A Note on Power Values of Derivation in Prime and Semiprime Rings

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Abstract. Let $R$ be a ring with derivation $d$, such that $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geq 1$ a fixed integer. In this paper, we show that if $R$ is prime, then $d = 0$ or $R$ is commutative. If $R$ is semiprime, then $d$ maps $R$ into its center. Moreover in semiprime case let $A = O(R)$ be the orthogonal completion of $R$ and $B = B(C)$ be the Boolean ring of $C$, where $C$ is the extended centroid of $R$. Then there exists an idempotent $e \in B$ such that $eA$ is a commutative ring and $d$ induces a zero derivation on $(1 - e)A$.

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1. Introduction

Let $R$ be an associative ring with center $Z(R)$. Recall that an additive map $d : R \rightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Many results in literature indicate that global structure of a prime (semiprime) ring $R$ is often lightly connected to the behaviour of additive mappings defined on $R$. A well-known result of Herstein [13] stated that if $R$ is a prime ring and $d$ is an inner derivation of $R$ such that $d(x)^n = 0$ for all $x \in R$ and $n \geq 1$ fixed integer, then $d = 0$.

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The number of authors extended this theorem in several ways. In [12] Giambruno and Herstein extended this result to arbitrary derivations in semiprime rings. In [5] Carini and Giambruno proved that if $R$ is a prime ring with derivation $d$ such that $d(x)^n(x) = 0$ for all $x \in L$, a Lie ideal of $R$, then $d(L) = 0$ when $R$ has no non-zero nil right ideal and $\text{char } R \neq 2$. The same conclusion holds when $n(x) = n$ is fixed and $R$ is a 2-torsion free semiprime ring. Using the ideas in [5] and the methods in [10] Lanski [16] removed both the bound on the indices of nilpotence and the characteristic assumptions on $R$. In [4] Bresar gave a generalization of the result due to Herstein and Giambruno [12] in another direction. Explicitly, he proved in semiprime ring $R$ with derivation $d$ and $a \in R$, if $ad(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $ad(R) = 0$ when $R$ is an $(n-1)!$-torsion free ring. In recent years, a number of articles discussed derivations in the context of prime and semiprime rings (see [6, 11, 20, 8, 1, 9]). But here we will extend Herstein result’s [13] when the condition is more widespread. Indeed, we consider the situation when $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geq 1$ is a fixed integer.

The main results in this paper are as follows:

**Theorem 1.1.** Let $R$ be a prime ring and $d$ a derivation of $R$. Suppose $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geq 1$ is a fixed integer. Then $d = 0$ or $R$ is commutative. When $R$ is a semiprime ring, we prove:

**Theorem 1.2.** Let $R$ be a semiprime ring and $d$ a non-zero derivation of $R$. Suppose $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geq 1$ is a fixed integer. Then $d$ maps $R$ into its center.

**Theorem 1.3.** Let $R$ be a semiprime ring with derivation $d$. Consider $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geq 1$ is a fixed integer. Further, let $A = O(R)$ be the orthogonal completion of $R$ and $B = B(C)$ where $C$ the extended centroid of $R$. Then there exists idempotent $e \in B$ such that $eA$ is a commutative ring and $d$ induce a zero derivation on $(1 - e)A$. 
Throughout the paper we use the standard notation from [3]. In particular, we denote by $Q$ the two sided Martindale quotient of prime (semiprime) ring $R$ and $C$ the center of $Q$. We call $C$ the extended centroid of $R$.

2. Main Results

First, we consider the case when $R$ is a prime ring. The following results are useful tools needed in the proof of Theorem 1.1.

**Lemma 2.1.** (see [7, Theorem 2]). Let $R$ be a prime ring and $I$ a non-zero ideal of $R$. Then $I$, $R$ and $Q$ satisfy the same generalized polynomial identities with coefficient in $Q$.

**Lemma 2.2.** (see [18, Theorem 2]). Let $R$ be a prime ring and $I$ a non-zero ideal of $R$. Then $I$, $R$ and $Q$ satisfy the same differential identities.

**Theorem 2.3.** (Kharchenko [15]). Let $R$ be a prime ring, $d$ a nonzero derivation of $R$ and $I$ a nonzero ideal of $R$. If $I$ satisfies the differential identity

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0,$$

for any $r_1, r_2, \ldots, r_n \in I$, then one of the following holds:

(i) satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0.$$

(ii) $d$ is $Q$-inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and $I$ satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.$$

We establish the following technical result required in the proof of Theorem 1.1.

**Lemma 2.4.** Let $R$ be a prime ring with extended centroid $C$. Suppose

$$([a, x][y + x[a, y]])^n - [a, x]^n[a, y]^n = 0,$$

for all $x, y \in R$ and some $a \in R$. Then $R$ is commutative or $a \in C$. 

**Proof.** If $R$ is commutative there is nothing to prove. Suppose $R$ is not commutative. Set

$$f(x, y) = ([a, x]y + x[a, y])^n - [a, x]^n[a, y]^n.$$ 

Since $R$ is not commutative, then by Lemma 2.1, $f(x, y)$ is a nontrivial generalized polynomial identity for $R$ and so for $Q$.

In case $C$ is infinite, we have $f(x, y) = 0$ for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $Q$ and $Q \otimes_C \overline{C}$ are prime and centrally closed [14], we may replace $R$ by $Q$ or $Q \otimes_C \overline{C}$ according to $C$ finite or infinite. Thus we may assume that $R$ is a centrally closed over $C$ which is either finite or algebraically closed and $f(x, y) = 0$ for all $x, y \in R$. By Martindale’s Theorem [19], $R$ is then a primitive ring having nonzero socle $H$ with $C$ as associated division ring. Hence by Jacobson’s Theorem [14] $R$ is isomorphic to a dense ring of linear transformations of some vector space $V$ over $C$, and $H$ consists of the linear transformations in $R$ of finite rank. Let $\dim_C V = k$. Then the density of $R$ on $V$ implies that $R \cong M_k(C)$. If $\dim_C V = 1$, then $R$ is commutative, which is a contradiction.

Suppose that $\dim_C V \geq 2$. We show that for any $v \in V$, $v$ and $av$ are linearly dependent over $C$. Suppose $v$ and $av$ are linearly independent for some $v \in V$. By density of $R$, there exist $x, y \in R$ such that

$$xv = 0, \quad xav = v,$$

$$yv = 0, \quad yav = v.$$ 

Since $[a, y]^nv = [a, x]^nv = (-1)^nv$, hence we get the following contradiction

$$0 = (([a, x]y + x[a, y])^n - [a, x]^n[a, y]^n)v = -v.$$ 

So we conclude that $\{v, av\}$ are linearly $C$-dependent. Hence for each $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in C$. Now we prove $\alpha_v$ is not depending on the choice of $v \in V$.

Since $\dim_C V \geq 2$ there exists $w \in V$ such that $v$ and $w$ are linearly independent over $C$. Now there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in C$ such that

$$av = v\alpha_v, \quad aw = w\alpha_w, \quad a(v + w) = (v + w)\alpha_{v+w}.$$
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Which implies

\[ v(\alpha_v - \alpha_{(v+w)}) + w(\alpha_w - \alpha_{(v+w)}) = 0, \]

and since \( \{v, w\} \) are linearly \( C \)-independent, it follows \( \alpha_v = \alpha_{(v+w)} = \alpha_w \). Therefore there exists \( \alpha \in C \) such that \( av = v\alpha \) for all \( v \in V \).

Now let \( r \in R, v \in V \). Since \( av = v\alpha \),

\[ [a, r]v = (ar)v - (ra)v = a(rv) - r(av) = (rv)\alpha - r(v\alpha) = 0, \]

that is \( [a, r]V = 0 \). Hence \( [a, r] = 0 \) for all \( r \in R \), implying \( a \in C \). \( \Box \)

Now we can prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( R \) be not commutative. By the given hypothesis, \( R \) satisfies the generalized differential identity

\[ (d(x)y + xd(y))^n = (d(x))^n(d(y))^n. \] (1)

By Lemma 2.2, \( R \) and \( Q \) satisfy the same differential identities, thus \( Q \) satisfies (1). We divide the proof in two cases:

**Case 1.** \( d \) is a \( Q \)-inner derivation. In the case, there exists an element \( a \in Q \) such that \( d(x) = [a, x] \) and \( d(y) = [a, y] \) for all \( x, y \in Q \). Notice that \( Q \) satisfies the generalized polynomial identity \( ([a, x]y + x[a, y])^n = [a, x]^n[a, y]^n \). In this case the conclusion follows from Lemma 1. Thus we have \( a \in C \) and so \( d = 0 \).

**Case 2.** \( d \) is not a \( Q \)-inner derivation. Applying Theorem 2.2, then (1) becomes

\[ (zy + xw)^n - (z)^n(w)^n, \]

for all \( x, y, z, w \in Q \). If \( z = w \), then \( Q \) satisfies

\[ (zy + xz)^n - z^{2n} = 0. \]

This is a polynomial identity. Hence there exists a field \( F \) such that \( Q \subseteq M_k(F) \), the ring of \( k \times k \) matrices over field \( F \), where \( k > 1 \). Moreover \( Q \) and \( M_k(F) \) satisfy the same polynomial identity [17, Lemma 1]. Choose

\[ x = z = e_{ij}, \quad y = e_{ji}, \]
for all \( i \neq j \). This leads to the contradiction

\[
0 = (zy + xz)^n - z^{2n} = e_{ii}.
\]

This completes the proof. \( \square \)

The following example shows the hypothesis of primeness is essential in Theorem 1.1.

**Example 2.5.** Let \( S \) be any ring, and \( R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\} \).

Define \( d : R \to R \) as follows:

\[
d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then \( 0 \neq d \) is a derivation of \( R \) such that \( (d(xy))^n = (d(x))^n(d(y))^n \) for all \( x, y \in R \), where \( n \geq 1 \) is a fixed integer, however \( R \) is not commutative.

Now let \( R \) be a semiprime ring.

We establish the following technical result required in the proof of Theorem 1.2.

**Lemma 2.6.** (see [2, Lemma 1 and Theorem 1] or [18, pages 31-32]).

Let \( R \) be a semiprime ring and \( P \) a maximal ideal of \( C \). Then \( PQ \) is a prime ideal of \( Q \) invariant under all derivations of \( Q \). Moreover

\[
\cap\{P \mid PQ \text{ is maximal ideal of } C\} = 0.
\]

Now we can prove Theorem 1.2.

**Proof.** Since any derivation \( d \) can be uniquely extended to a derivation in \( Q \), and \( R, Q \) satisfy the same differential identities [18, Theorem 3], we have

\[
(d(xy))^n = (d(x))^n(d(y))^n,
\]
for all $x, y \in Q$. Let $P$ be any maximal ideal of $C$ by Lemma 2.6, $PQ$ is prime ideal of $Q$ invariant under $d$. Set $\overline{Q} = Q/PQ$. Then derivation $d$ canonically induces a derivation $\overline{d}$ on $\overline{Q}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in Q$. Therefore,
\[(\overline{d(xy)})^n = (\overline{d(x)})^n(\overline{d(y)})^n,\]
for all $\overline{x}, \overline{y} \in \overline{Q}$. By Theorem 1.1 $d(Q) \subseteq PQ$ or $[Q, Q] \subseteq PQ$. Hence $d(Q)[Q, Q] \subseteq PQ$ for any maximal ideal $P$ of $C$. By Lemma 2.6, $d(Q)[Q, Q] = 0$. Without loss of generality we have $d(R)[R, R] = 0$. This implies that $d(R^2)[R, R] = d(R)R[R, R]$.

Therefore
\[[R, d(R)]R[R, d(R)] = 0.\]

By semiprimeness of $R$, we have $[R, d(R)] = 0$. This complete the proof. □

Now let $R$ be a semiprime orthogonally complete ring with extended centeroid $C$. The notations $B = B(C)$ and $\text{spec}(B)$ denotes Boolean ring of $C$ and the set of all maximal ideal of $B$, respectively. It is well known that if $M \in \text{spec}(B)$ then $R_M = R/RM$ is prime [3, Theorem 3.2.7]. We use the notations $\Omega$-$\Delta$-ring, Horn formulas and Hereditary formulas. We refer the reader to [3, pages 37, 38, 43, 120] for the definitions and the related properties of these objects.

We establish the following technical result required in the proof of Theorem 1.3.

**Lemma 2.7.** [3, Theorem 3.2.18]. Let $R$ be an orthogonally complete $\Omega$-$\Delta$-ring with extended centeroid $C$, $\Psi_i(x_1, x_2, \ldots, x_n)$ Horn formulas of signature $\Omega$-$\Delta$, $i = 1, 2, \ldots$ and $\Phi(y_1, y_2, \ldots, y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \ldots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \ldots, c_m) \in R^{(m)}$. Suppose $R \models \Phi(\vec{c})$ and for every $M \in \text{spec}(B)$ there exists a natural number $i = i(M) > 0$ such that
\[R_M \models \Phi(\phi_M(\vec{c})) \implies \Psi_i(\phi_M(\vec{a})) ,\]
where $\phi_M : R \to R_M = R/RM$ is the canonical projection. Then there exists a natural number $k > 0$ and pairwise orthogonal idempotents
\( e_1, e_2, \ldots, e_k \in B \) such that \( e_1 + e_2 + \ldots + e_k = 1 \) and \( e_i R \models \Psi_i(e_i A) \) for all \( e_i \neq 0 \).

We denote \( O(R) \) the orthogonal completion of \( R \) which is defined as the intersection of all orthogonally complete subset of \( Q \) containing \( R \). Now we can prove Theorem 1.3.

**Proof.** By assumption we have \( R \) satisfies

\[
(d(xy))^n = (d(x))^n(d(y))^n.
\]

According to [3, Theorem 3.1.16] \( d(A) \subseteq A \) and \( d(e) = 0 \) for all \( e \in B \). Therefore, \( A \) is an orthogonally complete \( \Omega-\Delta \)-ring, where \( \Omega = \{ o, +, -, \cdot, d \} \). Consider formulas

\[
\Phi = (\forall x)(\forall y)\|(d(xy))^n = (d(x))^n(d(y))^n\|,
\]

\[
\Psi_1 = (\forall x)\|d(x) = 0\|,
\]

\[
\Psi_2 = (\forall x)(\forall y)\|xy = yx\|.
\]

One can easily check that \( \Phi \) is a hereditary first order formula and \( \neg \Phi, \Psi_1, \Psi_2 \) are Horn formulas. So using Theorem 1.1 shows that all conditions of Lemma 2.7 are fulfilled. Hence there exist two orthogonal idempotent \( e_1 \) and \( e_2 \) such that \( e_1 + e_2 = 1 \) and if \( e_i \neq 0 \), then \( e_i A \models \Psi_i \), \( i = 1, 2 \). The proof is complete. \( \square \)

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**References**

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