Journal of Mathematical Extension
Vol. 16, No. 5, (2022) (10)1-20
URL: https://doi.org/10.30495/JME.2022.1478
ISSN: 1735-8299
Original Research Paper

The (k, s, h)-Riemann-Liouville and the (k, s)-Hadamard Operators

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Abstract. This paper deals with new results on Gruss inequality by using recent fractional integral operators. In fact, based on the (k, s, h) −Riemann-Liouville and the (k, h) −Hadamard fractional operators, we establish several integral results. For our results, some very recent results on the paper: [A Grüss type inequality for two weighted functions. J. Math. Computer Sci., 2018.] can be deduced as some special cases.

AMS Subject Classification: 26A33; 26D10; 24D15.

Keywords and Phrases: (k, s, h) −Riemann-Liouville fractional integral, (k, h) −Hadamard fractional operator, Chebyshev's functional, Gruss inequality.

Received: December 2019; Accepted: August 2020
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1 Introduction

The integral inequalities are very important in many areas of science, especially in mathematics, physics, chemistry, biology. Many researchers have given a lot of attention to the generalization of fractional integral inequalities related to weighted Chebyshev functional. In fact, they established many results to Grüss and Chebyshev inequalities. For more details, we refer the reader to [2, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15] and the references therein.

Let us now cite some recent work that have motivated the present paper. We begin by the paper [1], where the authors introduced two new fractional integral operators: the first one is the (k, s, h)-Riemann-Liouville fractional integral (for a function $f \in L^1([a, b])$ with respect to another measurable, increasing, positive function h with $h' \in C^1([a, b])$). It is given by

$$\begin{aligned} & {}_{k}^{s}J_{a,h}^{\alpha}\left(f\left(t\right)\right) \\ = & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)\,h'(\tau)\,f(\tau)\,d\tau, \end{aligned} \tag{1}$$

where $\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt,\ \alpha>0,\ k>0,\ s\in\mathbb{R}-\{-1\}$.

The second introduced operator of the paper [1] is the (k, h)-Hadamard fractional integral (of $f \in L^1([a, b])$ with respect to h). It is defined for $k > 0$ by

$$ {}_{k}I_{a,h}^{\alpha}\left(f(t)\right)=\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\tau)}{h(\tau)}f(\tau)\,d\tau. \tag{2}$$

It is important to note that based on these two operators, we can state that:

Proposition 1.1.

$$\lim_{s\longrightarrow -1^+} {}_{k}^{s}J_{a,h}^{\alpha}\left(f(t)\right)={}_{k}I_{a,h}^{\alpha}\left(f(t)\right) \tag{3}$$

Now, by considering the weighted functional (see [12]):

$$T(f,g,p,q)=\int_a^b p(x)\int_a^b q(x)f(x)g(x)dx \tag{4}$$

$$+\int_a^b q(x)\int_a^b p(x)f(x)g(x)dx$$
$$-\int_a^b p(x)f(x)dx\int_a^b q(x)g(x)dx-\int_a^b q(x)f(x)dx\int_a^b p(x)g(x)dx$$

where f and g are two real-valued integrable functions which are synchronous on $[a,b]$, i.e.:

$$(f(x)-f(y))\,(g(x)-g(y))\geq 0,\ \text{for any } x,y\in[a,b]\,, \tag{5}$$

and p,q are two positive integrable functions on a finite interval $[a,b]$. By considering the above functional, we can observe that

$$T\,(f,g,p,q)=\int_a^b\int_a^b\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right)p\,(\tau)\,q\,(\rho)\,d\tau d\rho. \tag{6}$$

We continue by citing the work that has motivated this paper. We observe that in [3], J. Choi proved the following interesting result:

Theorem 1.2. *Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions, such that*

$$m\leq f(x)\leq M, n\leq g(x)\leq N; m,M,n,N\in\mathbb{R},\ x\in[a,b]\,. \tag{7}$$

If $p,q:[a,b]\to[0,\infty[$ are two integrable functions on $[a,b]$ that satisfy

$$\min\left\{\int_a^b p(t)dt,\int_a^b q(t)dt\right\}>0,$$

then,

$$\begin{aligned}&\left|T\,(f,g,p,q)\right| \\ \leq\ &\left[\left(\frac{(M-m)^2}{2}+2\left\|f\right\|_\infty\right)\left(\frac{(N-n)^2}{2}+2\left\|g\right\|_\infty\right)\right]^{\frac{1}{2}} \\ &\times\left(\int_a^b p(t)dt\right)\left(\int_a^b q(t)dt\right),\end{aligned} \tag{8}$$

where

$$\left\|f\right\|_\infty=\sup_{x\in[a,b]}\left|f\,(x)\right|\ \text{ and }\ \left\|g\right\|_\infty=\sup_{x\in[a,b]}\left|g\,(x)\right|.$$

Based on the two fractional operators of [1], we establish several integral results related to Grüss inequality. We begin by proving an α-theorem. Then, using two α-β-auxiliary results (lemmas), we establish two more general theorems. For our results, the above theorem of [3] is deduced as a special case.

2 Main Results

The first main result is given by the following α-theorem. It depends on one fractional integral parameter.

Theorem 2.1. *Let f and g be two integrable functions on $[a,b]$ satisfying the condition (7), let p and q be two positive functions on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ and $h \in C^1([a,b])$. Then, the following inequality holds*

$$\begin{aligned} & \Big| {}_kI_{a,h}^{\alpha}[p(t)]\ {}_kI_{a,h}^{\alpha}[qfg(t)] + {}_kI_{a,h}^{\alpha}[q(t)]\ {}_kI_{a,h}^{\alpha}[pfg(t)] \\ & - {}_kI_{a,h}^{\alpha}[pf(t)]\, {}_kI_{a,h}^{\alpha}[qg(t)] - {}_kI_{a,h}^{\alpha}[qf(t)]\, {}_kI_{a,h}^{\alpha}[pg(t)] \Big| \qquad (9) \\ \leq & \left[\left(\frac{(M-m)^2}{2} + 2\|f\|_\infty\right)\left(\frac{(N-n)^2}{2} + 2\|g\|_\infty\right)\right]^{\frac{1}{2}} \\ & \times \left({}_kI_{a,h}^{\alpha}[p(t)]\right)\left({}_kI_{a,h}^{\alpha}[q(t)]\right), \end{aligned}$$

where $f,g \in L_\infty[a,b]$, $\alpha > 0$ and $k > 0$.

We need the following lemma to prove Theorem 2.1.

Lemma 2.2. *Let Φ be an integrable function on $[a,b]$ satisfying the condition $m \leq \Phi(x) \leq M$ on $[a,b]$ and let p,q be two positive functions on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h \in C^1([a,b])$. Then for all $t > 0$, we have*

$$\begin{aligned} & {}_kI_{a,h}^{\alpha}[p(t)]\ {}_kI_{a,h}^{\alpha}\left[q\Phi^2(t)\right] + {}_kI_{a,h}^{\alpha}[q(t)]\ {}_kI_{a,h}^{\alpha}\left[p\Phi^2(t)\right] \\ & -2\,{}_kI_{a,h}^{\alpha}[q\Phi(t)]\ {}_kI_{a,h}^{\alpha}[p\Phi(t)] \\ = & \left({}_kI_{a,h}^{\alpha}[(M-\Phi(t))\,q(t)]\right)\left({}_kI_{a,h}^{\alpha}[(\Phi(t)-m)\,p(t)]\right) \qquad (10) \\ & + \left({}_kI_{a,h}^{\alpha}[(M-\Phi(t))\,p(t)]\right)\left({}_kI_{a,h}^{\alpha}[(\Phi(t)-m)\,q(t)]\right) \\ & - \left({}_kI_{a,h}^{\alpha}[q(t)]\right)\left({}_kI_{a,h}^{\alpha}[(M-\Phi(t))(\Phi(t)-m)\,p(t)]\right) \end{aligned}$$

$$-\left({}_{k}I_{a,h}^{\alpha}\left[p(t)\right]\right)\left({}_{k}I_{a,h}^{\alpha}\left[\left(M-\Phi(t)\right)\left(\Phi(t)-m\right)q(t)\right]\right),$$

where $k>0,\ s\in\mathbb{R}-\{-1\},\ \alpha>0.$

Proof. Let Φ be an integrable function on $[a,b]$ satisfying the condition (7) on $[a,b]$. For any $\tau,\rho\in[a,b]$, we have the following identity

$$\begin{aligned}
&\Phi^{2}(\tau)+\Phi^{2}(\rho)-2\Phi(\tau)\Phi(\rho)\\
=\;&[M-\Phi(\rho)][\Phi(\tau)-m]+[M-\Phi(\tau)][\Phi(\rho)-m]\\
&-[M-\Phi(\tau)][\Phi(\tau)-m]-[M-\Phi(\rho)][\Phi(\rho)-m].
\end{aligned}\tag{11}$$

We also consider the quantities:

$$\left\{\begin{array}{l}
{}_{k}^{s}H_{\alpha,h}(t,\tau)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)h'(\tau)p(\tau)\\
{}_{k}^{s}H_{\alpha,h}(t,\rho)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}\left(h^{s+1}(t)-h^{s+1}(\rho)\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)h'(\rho)q(\rho).
\end{array}\right.\tag{12}$$

Multiplying (11) by ${}_{k}^{s}H_{\alpha,h}(t,\tau)\times{}_{k}^{s}H_{\alpha,h}(t,\rho)$, $(\tau,\rho)\in(a,t)^{2}$, $s\in\mathbb{R}-\{-1\}$ and integrating with respect to τ and ρ over $(a,t)^{2}$, we get

$$\begin{aligned}
&\frac{1}{k^{2}\Gamma_{k}^{2}(\alpha)}\\
&\times\left\{\int_{a}^{t}\int_{a}^{t}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)h'(\tau)p(\tau)\Phi^{2}(\tau)\right.\right.\\
&\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)h'(\rho)q(\rho)\right]d\tau d\rho\\
&+\int_{a}^{t}\int_{a}^{t}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)h'(\tau)p(\tau)\right.\\
&\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)h'(\rho)q(\rho)\Phi^{2}(\rho)\right]d\tau d\rho\\
&-2\int_{a}^{t}\int_{a}^{t}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)h'(\tau)p(\tau)\Phi(\tau)\right.\\
&\left.\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)h'(\rho)q(\rho)\Phi(\rho)\right]d\tau d\rho\right\}
\end{aligned}$$

$$
\begin{aligned}
= & \frac{1}{k^2\Gamma_k^2(\alpha)} \\
& \times\left\{\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)\,h'(\tau)\,(\Phi(\tau)-m)\,p(\tau)\right.\right. \\
& \times\left.\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)\,h'(\rho)\,(M-\Phi(\rho))\,q(\rho)\right] d\tau d\rho \\
& +\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)\,h'(\tau)\,(M-\Phi(\tau))\,p(\tau)\right. \\
& \times\left.\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)\,h'(\rho)\,(\Phi(\rho)-m)\,q(\rho)\right] d\tau d\rho \\
& -\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right. \\
& \times h^s(\tau)\,h'(\tau)\,(M-\Phi(\tau))\,(\Phi(\tau)-m)\,p(\tau) \\
& \times\left.\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)\,h'(\rho)\,q(\rho)\right] d\tau d\rho \\
& -\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)\,h'(\tau)\,p(\tau)\right. \\
& \times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)\,h'(\rho) \\
& \times\left.\left.(M-\Phi(\rho))\,(\Phi(\rho)-m)\,q(\rho)\right] d\tau d\rho\right\}.
\end{aligned}
$$

Therefore, it yields that

$$
\begin{aligned}
& \frac{1}{k^2\Gamma_k^2(\alpha)}\left\{\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1} h^s(\tau)\,h'(\tau)\,p(\tau)\,\Phi^2(\tau)\right.\right. \\
& \times\left.\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1} h^s(\rho)\,h'(\rho)\,q(\rho)\right] d\tau d\rho \\
& +\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1} h^s(\tau)\,h'(\tau)\,p(\tau)\right.
\end{aligned}
$$

$$\times\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)\,h'(\rho)\,q(\rho)\,\Phi^{2}(\rho)\Bigg]\,d\tau d\rho$$

$$-2\int_{a}^{t}\int_{a}^{t}\Bigg[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)\,h'(\tau)\,p(\tau)\,\Phi(\tau)$$

$$\times\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)\,h'(\rho)\,q(\rho)\,\Phi(\rho)\Bigg]\,d\tau d\rho\Bigg\}$$

$$=\frac{1}{k^{2}\Gamma_{k}^{2}(\alpha)}\Bigg\{\int_{a}^{t}\int_{a}^{t}\Bigg[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)\,h'(\tau)\,(\Phi(\tau)-m)\,p(\tau)$$

$$\times\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)\,h'(\rho)\,(M-\Phi(\rho))\,q(\rho)\Bigg]\,d\tau d\rho$$

$$+\int_{a}^{t}\int_{a}^{t}\Bigg[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)\,h'(\tau)\,(M-\Phi(\tau))\,p(\tau)$$

$$\times\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)\,h'(\rho)\,(\Phi(\rho)-m)\,q(\rho)\Bigg]\,d\tau d\rho$$

$$-\int_{a}^{t}\int_{a}^{t}\Bigg[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)\,h'(\tau)\,(M-\Phi(\tau))\,(\Phi(\tau)-m)\,p(\tau)$$

$$\times\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)\,h'(\rho)\,q(\rho)\Bigg]\,d\tau d\rho$$

$$-\int_{a}^{t}\int_{a}^{t}\Bigg[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}h^{s}(\tau)\,h'(\tau)\,p(\tau)$$

$$\times\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}h^{s}(\rho)\,h'(\rho)\,(M-\Phi(\rho))\,(\Phi(\rho)-m)\,q(\rho)\Bigg]\,d\tau d\rho\Bigg\}.$$

This completes the proof of the above lemma. □

Let us now prove Theorem 2.1.

Proof. We consider the functional

$$G(\tau,\rho)=(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\,,\tau,\rho\in(a,t)\,,t\in(a,b]\,,\quad(13)$$

where f and g be two integrable functions on $[a,b]$ satisfying the condition (7).
Multiplying (13) by ${}_{k}^{s}H_{\alpha,h}(t,\tau)\times {}_{k}^{s}H_{\alpha,h}(t,\rho)$, $\tau,\rho\in(a,t)$, and integrating the resulting identity with respect to τ and ρ over $(a,t)^2$, we have

$$\begin{aligned}
&\int_a^t\int_a^t {}_{k}^{s}H_{\alpha,h}(t,\tau)\,{}_{k}^{s}H_{\alpha,h}(t,\rho)\,(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\,d\tau d\rho \\
=\;& {}_{k}^{s}J_{a,h}^{\alpha}[p(t)]\,{}_{k}^{s}J_{a,h}^{\alpha}[qfg(t)]+{}_{k}^{s}J_{a,h}^{\alpha}[q(t)]\,{}_{k}^{s}J_{a,h}^{\alpha}[pfg(t)] \qquad (14)\\
&-{}_{k}^{s}J_{a,h}^{\alpha}[pf(t)]\,{}_{k}^{s}J_{a,h}^{\alpha}[qg(t)]-{}_{k}^{s}J_{a,h}^{\alpha}[qf(t)]\,{}_{k}^{s}J_{a,h}^{\alpha}[pg(t)].
\end{aligned}$$

Thanks to the Cauchy Schwarz integral inequality for double integrals, we get

$$\begin{aligned}
&\Big[\int_a^t\int_a^t {}_{k}^{s}H_{\alpha,h}(t,\tau)\,{}_{k}^{s}H_{\alpha,h}(t,\rho)\\
&\times(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\,d\tau d\rho\Big]^2\\
\leq\;&\int_a^t\int_a^t {}_{k}^{s}H_{\alpha,h}(t,\tau)\,{}_{k}^{s}H_{\alpha,h}(t,\rho)\,(f(\tau)-f(\rho))^2\,d\tau d\rho \qquad (15)\\
&\times\int_a^t\int_a^t {}_{k}^{s}H_{\alpha,h}(t,\tau)\,{}_{k}^{s}H_{\alpha,h}(t,\rho)\,(g(\tau)-g(\rho))^2\,d\tau d\rho.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\int_a^t\int_a^t {}_{k}^{s}H_{\alpha,h}(t,\tau)\,{}_{k}^{s}H_{\alpha,h}(t,\rho)\,[f(\tau)-f(\rho)]^2\,d\tau d\rho \qquad (16)\\
=\;&{}_{k}^{s}J_{a,h}^{\alpha}[q(t)]\,{}_{k}^{s}J_{a,h}^{\alpha}\left[pf^2(t)\right]+{}_{k}^{s}J_{a,h}^{\alpha}[p(t)]\,{}_{k}^{s}J_{a,h}^{\alpha}\left[qf^2(t)\right]\\
&-2\left({}_{k}^{s}J_{a,h}^{\alpha}[pf(t)]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}[qf(t)]\right)\\
\\
=\;&\left({}_{k}^{s}J_{a,h}^{\alpha}[(M-f(t))\,q(t)]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}[(f(t)-m)\,p(t)]\right)\\
&+\left({}_{k}^{s}J_{a,h}^{\alpha}[(M-f(t))\,p(t)]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}[(f(t)-m)\,q(t)]\right)\\
&-\left({}_{k}^{s}J_{a,h}^{\alpha}[q(t)]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}[(M-f(t))\,(f(t)-m)\,p(t)]\right)\\
&-\left({}_{k}^{s}J_{a,h}^{\alpha}[p(t)]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}[(M-f(t))\,(f(t)-m)\,q(t)]\right),
\end{aligned}$$

where

$$\left({}_{k}^{s}J_{a,h}^{\alpha}[q(t)]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}[(M-f(t))\,(f(t)-m)\,p(t)]\right) \qquad (17)$$

$$+\left({}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-f(t)\right)\left(f(t)-m\right)q(t)\right]\right)\geq 0,$$

and

$$\begin{aligned}
&\int_a^t\int_a^t {}_{k}^{s}H_{\alpha,h}\left(t,\tau\right)\ {}_{k}^{s}H_{\alpha,h}\left(t,\rho\right)\left[g\left(\tau\right)-g\left(\rho\right)\right]^2 d\tau d\rho \qquad (18)\\
=\ & {}_{k}^{s}J_{a,h}^{\alpha}\left[q(t)\right]\ {}_{k}^{s}J_{a,h}^{\alpha}\left[pg^2(t)\right]+{}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right]\ {}_{k}^{s}J_{a,h}^{\alpha}\left[qg^2(t)\right]\\
&-2\left({}_{k}^{s}J_{a,h}^{\alpha}\left[pg(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[qg(t)\right]\right)
\end{aligned}$$

$$\begin{aligned}
=\ &\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-g(t)\right)q(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(g(t)-m\right)p(t)\right]\right)\\
&+\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-g(t)\right)p(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(g(t)-m\right)q(t)\right]\right)\\
&-\left({}_{k}^{s}J_{a,h}^{\alpha}\left[q(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-g(t)\right)\left(g(t)-m\right)p(t)\right]\right)\\
&-\left({}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-g(t)\right)\left(g(t)-m\right)q(t)\right]\right),
\end{aligned}$$

such that

$$\begin{aligned}
&\left({}_{k}^{s}J_{a,h}^{\alpha}\left[q(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-g(t)\right)\left(g(t)-m\right)p(t)\right]\right) \qquad (19)\\
&+\left({}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right]\right)\left({}_{k}^{s}J_{a,h}^{\alpha}\left[\left(M-g(t)\right)\left(g(t)-m\right)q(t)\right]\right)\geq 0.
\end{aligned}$$

Now, by Lemma 4 and thanks to (7) and (17), the following inequality holds:

$$\begin{aligned}
&\frac{{}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}q(t)}{2}\\
&\times\left\{{}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right]\ {}_{k}^{s}J_{a,h}^{\alpha}\left[qf^2(t)\right]+{}_{k}^{s}J_{a,h}^{\alpha}\left[q(t)\right]\ {}_{k}^{s}J_{a,h}^{\alpha}\left[pf^2(t)\right]\right.\\
&\left.-2\ {}_{k}^{s}J_{a,h}^{\alpha}\left[qf(t)\right]\ {}_{k}^{s}J_{a,h}^{\alpha}\left[pf(t)\right]\right\} \qquad (20)\\
\leq\ &-Mm\left[{}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}q(t)\right]^2\\
&+\frac{M\left[{}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}q(t)\right]}{2}\\
&\times\left[{}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}qf(t)+{}_{k}^{s}J_{a,h}^{\alpha}q(t)\ {}_{k}^{s}J_{a,h}^{\alpha}pf(t)\right]\\
&+\frac{m\left[{}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}q(t)\right]}{2}\\
&\times\left[{}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}qf(t)+{}_{k}^{s}J_{a,h}^{\alpha}q(t)\ {}_{k}^{s}J_{a,h}^{\alpha}pf(t)\right]
\end{aligned}$$

$$- {}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t).$$

Therefore, we obtain

$$\begin{aligned}
& \frac{{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)}{2} \\
& \times \left\{ {}_k^sJ_{a,h}^{\alpha}\left[p(t)\right]\ {}_k^sJ_{a,h}^{\alpha}\left[qf^2(t)\right] + {}_k^sJ_{a,h}^{\alpha}\left[q(t)\right]\ {}_k^sJ_{a,h}^{\alpha}\left[pf^2(t)\right] \right. \\
& \left. -2\ {}_k^sJ_{a,h}^{\alpha}\left[qf(t)\right]\ {}_k^sJ_{a,h}^{\alpha}\left[pf(t)\right]\right\} \\
\leq\ & \Big(M\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\right] \\
& \left. -\frac{1}{2}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) + {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\right]\right) \\
& \times \left(\frac{1}{2}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) + {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\right]\right. \\
& -m\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\right]\Big) \\
& +\frac{1}{4}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) + {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\right]^2 \\
& - {}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) \\
\leq\ & \Big(M\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\right] \\
& \left. -\frac{1}{2}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) + {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\right]\right) \\
& \times \left(\frac{1}{2}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) + {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\right]\right. \\
& -m\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\right]\Big) \\
& +\frac{1}{2}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t) + {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\right]^2 \\
& - {}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}pf(t)\ {}_k^sJ_{a,h}^{\alpha}q(t)\ {}_k^sJ_{a,h}^{\alpha}qf(t).
\end{aligned}$$

Then, we get

$$\begin{aligned}
& \frac{{}_k^sJ_{a,h}^{\alpha}p(t){}_k^sJ_{a,h}^{\alpha}q(t)}{2} \\
& \times \left\{{}_k^sJ_{a,h}^{\alpha}\left[p(t)\right]\ {}_k^sJ_{a,h}^{\alpha}\left[qf^2(t)\right] + {}_k^sJ_{a,h}^{\alpha}\left[q(t)\right]\ {}_k^sJ_{a,h}^{\alpha}\left[pf^2(t)\right]\right. \\
& \left. -2{}_k^sJ_{a,h}^{\alpha}\left[qf(t)\right]\ {}_k^sJ_{a,h}^{\alpha}\left[pf(t)\right]\right\} \qquad (21) \\
\leq\ & \left(M\left[{}_k^sJ_{a,h}^{\alpha}p(t){}_k^sJ_{a,h}^{\alpha}q(t)\right] - \frac{1}{2}\right.
\end{aligned}$$

$$
\begin{aligned}
&\left[{}^s_kJ^{\alpha}_{a,h}p(t){}^s_kJ^{\alpha}_{a,h}qf(t)+{}^s_k J^{\alpha}_{a,h}q(t){}^s_kJ^{\alpha}_{a,h}pf(t)\right]\big) \\
&\times\left(\frac{1}{2}\left[{}^s_kJ^{\alpha}_{a,h}p(t){}^s_kJ^{\alpha}_{a,h}qf(t)+{}^s_k J^{\alpha}_{a,h}q(t){}^s_kJ^{\alpha}_{a,h}pf(t)\right]\right. \\
&\left.-m\left[{}^s_kJ^{\alpha}_{a,h}p(t){}^s_kJ^{\alpha}_{a,h}q(t)\right]\right) \\
&+\frac{1}{2}\left[{}^s_kJ^{\alpha}_{a,h}p(t){}^s_kJ^{\alpha}_{a,h}qf(t)\right]^2+\frac{1}{2}\left[{}^s_kJ^{\alpha}_{a,h}q(t){}^s_kJ^{\alpha}_{a,h}pf(t)\right]^2.
\end{aligned}
$$

Now, using the fact that $4xy \le (x+y)^2$ for all $x,y\in\mathbb{R}$ and using also the two following inequalities

$$
\begin{aligned}
\left[\,{}^s_kJ^{\alpha}_{a,h}p(t)\;{}^s_kJ^{\alpha}_{a,h}qf(t)\right]^2 &\le \|f\|^2_{\infty}\left[\,{}^s_kJ^{\alpha}_{a,h}p(t)\;{}^s_kJ^{\alpha}_{a,h}q(t)\right]^2, \\
\left[\,{}^s_kJ^{\alpha}_{a,h}q(t)\;{}^s_kJ^{\alpha}_{a,h}pf(t)\right]^2 &\le \|f\|^2_{\infty}\left[\,{}^s_kJ^{\alpha}_{a,h}p(t)\;{}^s_kJ^{\alpha}_{a,h}q(t)\right]^2,
\end{aligned}
\tag{22}
$$

we can write

$$
\begin{aligned}
&{}^s_kJ^{\alpha}_{a,h}\left[p(t)\right]\;{}^s_kJ^{\alpha}_{a,h}\left[qf^2(t)\right]+\;{}^s_kJ^{\alpha}_{a,h}\left[q(t)\right]\;{}^s_kJ^{\alpha}_{a,h}\left[pf^2(t)\right] \\
&-2\;{}^s_kJ^{\alpha}_{a,h}\left[qf(t)\right]\;{}^s_kJ^{\alpha}_{a,h}\left[pf(t)\right] \\
\le\;&\left(\frac{(M-m)^2}{2}+2\left\|f\right\|^2_{\infty}\right)\;{}^s_kJ^{\alpha}_{a,h}p(t)\;{}^s_kJ^{\alpha}_{a,h}q(t).
\end{aligned}
\tag{23}
$$

Similarly, we have

$$
\begin{aligned}
&{}^s_kJ^{\alpha}_{a,h}\left[p(t)\right]\;{}^s_kJ^{\alpha}_{a,h}\left[qg^2(t)\right]+\;{}^s_kJ^{\alpha}_{a,h}\left[q(t)\right]\;{}^s_kJ^{\alpha}_{a,h}\left[pg^2(t)\right] \\
&-2\;{}^s_kJ^{\alpha}_{a,h}\left[qg(t)\right]\;{}^s_kJ^{\alpha}_{a,h}\left[pg(t)\right] \\
\le\;&\left(\frac{(N-n)^2}{2}+2\left\|g\right\|^2_{\infty}\right)\left(\,{}^s_kJ^{\alpha}_{a,h}\left[p(t)\right]\right)\left(\,{}^s_kJ^{\alpha}_{a,h}\left[q(t)\right]\right).
\end{aligned}
\tag{24}
$$

Consequently, by (23), (24) and (3), we end the proof of Theorem 2.1.
□

Corollary 2.3. *Let f be an integrable functions on $[a,b]$ satisfying the condition (7), p be a positive functions on $[a,b]$, h be a measurable, increasing and positive function on $(a,b]$ and $h\in C^1([a,b])$. Then for all $t>0$, the following inequality is valid:*

$$
\left|\;{}_kI^{\alpha}_{a,h}\left[p(t)\right]\;{}_kI^{\alpha}_{a,h}\left[pf^2(t)\right]-\left(\,{}_kI^{\alpha}_{a,h}\left[pf(t)\right]\right)^2\right| \tag{25}
$$

$$\leq \left(\frac{(M-m)^2}{4}+\|f\|_\infty\right)\left[\,{}_kI^{\alpha}_{a,h}p(t)\right]^2,$$

where $f\in L_\infty[a,b]$, $\alpha>0$ *and* $k>0$.

Proof. Applying Theorem 2.1 for $f(x)=g(x)$ and $p(x)=q(x)$, we obtain (25). $\square$

Remark 2.4. Taking $\alpha=k=1$ and $h(x)=e^x$, $t=b$ in Theorem 2.1, we obtain Theorem 1.2.

Now we use two real positive parameters to prove the following α-β-theorem.

Theorem 2.5. *Let f and g be two integrable functions on $[a,b]$ satisfying the condition (7), let p and q be two positive functions on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^1([a,b])$. Then, we have*

$$\begin{aligned}
&\Big|\,{}_kI^{\alpha}_{a,h}[p(t)]\;{}_kI^{\beta}_{a,h}[qfg(t)]+{}_kI^{\beta}_{a,h}[p(t)]\;{}_kI^{\alpha}_{a,h}[qfg(t)]\\
&+{}_kI^{\alpha}_{a,h}[q(t)]\;{}_kI^{\beta}_{a,h}[pfg(t)]+{}_kI^{\beta}_{a,h}[q(t)]\;{}_kI^{\alpha}_{a,h}[pfg(t)]\\
&-{}_kI^{\alpha}_{a,h}[pf(t)]\;{}_kI^{\beta}_{a,h}[qg(t)]-{}_kI^{\beta}_{a,h}[pf(t)]\;{}_kI^{\alpha}_{a,h}[qg(t)] \qquad (26)\\
&-{}_kI^{\alpha}_{a,h}[qf(t)]_k\,I^{\beta}_{a,h}[pg(t)]-{}_kI^{\beta}_{a,h}[qf(t)]_k\,I^{\alpha}_{a,h}[pg(t)]\Big|\\
\leq\;&\left[\left(\frac{(M-m)^2}{2}+2\|f\|_\infty\right)\left(\frac{(N-n)^2}{2}+2\|g\|_\infty\right)\right]^{\frac{1}{2}}\\
&\times\Big[\,{}_kI^{\alpha}_{a,h}[p(t)]\;{}_kI^{\beta}_{a,h}[q(t)]+{}_kI^{\beta}_{a,h}[p(t)]\;{}_kI^{\alpha}_{a,h}[q(t)]\Big],
\end{aligned}$$

where $\alpha,\beta>0$, $k>0$ *and* $f,g\in L_\infty[a,b]$.

To prove the above result, we prove the following auxiliary result:

Lemma 2.6. *Let f and g be two integrable functions on $[a,b]$ satisfying the condition (7), let p and q be two positive functions on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^1([a,b])$. Then for all $\alpha,\beta>0$, we have*

$$\Big\{\,{}_kI^{\alpha}_{a,h}[p(t)]\;{}_kI^{\beta}[qfg(t)]+{}_kI^{\beta}_{a,h}[p(t)]\;{}_kI^{\alpha}[qfg(t)]$$

$$\begin{aligned}
&+ {}_kI^{\alpha}_{a,h}\left[q(t)\right]\ {}_kI^{\beta}\left[pfg(t)\right] + {}_kI^{\beta}_{a,h}\left[q(t)\right]\ {}_kI^{\alpha}\left[pfg(t)\right] \\
&- {}_kI^{\alpha}_{a,h}\left[pf(t)\right]\ {}_kI^{\beta}_{a,h}\left[qg(t)\right] - {}_kI^{\beta}_{a,h}\left[pf(t)\right]\ {}_kI^{\alpha}_{a,h}\left[qg(t)\right] \\
&- {}_kI^{\alpha}_{a,h}\left[qf(t)\right]\ {}_kI^{\beta}_{a,h}\left[pg(t)\right] - {}_kI^{\beta}_{a,h}\left[qf(t)\right]\ {}_kI^{\alpha}_{a,h}\left[pg(t)\right]\Big\}^2 \qquad (27) \\
\le\ &\Big\{ {}_kI^{\alpha}_{a,h}\left[p(t)\right]\ {}_kI^{\beta}\left[qf^2(t)\right] + {}_kI^{\beta}_{a,h}\left[p(t)\right]\ {}_kI^{\alpha}\left[qf^2(t)\right] \\
&+ {}_kI^{\alpha}_{a,h}\left[q(t)\right]\ {}_kI^{\beta}\left[pf^2(t)\right] + {}_kI^{\beta}_{a,h}\left[q(t)\right]\ {}_kI^{\alpha}\left[pf^2(t)\right] \\
&- 2\ {}_kI^{\alpha}_{a,h}\left[pf(t)\right]\ {}_kI^{\beta}_{a,h}\left[qf(t)\right] - 2\ {}_kI^{\beta}_{a,h}\left[pf(t)\right]\ {}_kI^{\alpha}_{a,h}\left[qf(t)\right]\Big\} \\
&\times\Big\{ {}_kI^{\alpha}_{a,h}\left[p(t)\right]\ {}_kI^{\beta}\left[qg^2(t)\right] + {}_kI^{\beta}_{a,h}\left[p(t)\right]\ {}_kI^{\alpha}\left[qg^2(t)\right] \\
&+ {}_kI^{\alpha}_{a,h}\left[q(t)\right]\ {}_kI^{\beta}\left[pg^2(t)\right] + {}_kI^{\beta}_{a,h}\left[q(t)\right]\ {}_kI^{\alpha}\left[pg^2(t)\right] \\
&- 2\ {}_kI^{\alpha}_{a,h}\left[pg(t)\right]\ {}_kI^{\beta}_{a,h}\left[qg(t)\right] - 2\ {}_kI^{\beta}_{a,h}\left[pg(t)\right]\ {}_kI^{\alpha}_{a,h}\left[qg(t)\right]\Big\}.
\end{aligned}$$

Proof. Using the functional $G(\tau,\rho)$ which gives in (13) and multiplying (11) by ${}^s_kH_{\alpha,h}(t,\tau)\times{}^s_kH_{\beta,h}(t,\rho)$, $(\tau,\rho)\in(a,t)^2$, $s\in\mathbb{R}-\{-1\}$ and integrating with respect to τ and ρ over $(a,t)^2$, we get

$$\begin{aligned}
&\int_a^t\int_a^t {}^s_kH_{\alpha,h}(t,\tau)\ {}^s_kH_{\beta,h}(t,\rho) \\
&\times(f(\tau)-f(\rho))(g(\tau)-g(\rho))\,d\tau d\rho \\
=\ &{}^s_kJ^{\alpha}_{a,h}p(t)\ {}^s_kJ^{\beta}_{a,h}(qfg)(t) + {}^s_kJ^{\beta}_{a,h}p(t)\ {}^s_kJ^{\alpha}_{a,h}(qfg)(t) \qquad (28) \\
&+ {}^s_kJ^{\alpha}_{a,h}q(t)\ {}^s_kJ^{\beta}_{a,h}(pfg)(t) + {}^s_kJ^{\beta}_{a,h}q(t)\ {}^s_kJ^{\alpha}_{a,h}(pfg)(t) \\
&- {}^s_kJ^{\alpha}_{a,h}(pf)(t)\ {}^s_kJ^{\beta}_{a,h}(qg)(t) - {}^s_kJ^{\beta}_{a,h}(pf)(t)\ {}^s_kJ^{\alpha}_{a,h}(qg)(t) \\
&- {}^s_kJ^{\alpha}_{a,h}(qf)(t)\ {}^s_kJ^{\beta}_{a,h}(pg)(t) - {}^s_kJ^{\beta}_{a,h}(qf)(t)\ {}^s_kJ^{\alpha}_{a,h}(pg)(t).
\end{aligned}$$

Now, by using Cauchy Schwarz integral inequalities, we have

$$\begin{aligned}
&\Big[\left({}^s_kJ^{\alpha}_{a,h}\left[p(t)\right]\right)\ \left({}^s_kJ^{\beta}_{a,h}\left[qfg(t)\right]\right) \\
&+\left({}^s_kJ^{\beta}_{a,h}\left[p(t)\right]\right)\ \left({}^s_kJ^{\alpha}_{a,h}\left[qfg(t)\right]\right) \\
&+ {}^s_kJ^{\alpha}_{a,h}q(t)\ {}^s_kJ^{\beta}_{a,h}(pfg)(t) + {}^s_kJ^{\beta}_{a,h}q(t)\ {}^s_kJ^{\alpha}_{a,h}(pfg)(t) \\
&- {}^s_kJ^{\alpha}_{a,h}(pf)(t)\ {}^s_kJ^{\beta}_{a,h}(qg)(t) - {}^s_kJ^{\beta}_{a,h}(pf)(t)\ {}^s_kJ^{\alpha}_{a,h}(qg)(t)
\end{aligned}$$

$$
\begin{aligned}
& -\left({}_{k}^{s}J_{a,h}^{\alpha}\left[qf(t)\right] \right) \left({}_{k}^{s}J_{a,h}^{\beta}\left[pg(t)\right] \right) \\
& -\left. \left({}_{k}^{s}J_{a,h}^{\beta}\left[qf(t)\right] \right) \left({}_{k}^{s}J_{a,h}^{\alpha}\left[pg(t)\right] \right) \right] ^{2} \\
\leq & \left\{ {}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right] {}_{k}^{s}J_{a,h}^{\beta}\left[qf^{2}(t)\right] +{}_{k}^{s}J_{a,h}^{\beta}\left[p(t)\right] {}_{k}^{s}J_{a,h}^{\alpha}\left[qf^{2}(t)\right] \right. \qquad (29) \\
& +{}_{k}^{s}J_{a,h}^{\alpha}q(t){}_{k}^{s}J_{a,h}^{\beta}\left(pf^{2}\right) (t)+{}_{k}^{s}J_{a,h}^{\beta}q(t){}_{k}^{s}J_{a,h}^{\alpha}\left(pf^{2}\right) (t) \\
& \left. -2{}_{k}^{s}J_{a,h}^{\alpha}\left(pf\right) (t){}_{k}^{s}J_{a,h}^{\beta}\left(qf\right) (t)-2{}_{k}^{s}J_{a,h}^{\beta}\left[pf(t)\right] {}_{k}^{s}J_{a,h}^{\alpha}\left[qf(t)\right] \right\} \\
& \times \left\{ {}_{k}^{s}J_{a,h}^{\alpha}\left[p(t)\right] {}_{k}^{s}J_{a,h}^{\beta}\left[qg^{2}(t)\right] +{}_{k}^{s}J_{a,h}^{\beta}\left[p(t)\right] {}_{k}^{s}J_{a,h}^{\alpha}\left[qg^{2}(t)\right] \right. \\
& +{}_{k}^{s}J_{a,h}^{\alpha}q(t){}_{k}^{s}J_{a,h}^{\beta}\left(pg^{2}\right) (t)+{}_{k}^{s}J_{a,h}^{\beta}q(t){}_{k}^{s}J_{a,h}^{\alpha}\left(pg^{2}\right) (t) \\
& \left. -2{}_{k}^{s}J_{a,h}^{\alpha}\left[pg(t)\right] {}_{k}^{s}J_{a,h}^{\beta}\left[qg(t)\right] -2{}_{k}^{s}J_{a,h}^{\beta}\left[pg(t)\right] {}_{k}^{s}J_{a,h}^{\alpha}\left[qg(t)\right] \right\} .
\end{aligned}
$$

At the end, applying (3), we get (27). □

Lemma 2.7. *Let φ be an integrable function on $[a,b]$ satisfying the condition (7) on $[a,b]$, let p and q be two positive functions on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^{1}([a,b])$. Then for all $\alpha,\beta>0$, we have*

$$
\begin{aligned}
& {}_{k}I_{a,h}^{\alpha}\left[p(t)\right] {}_{k}I_{a,h}^{\beta}\left[q\varphi^{2}(t)\right] +{}_{k}I_{a,h}^{\beta}\left[p(t)\right] {}_{k}I_{a,h}^{\alpha}\left[q\varphi^{2}(t)\right] \\
& +{}_{k}I_{a,h}^{\alpha}\left[q(t)\right] {}_{k}I_{a,h}^{\beta}\left[p\varphi^{2}(t)\right] +{}_{k}I_{a,h}^{\beta}\left[q(t)\right] {}_{k}I_{a,h}^{\alpha}\left[p\varphi^{2}(t)\right] \\
& -2\left({}_{k}I_{a,h}^{\alpha}\left[p\varphi(t)\right] \right) \left({}_{k}I_{a,h}^{\beta}\left[q\varphi(t)\right] \right) \\
& -2\left({}_{k}I_{a,h}^{\beta}\left[p\varphi(t)\right] \right) \left({}_{k}I_{a,h}^{\alpha}\left[q\varphi(t)\right] \right) \\
= & \left({}_{k}I_{a,h}^{\alpha}\left[\left(M-\varphi(t)\right) q(t)\right] \right) \left({}_{k}I_{a,h}^{\beta}\left[\left(\varphi(t)-m\right) p(t)\right] \right) \\
& +\left({}_{k}I_{a,h}^{\beta}\left[\left(M-\varphi(t)\right) q(t)\right] \right) \left({}_{k}I_{a,h}^{\alpha}\left[\left(\varphi(t)-m\right) p(t)\right] \right) \qquad (30) \\
& +\left({}_{k}I_{a,h}^{\alpha}\left[\left(M-\varphi(t)\right) p(t)\right] \right) \left({}_{k}I_{a,h}^{\beta}\left[\left(\varphi(t)-m\right) q(t)\right] \right) \\
& +\left({}_{k}I_{a,h}^{\beta}\left[\left(M-\varphi(t)\right) p(t)\right] \right) \left({}_{k}I_{a,h}^{\alpha}\left[\left(\varphi(t)-m\right) q(t)\right] \right) \\
& -\left({}_{k}I_{a,h}^{\alpha}\left[q(t)\right] \right) \left({}_{k}I_{a,h}^{\beta}\left[\left(M-\varphi(t)\right) \left(\varphi(t)-m\right) p(t)\right] \right) \\
& -\left({}_{k}I_{a,h}^{\beta}\left[q(t)\right] \right) \left({}_{k}I_{a,h}^{\alpha}\left[\left(M-\varphi(t)\right) \left(\varphi(t)-m\right) p(t)\right] \right)
\end{aligned}
$$

$$-\left({}_kI^{\alpha}_{a,h}\left[p(t)\right]\right)\left({}_kI^{\beta}_{a,h}\left[\left(M-\varphi(t)\right)\left(\varphi(t)-m\right)q(t)\right]\right)$$
$$-\left({}_kI^{\beta}_{a,h}\left[p(t)\right]\right)\left({}_kI^{\alpha}_{a,h}\left[\left(M-\varphi(t)\right)\left(\varphi(t)-m\right)q(t)\right]\right)$$

where $k>0$.

Proof. Let us multiplying (11) by ${}^s_kH_{\alpha,h}(t,\tau)\times {}^s_kH_{\beta,h}(t,\rho)$, $(\tau,\rho)\in(a,t)^2$, $s\in\mathbb{R}-\{-1\}$ witch gives in (12) and integrating with respect to τ and ρ over $(a,t)^2$, we get

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$$
$$\times\left\{\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\tau)h'(\tau)p(\tau)\varphi^2(\tau)\right.\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\rho)h'(\rho)q(\rho)\right]d\tau d\rho$$
$$+\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\tau)h'(\tau)p(\tau)\varphi^2(\tau)\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\rho)h'(\rho)q(\rho)\right]d\tau d\rho$$
$$+\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\tau)h'(\tau)p(\tau)\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\rho)h'(\rho)q(\rho)\varphi^2(\rho)\right]d\tau d\rho$$
$$+\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\tau)h'(\tau)p(\tau)\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\rho)h'(\rho)q(\rho)\varphi^2(\rho)\right]d\tau d\rho$$
$$-2\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\tau)h'(\tau)p(\tau)\varphi(\tau)\right.$$

$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^{s}(\rho)\, h'(\rho)\, q(\rho)\, \Phi^{2}(\rho)\Bigg] d\tau d\rho$$

$$-2\int_a^t\int_a^t\Bigg[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1} h^{s}(\tau)\, h'(\tau)\, p(\tau)\, \varphi(\tau)$$

$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}(\rho)\, h'(\rho)\, q(\rho)\, \varphi^{2}(\rho)\Bigg] d\tau d\rho\Bigg\}$$

$$= \frac{1}{k^{2}\Gamma_k(\alpha)\Gamma_k(\beta)}$$

$$\times\Bigg\{\int_a^t\int_a^t\Bigg[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}(\tau)\, h'(\tau)\, (\varphi(\tau)-m)\, p(\tau)$$

$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^{s}(\rho)\, h'(\rho)\, (M-\varphi(\rho))\, q(\rho)\Bigg] d\tau d\rho$$

$$+\int_a^t\int_a^t\Bigg[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1} h^{s}(\tau)\, h'(\tau)\, (\varphi(\tau)-m)\, p(\tau)$$

$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}(\rho)\, h'(\rho)\, (M-\varphi(\rho))\, q(\rho)\Bigg] d\tau d\rho$$

$$+\int_a^t\int_a^t\Bigg[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}(\tau)\, h'(\tau)\, (M-\varphi(\tau))\, p(\tau)$$

$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^{s}(\rho)\, h'(\rho)\, (\varphi(\rho)-m)\, q(\rho)\Bigg] d\tau d\rho$$

$$+\int_a^t\int_a^t\Bigg[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1} h^{s}(\tau)\, h'(\tau)\, (M-\varphi(\tau))\, p(\tau)$$

$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^{s}(\rho)\, h'(\rho)\, (\varphi(\rho)-m)\, q(\rho)\Bigg] d\tau d\rho$$

$$-\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\tau)\,h'(\tau)\,(M-\varphi(\tau))\,(\varphi(\tau)-m)\,p(\tau)\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\rho)\,h'(\rho)\,q(\rho)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1}\right.$$
$$\times h^s(\tau)\,h'(\tau)\,(M-\varphi(\tau))\,(\varphi(\tau)-m)\,p(\tau)$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\rho)\,h'(\rho)\,q(\rho)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\tau)\,h'(\tau)\,p(\tau)\right.$$
$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\rho)\,h'(\rho)\,p(\rho)$$
$$\left.\times\,(M-\varphi(\rho))\,(\varphi(\rho)-m)\,q(\rho)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\beta}{k}-1}h^s(\tau)\,h'(\tau)\,p(\tau)\right.$$
$$\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\rho)\,h'(\rho)\,p(\rho)$$
$$\left.\times\,(M-\varphi(\rho))\,(\varphi(\rho)-m)\,q(\rho)\right]d\tau d\rho\}$$

Thanks to (3), we end the proof. □

Now, we are ready to prove Theorem 2.5.

Proof. By (7) and (1), we have

$$\begin{aligned}&-\left({}_k^sJ_{a,h}^{\alpha}\left[p(t)\right]\right)\left({}_k^sJ_{a,h}^{\beta}\left[(M-\varphi(t))\,(\varphi(t)-m)\,q(t)\right]\right)\\&-\left({}_k^sJ_{a,h}^{\beta}\left[p(t)\right]\right)\left({}_k^sJ_{a,h}^{\alpha}\left[(M-\varphi(t))\,(\varphi(t)-m)\,q(t)\right]\right)\\&-\left({}_k^sJ_{a,h}^{\alpha}\left[q(t)\right]\right)\left({}_k^sJ_{a,h}^{\beta}\left[(M-\varphi(t))\,(\varphi(t)-m)\,p(t)\right]\right)\\&-\left({}_k^sJ_{a,h}^{\beta}\left[q(t)\right]\right)\left({}_k^sJ_{a,h}^{\alpha}\left[(M-\varphi(t))\,(\varphi(t)-m)\,p(t)\right]\right)\leq 0.\end{aligned}\tag{31}$$

Now, from Lemma 2.6, by using (1) and Cauchy Schwarz integral inequality for double integrals with two parameters $\alpha, \beta > 0$, applying also the Lemma 2.7, (31) and (3), we get (26). This completes the proof of the Theorem 2.5. □

Remark 2.8. Taking $\alpha = \beta$ in Theorem 2.5, we obtain Theorem 2.1.

Theorem 2.9. *Let f be integrable function on $[a, b]$ satisfying the condition (7), p and q be two positive functions on $[a, b]$ and let h be a measurable, increasing and positive function on $(a, b]$ with $h \in C^1([a, b])$. Then for all $t > 0$, the following inequality holds*

$$\begin{aligned} & \Big| {}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}_{a,h}\left[qf^2(t)\right] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}\left[qf^2(t)\right] \\ & + {}_kI^{\alpha}_{a,h}[q(t)]\ {}_kI^{\beta}_{a,h}\left[pf^2(t)\right] + {}_kI^{\beta}_{a,h}[q(t)]\ {}_kI^{\alpha}_{a,h}\left[pf^2(t)\right] \\ & - 2\ {}_kI^{\alpha}_{a,h}[pf(t)]\ {}_kI^{\beta}_{a,h}[qf(t)] - 2\ {}_kI^{\beta}_{a,h}[pf(t)]\ {}_kI^{\alpha}_{a,h}[qf(t)] \Big| \\ \leq & \left(\frac{(M-m)^2}{2} + 2\|f\|_{\infty} \right) \\ & \times \left[{}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}_{a,h}[q(t)] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}[q(t)] \right], \end{aligned} \tag{32}$$

where $\alpha, \beta > 0$, $k > 0$.

Proof. Applying Theorem 2.5 for $f(x) = g(x)$, we obtain (32). □

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