x) = \varpi_\lambda(y) = F_\lambda(A, B) < \infty$. □

Note: If $\lambda = 0$ in (7), then $\varpi_\lambda(x) = D(A, B)$ which is given in [3, Lemma 1.2].

Theorem 2.2. *Let $T_1, T_2 : L^2(0, \infty) \to L^2(0, \infty)$ be defined as follows:*
i) *for any $\lambda > 0$, $\lambda \neq \frac{1}{2}, 1$, and $A > B \geqslant 0$,*

$$(T_1 f)(y) := \int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f(x)dx \quad , \quad (y \in (0, \infty)); \quad (8)$$

ii) *for any* $\lambda \geqslant 0$, *and* $A > 0$, $B \geqslant 0$,

$$(T_2 f)(y) := \int_0^\infty \int_0^\infty \frac{\left(min\{(\frac{x}{y}),(\frac{y}{x})\}\right)^{\frac{\lambda}{2}}}{A\min\{x,y\}+B\max\{x,y\}} f(x)dx, \quad (y \in (0,\infty)). \quad (9)$$

Then, $\|T_1\| = E_\lambda(A,B)$, *and* $\|T_2\| = F_\lambda(A,B)$; *furthermore, if* $f, g \in L^2(0,\infty)$ *and* $f(x), g(x) \geqslant 0$, *then*

$$(T_1 f, g) < E_\lambda(A,B)\|f\|_2\|g\|_2; \quad (10)$$

and

$$(T_2 f, g) < F_\lambda(A,B)\|f\|_2\|g\|_2; \quad (11)$$

where the constant factors $E_\lambda(A,B)$ *and* $F_\lambda(A,B)$ *are the best possible choices.*

Proof. For $A > B \geqslant 0$, , applying Holder's inequality, we obtain

$$(T_1 f, g) = \left(\int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda} f(x)dx, g(y)\right)$$

$$\leqslant \left\{\int_0^\infty E_\lambda(A,B) f^2(x)dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty E_\lambda(A,B) g^2(y)dy\right\}^{\frac{1}{2}}$$

$$= E_\lambda(A,B)\|f\|_2\|g\|_2; \quad (12)$$

hence $\|T_1\| \leqslant E_\lambda(A,B)$.
For $A > 0$, $B \geqslant 0$, Applying Holder's inequality and using an argument similar to the preceding paragraph, we obtain

$$(T_2 f, g) = \left(\int_0^\infty \frac{\left(min\{(\frac{x}{y}),(\frac{y}{x})\}\right)^{\frac{\lambda}{2}}}{A\min\{x,y\}+B\max\{x,y\}} f(x)dx, g(y)\right)$$

$$\leqslant F_\lambda(A,B)\|f\|_2\|g\|_2; \quad (13)$$

and hence $\|T_2\| \leqslant F_\lambda(A,B)$.
If (12)and (13) take the form of the equality, then by [2] there exist constants α and β , not both zero such that

$$\alpha f^2(x)\left(\frac{x}{y}\right)^{\frac{1}{2}} = \beta g^2(y)\left(\frac{y}{x}\right)^{\frac{1}{2}}. \tag{14}$$

Therefore, we have

$$\alpha f^2(x)x = \beta g^2(y)y \qquad \text{a.e. on } (0,\infty)\times(0,\infty).$$

That is there exists a constant c, such that

$$\alpha f^2(x)x = \beta g^2(y)y = c \qquad \text{a. e. on } (0,\infty)\times(0,\infty).$$

Without loss of generality, suppose $\alpha \neq 0$. Then we obtain $f^2(x) = \frac{c}{\alpha x}$, a.e. on $(0,\infty)$, which contradicts the fact that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (12) and (13) take the form of a strict inequality.
For any $a, b \geqslant 1$, and a sufficiently small $\varepsilon > 0$, set $f_\varepsilon(x) = a^{\frac{\varepsilon}{2}} x^{\frac{-(1+\varepsilon)}{2}}$, if $x \in [a,\infty)$, and $f_\varepsilon(x) = 0$, if $x \in (0,a)$. Similarly, $g_\varepsilon(y) = b^{\frac{\varepsilon}{2}} y^{\frac{-(1+\varepsilon)}{2}}$, if $y \in [b,\infty)$, and $g_\varepsilon(y) = 0$, if $y \in (0,b)$. Assume that the constant factors $E_\lambda(A,B)$, $F_\lambda(A,B)$ in (10) ,(11) are not the best possible choices, then there exist positive real numbers k, k' with $k < E_\lambda(A,B)$ and $k' < F_\lambda(A,B)$ such that (10),(11) are valid by changing $E_\lambda(A,B)$, $F_\lambda(A,B)$ to k, k'. On one hand,

$$\int_0^\infty\int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f_\varepsilon(x)g_\varepsilon(y)dxdy <$$

$$k\|f_\varepsilon\|_2\|g_\varepsilon\|_2 = \frac{k}{\varepsilon}. \tag{15}$$

Similarly

$$\int_0^\infty\int_0^\infty \frac{\left(min\{(\frac{x}{y}),(\frac{y}{x})\}\right)^{\frac{\lambda}{2}}}{A\min\{x,y\} + B\max\{x,y\}} f_\varepsilon(x)g_\varepsilon(y)dxdy < \frac{k'}{\varepsilon}. \tag{16}$$

On the other hand, setting $t = \frac{y}{x}$, we have

$$\int_0^\infty\int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} f_\varepsilon(x)g_\varepsilon(y)dxdy$$

$$= (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_{\frac{b}{x}}^\infty \frac{|1-t^{\lambda-1}|}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dtdx$$

$$= (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^\infty \frac{|1-t^{\lambda-1}|}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dtdx$$

$$-(ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{|1-t^{\lambda-1}|}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dtdx.$$

For $x \geqslant b$ and $0 < \varepsilon < \frac{1}{2}$, we get

$$\int_0^{\frac{b}{x}} \frac{|1-t^{\lambda-1}|}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dt = \int_0^{\frac{b}{x}} \frac{(1-t^{\lambda-1})}{[At+B]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dt,$$

$$\leqslant \left(\frac{4}{B^\lambda}\right)\left(b^{\frac{1}{2}} x^{\frac{-1}{4}} - \frac{b^{\frac{2\lambda-1}{2}} x^{\frac{4\lambda-3}{4}}}{4\lambda-3}\right).$$

Thus

$$0 < (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{|1-t^{\lambda-1}|}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dtdx$$

$$< \left(\frac{16}{B^\lambda}\right)\left(a^{\frac{1}{4}} b + \frac{a^{\frac{4\lambda-1}{4}} b^\lambda}{(4\lambda-3)^2}\right) < \infty.$$

Note that

$$(ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{|1-t^{\lambda-1}|}{[A\min\{1,t\}+B\max\{1,t\}]^\lambda} t^{-\frac{(1+\varepsilon)}{2}} dtdx = O(1).$$

So we have

$$\int_0^\infty \int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda} f_\varepsilon(x) g_\varepsilon(y) dxdy$$

$$= \frac{a^{\frac{-\varepsilon}{2}} b^{\frac{\varepsilon}{2}}}{\varepsilon} [\omega_\lambda(x) + o(1)] - O(1)$$

$$= \frac{a^{\frac{-\varepsilon}{2}} b^{\frac{\varepsilon}{2}}}{\varepsilon} [\omega_\lambda(x) + o(1)]. \tag{17}$$

Also by setting $t = \frac{y}{x}$, we have

$$\int_0^\infty \int_0^\infty \frac{\Big(min\{(\frac{x}{y}),(\frac{y}{x})\}\Big)^{\frac{\lambda}{2}}}{A\min\{x,y\}+B\max\{x,y\}} f_\varepsilon(x) g_\varepsilon(y) dxdy$$

$$= (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_{\frac{b}{x}}^\infty \frac{\big(min\{t^{-1},t\}\big)^{\frac{\lambda}{2}}}{A\min\{1,t\}+B\max\{1,t\}} t^{-\frac{(1+\varepsilon)}{2}} dtdx$$

$$= (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^\infty \frac{\big(min\{t^{-1},t\}\big)^{\frac{\lambda}{2}}}{A\min\{1,t\}+B\max\{1,t\}} t^{-\frac{(1+\varepsilon)}{2}} dtdx$$

$$-(ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{\big(min\{t^{-1},t\}\big)^{\frac{\lambda}{2}}}{A\min\{1,t\}+B\max\{1,t\}} t^{-\frac{(1+\varepsilon)}{2}} dtdx.$$

For $x \geqslant b$ and $0 < \varepsilon < \frac{1}{2}$, we get

$$\int_0^{\frac{b}{x}} \frac{\big(min\{t^{-1},t\}\big)^{\frac{\lambda}{2}}}{A\min\{1,t\}+B\max\{1,t\}} t^{-\frac{(1+\varepsilon)}{2}} dt = \int_0^{\frac{b}{x}} \frac{t^{\frac{\lambda-1-\varepsilon}{2}}}{At+B} dt$$

$$\leqslant \left(\frac{4}{B}\right)\left(\frac{b^{\frac{\lambda+1}{2}} x^{\frac{-(2\lambda+1)}{4}}}{2\lambda+1}\right).$$

Thus

$$0 < (ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{\big(min\{t^{-1},t\}\big)^{\frac{\lambda}{2}}}{A\min\{1,t\}+B\max\{1,t\}} t^{-\frac{(1+\varepsilon)}{2}} dtdx$$

$$< \left(\frac{16}{B}\right)\left(\frac{a^{\frac{-2\lambda+1}{4}} b^{\frac{\lambda+2}{2}}}{(2\lambda+1)^2}\right) < \infty.$$

Note that

$$(ab)^{\frac{\varepsilon}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{\big(min\{t^{-1},t\}\big)^{\frac{\lambda}{2}}}{A\min\{1,t\}+B\max\{1,t\}} t^{-\frac{(1+\varepsilon)}{2}} dtdx = O(1).$$

So we have

$$\int_0^\infty \int_0^\infty \frac{\Big(min\{(\frac{x}{y}),(\frac{y}{x})\}\Big)^{\frac{\lambda}{2}}}{A\min\{x,y\}+B\max\{x,y\}} f_\varepsilon(x) g_\varepsilon(y) dxdy$$

$$= \frac{a^{\frac{-\varepsilon}{2}} b^{\frac{\varepsilon}{2}}}{\varepsilon}[\omega_\lambda(x) + o(1)]. \tag{18}$$

Now from (15), (17) we get $\frac{a^{\frac{-\varepsilon}{2}} b^{\frac{\varepsilon}{2}}}{\varepsilon}[E_\lambda(A,B)+o(1)] < \frac{k}{\varepsilon}$, and from (16), (18), $\frac{a^{\frac{-\varepsilon}{2}} b^{\frac{\varepsilon}{2}}}{\varepsilon}[F_\lambda(A,B)+o(1)] < \frac{\acute{k}}{\varepsilon}$,that is, $E_\lambda(A,B) < k$, $F_\lambda(A,B) < \acute{k}$ when ε is sufficiently small and $a, b \geqslant 1$, which contradicts the hypothesis. Hence the constant factors $E_\lambda(A,B)$, $F_\lambda(A,B)$ in (10), (11) are the best possible choices and $\|T_1\|_2 = E_\lambda(A,B)$, $\|T_2\|_2 = F_\lambda(A,B)$. This completes the proof. □

Note: If $A = B = 1$ and $\lambda = 0$ in (11), then by theorem 2.2 , one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}. \tag{19}$$

If $A = 0, B = 1$ and $\lambda = 0$, then one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}},$$

where the constant factors π and 4 are both the best possible choices. Inequality (19) is the Hilbert's integral inequality [5].

Theorem 2.3. *Suppose that* $0 < \int_0^\infty f^2(x)dx < \infty$. *Then*
i) for any $\lambda > 0$, $\lambda \neq \frac{1}{2}, 1$, *and* $A > B \geqslant 0$

$$\int_0^\infty \left[\int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}| f(x)}{[A\min\{x,y\} + B\max\{x,y\}]^\lambda} dx \right]^2 dy < E_\lambda^2(A,B)\|f\|_2^2; \tag{20}$$

ii) for any $\lambda \geqslant 0$, *and* $A > 0, B \geqslant 0$

$$\int_0^\infty \left[\int_0^\infty \frac{\left(min\{(\frac{x}{y}),(\frac{y}{x})\}\right)^{\frac{\lambda}{2}} f(x)}{A\min\{x,y\} + B\max\{x,y\}} dx \right]^2 dy < F_\lambda^2(A,B)\|f\|_2^2; \tag{21}$$

where the constant factors $E_\lambda^2(A,B)$ *and* $F_\lambda^2(A,B)$ *are the best possible choices. Furthermore, inequalities (20), (10) and similarly (21), (11) are equivalent.*

Proof. Let

$$g(y)=\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}dx.$$

Then , by (2.5), we get

$$\begin{aligned}0<\int_0^\infty g^2(y)dy &= \int_0^\infty \left[\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}dx\right]^2 dy\\ &= \int_0^\infty\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}f(x)g(y)dxdy\\ &\leqslant E_\lambda(A,B)\|f\|_2\|g\|_2; \qquad (22)\end{aligned}$$

Hence, we obtain

$$0<\int_0^\infty g^2(y)dy = E_\lambda^2(A,B)\|f\|_2^2<\infty. \qquad (23)$$

By (10), both (22) and (23) take the form of a strict inequality, so we have (20). On the other hand, suppose that (20) is valid. By Holder's inequality, we find

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|f(x)g(y)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}dxdy\\ &= \int_0^\infty \left[\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}dx\right] g(y)dy\\ &\leqslant \left\{\int_0^\infty \left[\int_0^\infty \frac{|x^{\lambda-1}-y^{\lambda-1}|f(x)}{[A\min\{x,y\}+B\max\{x,y\}]^\lambda}dx\right]^2 dy\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(y)dy\right\}^{\frac{1}{2}}.\end{aligned}$$

By (20), we obtain (10). Thus (10) and (20) are equivalent. If the constant $E_\lambda^2(A,B)$ in (20) is not the best possible choice, then so is $E_\lambda(A,B)$ in (10) . Also in the same way

$$\int_0^\infty \left[\int_0^\infty \frac{\left(min\{(\frac{x}{y}),(\frac{y}{x})\}\right)^{\frac{\lambda}{2}}f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2 dy < F_\lambda^2(A,B)\int_0^\infty f^2(x)dx,$$

and $F_\lambda^2(A,B)$ is the best possible. Inequality (21)is equivalent with (11). This completes the proof. $\square$

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