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On Nilpotent Elements of Skew Polynomial Rings

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Abstract. We study the structure of the set of nilpotent elements in skew polynomial ring $R[x;\alpha]$, when R is an α-Armendariz ring. We prove that if R is a nil α-Armendariz ring and $\alpha^t = I_R$, then the set of nilpotent elements of R is an α-compatible subrng of R. Also, it is shown that if R is an α-Armendariz ring and $\alpha^t = I_R$, then R is nil α-Armendariz. We give some examples of non α-Armendariz rings which are nil α-Armendariz. Moreover, we show that if $\alpha^t = I_R$ for some positive integer t and R is a nil α-Armendariz ring and $nil(R[x][y;\alpha]) = nil(R[x])[y]$, then $R[x]$ is nil α-Armendariz. Some results of [3] follow as consequences of our results.

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1. Introduction

Rege and Chhawchharia ([17]) called a ring R an *Armendariz* ring if whenever any polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for any i and j. The name of the ring was given due to Armendariz who proved [4] that *reduced* rings (i.e. rings without nonzero nilpotent elements) satisfy this condition. Armendariz rings are thus a generalization of reduced rings,

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(see [4, Lemma 1]), and therefore, nilpotent elements play an important role in this class of rings (see [3]). Some properties of Armendariz rings have been studied in [1, 2, 3, 4, 10, 12, 13, 11, 16, 17]. For a ring R with a ring endomorphism $\alpha : R \to R$, a *skew polynomial ring* (also called an Ore extension of endomorphism type) $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R, the new multiplication $xr = \alpha(r)x$ for all $r \in R$ (see [14, Example 1.7]).

The Armendariz property of rings mentioned earlier was extended to skew polynomial rings in [10]: For an endomorphism α of a ring R, R is called α-*Armendariz* ring if for $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_ib_j = 0$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$.

Recall that an endomorphism α of a ring R is called *rigid* (see [11, 13]) if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced by [9, Proposition 5], and according to [7], an endomorphism α of a ring R is called *compatible* whenever $ab = 0 \Leftrightarrow a\alpha(b) = 0$, for each $a, b \in R$. Note that R is α-rigid if and only if R is α-compatible and reduced, by [7]. If R is an α-rigid ring, then for $p = a_0 + a_1x + \cdots + a_mx^m$ and $q = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$, $pq = 0$ if and only if $a_ib_j = 0$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$ ([9, Proposition 6]). Hence α-rigid rings are α-Armendariz by [7, Lemma 2.2].

Now, we establish our general notations. All rings considered here are associative and unitary and *subrng* will denote a subring without unit. If R is a ring, $nil(R)$ denotes the set of nilpotent elements in R, $R[x]$ denotes the polynomial ring over R, and if $f(x) \in R[x]$, $coef(f(x))$ denotes the subset of R of the coefficients of $f(x)$. Also, if I is a subset of R, $I[x]$ denotes the set of all polynomials whose coefficients belong to I.

According to Antoine ([3]), a ring R is called to be *nil-Armendariz* if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in nil(R)[x]$ then $ab \in nil(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$. Then he studied the conditions under which the polynomial ring over a nil-Armendariz ring is also nil-Armendariz. That conditions are strongly connected to the question of Amitsur of whether or not a polynomial

ring over a nil ring is nil.
Motivated by Antoine [3] and Hong, Kwak and Rizvi [10], we introduce the notion of a nil α-Armendariz ring for an endomorphism α of a ring R as follows:

Definition 1.1. *Let α be an endomorphism of a ring R. R is called nil α-Armendariz, if whenever two polynomials $f(x), g(x) \in R[x;\alpha]$ satisfy $f(x)g(x) \in nil(R)[x]$, then $ab \in nil(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$. Let α be an endomorphism of a ring R and X a nonempty subset of R. We say X is an α-*compatible *subset of R, whenever $ab \in X \Leftrightarrow a\alpha(b) \in X$. Clearly, R is an α-compatible ring if and only if $\{0\}$ is an α-compatible subset of R.*

Example 1.2. Let D be an integral domain and consider the trivial extension of D given by: $R = \left\{ \begin{pmatrix} a & d \\ 0 & a \end{pmatrix} \mid a, d \in D \right\}$. Clearly, R is a commutative ring. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} a & d \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & ud \\ 0 & a \end{pmatrix}$, where u is a fix unit element of D. Then:

1. R is α-compatible.
2. R is not α-rigid.
3. $nil(R)$ is an α-compatible ideal of R.
4. R is a nil α-Armendariz ring.

(1) Suppose that $\begin{pmatrix} a & d \\ 0 & a \end{pmatrix}\begin{pmatrix} b & d_1 \\ 0 & b \end{pmatrix} = 0$, hence $ab = 0 = ad_1 + db$. So $a = 0$ or $b = 0$. In each case, $aud_1 + db = 0$, hence $\begin{pmatrix} a & d \\ 0 & a \end{pmatrix}\alpha\left(\begin{pmatrix} b & d_1 \\ 0 & b \end{pmatrix}\right) = 0$. If $\begin{pmatrix} a & d \\ 0 & a \end{pmatrix}\alpha\left(\begin{pmatrix} b & d_1 \\ 0 & b \end{pmatrix}\right) = 0$, then by a similar argument we have $\begin{pmatrix} a & d \\ 0 & a \end{pmatrix}\begin{pmatrix} b & d_1 \\ 0 & b \end{pmatrix} = 0$. Therefore R is α-compatible.

(2) If $d \neq 0$, then $\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}\alpha\left(\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}\right) = 0$, but $\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \neq 0$. Thus R is not α-rigid.

(3) Since $nil(R) = \left\{\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \mid d \in D\right\}$, hence $nil(R)$ is an α-compatible ideal of R.

(4) Suppose that $f(x) = \sum_{i=0}^{m} A_i x^i$ and $g(x) = \sum_{j=0}^{n} B_j x^j \in R[x;\alpha]$, where $A_i = \begin{pmatrix} a_i & c_i \\ 0 & a_i \end{pmatrix}$ and $B_j = \begin{pmatrix} b_j & d_j \\ 0 & b_j \end{pmatrix}$ for each $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Assume that $f(x)g(x) \in nil(R)[x]$. Then we have:

$$\sum_{k=0}^{m+n} (\sum_{i+j=k} A_i \alpha^i(B_j)) x^k \in nil(R)[x])$$

We claim that $A_i \alpha^i(B_j) \in nil(R)$ for all i, j.

(i) Suppose that there is $A_k = \begin{pmatrix} a_k & c_k \\ 0 & a_k \end{pmatrix}$ with $a_k \neq 0$ and $A_0 = \cdots = A_{k-1} = 0$ where $0 \leqslant k$. From Eq.(†), $A_0 B_k + A_1 \alpha(B_{k-1}) + \cdots + A_{k-1}\alpha^{k-1}(B_1) + A_k \alpha^k(B_0) \in nil(R)$, so $A_k \alpha^k(B_0) \in nil(R)$. That is

$$\begin{pmatrix} a_k & c_k \\ 0 & a_k \end{pmatrix} \begin{pmatrix} b_0 & u^k d_0 \\ 0 & b_0 \end{pmatrix}$$

$= \begin{pmatrix} a_k b_0 & a_k u^k d_0 + c_k b_0 \\ 0 & a_k b_0 \end{pmatrix} \in nil(R)$. Thus $a_k b_0 = 0$ and so $b_0 = 0$, since D is a domain. Then $B_0 \in nil(R)$, which implies that $A_i \alpha^i(B_0) \in nil(R)$, for each $0 \leqslant i \leqslant m$, since $nil(R)$ is an α-compatible ideal of R. Since $A_0 B_{k+1} + A_1 \alpha(B_k) + \cdots + A_k \alpha^k(B_1) + A_{k+1}\alpha^{k+1}(B_0) \in nil(R)$, we have $A_k \alpha^k(B_1) \in nil(R)$ and so $b_1 = 0$, by a similar argument as above. Then $B_1 \in nil(R)$, which implies that $A_i \alpha^i(B_1) \in nil(R)$, for each $0 \leqslant i \leqslant m$, since $nil(R)$ is an α-compatible ideal of R. Continuing this process, we obtain $B_j \in nil(R)$ for all $0 \leqslant j \leqslant n$, which implies that $A_i \alpha^i(B_j) \in nil(R)$ for all i, j.

(ii) Suppose that there is $B_k = \begin{pmatrix} b_k & d_k \\ 0 & b_k \end{pmatrix}$ with $b_k \neq 0$ and $B_0 = \cdots = B_{k-1} = 0$, where $0 \leqslant k$. By a similar way as used in (i), we can show that $A_i \in nil(R)$ for each $0 \leqslant i \leqslant m$, which implies that $A_i \alpha^i(B_j) \in nil(R)$ for all i, j, since $nil(R)$ is an ideal of R.

(iii) Suppose that $A_i = \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix}$, $B_j = \begin{pmatrix} 0 & d_j \\ 0 & 0 \end{pmatrix}$ for all i, j.

Then
$A_i\alpha^i(B_j) = \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & u^i d_j \\ 0 & 0 \end{pmatrix} = 0 \in nil(R)$ for all i, j. Therefore R is a nil α-Armendariz ring, by (i), (ii) and (iii).
In this paper, we prove that if R is a nil α-Armendariz ring and $\alpha^t = I_R$, then the set of nilpotent elements of R is an α-compatible subrng of R. Also, it is shown that if R is an α-Armendariz ring and $\alpha^t = I_R$, then R is nil α-Armendariz. Some examples of nil α-Armendariz rings which are'nt α-Armendariz are given. Moreover, we show that if $\alpha^t = I_R$ for some positive integer t and R is a nil α-Armendariz ring and $nil(R[x][y;\alpha]) = nil(R[x])[y]$, then $R[x]$ is nil α-Armendariz. Some results of ([3]) follow as consequences of our results.

2. Polynomial Rings Over Nil α-Armendariz Rings

Recall that an ideal I of a ring R is called an *α-ideal* if $\alpha(I) \subseteq I$ (see [14, Page 47]). Clearly, if I is an α-ideal of R, then $\overline{\alpha} : R/I \to R/I$ defined by $\overline{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring R/I. Note that each α-compatible ideal is α-ideal, by [6, Proposition 2.1].
Note that the set of nilpotent elements of a ring is not ideal in general, (see [18, 3]). According to ([5]), a ring R is called *semi-commutative* if $ab = 0$ implies $aRb = 0$. If R is a semi-commutative ring, then $nil(R)$ is an ideal of R, by ([8, Lemma 2.10]). Also, Example 1.2, shows that there exists a ring R and an endomorphism α on R such that $nil(R)$ is an α-compatible ideal of R.

Proposition 2.1. *Let R be a ring such that $nil(R)$ is an α-compatible ideal of R. If $f(x), g(x) \in R[x;\alpha]$ satisfy $f(x)g(x) \in nil(R)[x]$, then $ab \in nil(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$.*

Proof. Observe that $R/nil(R)$ is reduced. Then, since $nil(R)$ is an α-compatible ideal of R, hence $R/nil(R)$ is an $\overline{\alpha}$-rigid ring, by [6]. Suppose that $f(x)g(x) \in nil(R)[x]$. If we denote by $\overline{f}(x), \overline{g}(x)$ the corresponding polynomials in $R/nil(R)[x;\overline{\alpha}]$, then $\overline{f}(x)\overline{g}(x) = \overline{0}$. Since $R/nil(R)$ is

$\overline{\alpha}$-rigid, $\overline{a}\overline{b} = \overline{0}$ for all $\overline{a} \in coef(\overline{f}(x))$ and $\overline{b} \in coef(\overline{g}(x))$, by [9]. Hence ab is nilpotent for all $a \in coef(f(x))$ and $b \in coef(g(x))$.
Observe that if $nil(R)$ is an α-compatible ideal of R, then by Proposition 2.1, R is nil α-Armendariz. More generally we obtain the following. □

Proposition 2.2. *Let α be an endomorphism of a ring R and I an α-compatible nil ideal of R. Then R is nil α-Armendariz if and only if R/I is nil $\overline{\alpha}$-Armendariz.*

Proof. We denote $\overline{R} = R/I$. Since I is nil, then $nil(\overline{R}) = \overline{nil(R)}$. Hence $f(x)g(x) \in nil(x)[x]$ if and only if $\overline{f}(x)\overline{g}(x) \in nil(\overline{R})[x]$, where $\overline{f}(x), \overline{g}(x) \in R/I[x; \overline{\alpha}]$. And, if $a \in coef(f(x)$ and $b \in coef(g(x))$, then $ab \in nil(R)$ if and only if $\overline{a}\overline{b} \in nil(\overline{R})$. Therefore R is nil α-Armendariz if and only if $\overline{R}$ is nil $\overline{\alpha}$-Armendariz. □

Lemma 2.3. *Let R be a nil α-Armendariz ring and $n \geqslant 2$. If $f_1(x), f_2(x), \cdots, f_n(x) \in R[x;\alpha]$ such that $f_1(x)f_2(x)\cdots f_n(x) \in nil(R)[x]$, then if $a_k \in coef(f_k(x))$ for $k = 1, \cdots, n$, we have $a_1a_2\cdots a_n \in nil(R)$.*

Proof. We use induction on n. The case $n = 2$ is clear by definition of nil α-Armendariz ring. Suppose that $n > 2$. Consider $h(x) = f_2(x)\cdots f_n(x)$. Then $f_1(x)h(x) \in nil(R)[x]$ and hence, since R is nil α-Armendariz, $a_1a_h \in nil(R)$ where $a_h \in coef(h(x))$ and $a_1 \in coef(f_1(x))$. Therefore, for all $a_1 \in coef(f_1(x))$, $(a_1f_2(x))(f_3(x)\cdots f_n(x)) = a_1h(x) \in nil(R)[x]$, and by induction, since the coefficients of $a_1f_2(x)$ are a_1a_2 where a_2 is a coefficient of $f_2(x)$, we obtain
$a_1a_2\cdots a_{n-1}a_n \in nil(R)$ for $a_k \in coef(f_k(x))$, $k = 1, \cdots, n$. □

Proposition 2.4. *Let R be a nil α-Armendariz ring. For $a, b \in R$, we have the following:*

1. *If $ab \in nil(R)$, then $\alpha^n(a)b$, $a\alpha^n(b)$ are nilpotent for any positive integer n.*
2. *If $\alpha^n(a)b \in nil(R)$ or $a\alpha^n(b) \in nil(R)$ for some positive integer n, then $ab \in nil(R)$.*
3. *$nil(R)$ is an α-compatible subset of R.*

Proof.

(1) Suppose that $ab \in nil(R)$. It is enough to show that $\alpha(a)b \in nil(R)$. Let $p = \alpha(a)x$ and $q = bx$ in $R[x;\alpha]$. Then $pq = \alpha(a)\alpha(b)x^2 = \alpha(ab)x^2 \in nil(R)[x]$. Since R is nil α-Armendariz, $\alpha(a)b \in nil(R)$. Since $ab \in nil(R)$, we have $ba \in nil(R)$. By a similar argument one can show that $\alpha(b)a \in nil(R)$, and hence $a\alpha(b) \in nil(R)$.

(2) Suppose that $a\alpha^n(b) \in nil(R)$, for some positive integer n. Let $p = ax^n$ and $q = bx$ in $R[x;\alpha]$. Then $pq = a\alpha^n(b)x^{n+1} \in nil(R)[x]$ and thus $ab \in nil(R)$, since R is nil α-Armendariz.

(3) It follows from (1) and (2). □

Theorem 2.5. *Let R be a nil α-Armendariz ring and $\alpha^t = I_R$, for some $t \geqslant 1$. Then we have the following:*

1. *$nil(R)$ is an α-compatible subrng of R.*
2. *R is an α-compatible ring.*

Proof.

(1) The idea of the proof comes from the proof of [3, Theorem 12].

(a) Suppose that a, b are nilpotent and $b^m = 0$. Then, since $\alpha^t = I_R$,

$$(a - abx^t)(1 + bx^t + b^2x^{2t} + \cdots + b^{m-1}x^{t(m-1)}) = a \in nil(R)[x].$$

Since R is nil α-Armendariz, $ab \in nil(R)$.

(b) Suppose a, b, c are nilpotent and $a^n = b^m = 0$. Then $(1+ax^t+\cdots+a^{(n-1)}x^{(n-1)t})(1-ax^t)(1-bx^t)(1+bx^t+\cdots+b^{(m-1)}x^{(m-1)t})c = c \in nil(R)[x]$. Hence $(1+ax^t+\cdots+a^{(n-1)}x^{(n-1)t})(1-(a+b)x^t+abx^{2t})(1+bx^t + \cdots + b^{(m-1)}x^{(m-1)t})c = c \in nil(R)[x]$. Now, since R is nil α-Armendariz, by Lemma 2.3, we can choose the appropriate coefficients from each polynomial to obtain $(a + b)c \in nil(R)$. Similarly we see that $c(a + b) \in nil(R)$.

(c) Suppose a, b, c are nilpotent. Then bc and $b(a+bc)$ are nilpotent. Hence $(1-bx^t)(c+(a+bc)x^t) = c+ax^t - b(a+bc)x^{2t} \in nil(R)[x]$. Now, since R is nil α-Armendariz, $1.(a + bc) = a + bc$ is nilpotent.

(d) Suppose that a, b are nilpotent. Now by applying (c) several times we can see that, since a^2, a and $-b$ are nilpotent, $a^2 - ab$ is nilpotent;

hence $a^2 - ab - ba$ is nilpotent; hence $a^2 - ab - ba + b^2$ is nilpotent. Therefore $(a-b)^2$ is nilpotent, which means that $a-b$ is nilpotent. By using (a), (b), (c) and (d) we have $nil(R)$ is a subrng of R.
(2) Suppose $ab = 0$. Let $f(x) = \alpha(a)x$ and $g(x) = bx$ in $R[x;\alpha]$. Then $f(x)g(x) = \alpha(a)\alpha(b)x^2 = \alpha(ab)x^2 = 0$. Since R is α-Armendariz, $\alpha(a)b = 0$. By using induction on m one can show that $\alpha^m(a)b = 0$.
Now, since $ab = 0$, we have $\alpha(a)b = 0$, and hence $a\alpha^{t-1}(b) = \alpha^t(a)\alpha^{t-1}(b) = \alpha^{t-1}(\alpha(a)b) = 0$. Then $\alpha^{t-2}(a)\alpha^{t-1}(b) = 0$, and so $a\alpha(b) = 0$, since α is monomorphism.
Suppose $a\alpha(b) = 0$. Then $\alpha(a)\alpha(b) = 0$, by the previous paragraph. Hence $ab = 0$, since α is monomorphism. Therefore R is α-compatible. $\square$

Lemma 2.6. *Let R be an α-Armendariz ring and $\alpha^t = I_R$ for some $t \geqslant 1$. Then $nil(R)[x] \subseteq nil(R[x;\alpha])$.*

Proof. Suppose that R is an α-Armendariz ring. Let $f = a_0 + a_1x + \cdots + a_nx^n \in nil(R)[x]$ and $k > 1$ such that $a_i^k = 0$ for all $i = 0, 1, \cdots, n$. We show that $f(x)^{(n+1)k} = 0$. The coefficients of $f(x)^{(n+1)k}$ can be written as sums of monomials of length $(n+1)k$ in $\alpha^j(a_i)$'s, where $j \geqslant 0$ and $i = 0, 1, \cdots, n$. Consider one of these monomials $\alpha^{j_1}(a_{i_1})\alpha^{j_2}(a_{i_2})\cdots\alpha^{j_{(n+1)k}}(a_{i_{(n+1)k}})$ where $0 \leqslant i_s \leqslant n$ and $j_s \geqslant 0$. Clearly there exists $\alpha^{j_{s_1}}(a_{i_{s_1}}), \cdots, \alpha^{j_{s_k}}(a_{i_{s_k}})$ where $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_k$ such that $a_{i_{s_1}} = a_{i_{s_2}} = \cdots = a_{i_{s_k}} = a_{j_0}$ for some $0 \leqslant j_0 \leqslant n$. Since $(a_{j_0})^k = 0$, hence $\alpha^{j_{s_1}}(a_{j_0})\alpha^{j_{s_2}}(a_{j_0})\cdots\alpha^{j_{s_k}}(a_{j_0}) = 0$, by Theorem 2.5. For $i_{r_m} \neq i_s$, let $f'_{i_{r_m}} = 1 - a_{i_{r_m}}x^t$ and $f''_{i_{r_m}} = 1 + a_{i_{r_m}}x^t + \cdots + a_{i_{r_m}}^{k-1}x^{t(k-1)}$. Since $\alpha^t = I_R$, we have $f'_{i_{r_m}}f''_{i_{r_m}} = 1$ and observe that $a_{i_{r_m}}$ is a product of coefficients of $f'_{i_{r_m}}$ and $f''_{i_{r_m}}$. Now we can write the monomial as $\alpha^{j_1}(a_{i_1})\cdots\alpha^{j_{s_1-1}}(a_{i_{s_1-1}})\alpha^{j_{s_1}}(a_{j_0})\alpha^{j_{s_1+1}}(a_{i_{s_1+1}})\cdots$
$\alpha^{j_{s_2-1}}(a_{i_{s_2-1}})\alpha^{j_{s_2}}(a_{j_0})\alpha^{j_{s_2+1}}(a_{i_{s_2+1}})\cdots\alpha^{j_{(n+1)k}}(a_{i_{(n+1)k}})$. By replacing each $\alpha^{j_{r_m}}(a_{i_{r_m}})$ by the product $f'_{i_{r_m}}(x)f''_{i_{r_m}}(x)$, and since
$\alpha^{j_{s_1}}(a_{j_0})\alpha^{j_{s_2}}(a_{j_0})\cdots\alpha^{j_{s_k}}(a_{j_0}) = 0$, we have that
$$f'_{i_1}(x)f''_{i_1}(x)\cdots f'_{i_{s_1-1}}(x)f''_{i_{s_1-1}}(x)\alpha^{j_{s_1}}(a_{j_0})f'_{i_{s_1+1}}(x)f''_{i_{s_1+1}}(x)\cdots$$
$$f'_{i_{s_k-1}}(x)f''_{i_{s_k-1}}(x)\alpha^{j_{s_k}}(a_{j_0})f'_{i_{s_k+1}}(x)f''_{i_{s_k+1}}(x)\cdots f'_{i_{(n+1)k}}(x)f''_{i_{(n+1)k}}(x) = 0.$$
Now, since R is α-Armendariz, by Lemma 2.3, we can choose a coefficient from each of the polynomials in the last equality and the product will be 0. Hence

$a_{i_1}a_{i_2}\cdots a_{i_{s_1-1}}\alpha^{j_{s_1}}(a_{j_0})a_{i_{s_1+1}}\cdots a_{i_{s_k-1}}\alpha^{j_{s_k}}(a_{j_0})a_{i_{s_k+1}}\cdots a_{i_{(n+1)k}}=0$. Thus $\alpha^{j_1}(a_{i_1})\alpha^{j_2}(a_{i_2})\cdots\alpha^{j_{s_1-1}}(a_{i_{s_1-1}})\alpha^{j_{s_1}}(a_{s_1})\alpha^{j_{s_1+1}}(a_{i_{s_1+1}})\cdots$ $\alpha^{j_{s_k-1}}(a_{i_{s_k-1}})\alpha^{j_{s_k}}(a_{s_k})\alpha^{j_{s_k+1}}(a_{i_{s_k+1}})\cdots\alpha^{j_{(n+1)k}}(a_{i_{(n+1)k}})=0$, since R is α-compatible and $a_{i_{s_1}}=a_{i_{s_2}}=\cdots=a_{i_{s_k}}=a_{j_0}$. Therefore, we have proved that all the monomials appearing in the coefficients of $f(x)^{(n+1)k}$ are 0. Hence $f(x)\in nil(R[x;\alpha])$. □

Proposition 2.7. *If R is an α-Armendariz ring and $\alpha^t=I_R$ for some $t\geqslant 1$, then R is nil α-Armendariz.*

Proof. Suppose that $f(x),g(x)\in R[x;\alpha]$ such that $f(x)g(x)\in nil(R)[x]$. By Lemma 2.6, $f(x)g(x)$ is nilpotent and there exists $k\geqslant 1$ such that $(f(x)g(x))^k=0$. Hence, since R is α-Armendariz, for all $a\in coef(f(x)$ and $b\in coef(g(x))$, by choosing the corresponding coefficient in each polynomial, we have $abab\cdots ab=0$ and thus $ab\in nil(R)$. Therefore R is nil α-Armendariz. □

Corollary 2.8. [3, Proposition 2.7] *If R is an Armendariz ring, then R is nil-Armendariz.*

Proof. It follows from Proposition 2.7, whenever $\alpha=id_R$.
The following examples show that there exists a ring R with an automorphism α such that R is nil α-Armendariz but not α-Armendariz. □

Example 2.9. Let $R=\begin{bmatrix} F & F \\ 0 & F\end{bmatrix}$, where F is a filed and an endomorphism of R defined by $\alpha(\begin{bmatrix} a & b \\ 0 & c\end{bmatrix})=\begin{bmatrix} a & -b \\ 0 & c\end{bmatrix}$. By [10, Example 1.12] R is not α-armendariz. We claim that R is nil α-Armendariz. Clearly $nil(R)=\begin{bmatrix} 0 & F \\ 0 & 0\end{bmatrix}$ is an ideal of R. Now we show that $nil(R)$ is α-compatible. Let $A=\begin{bmatrix} a & b \\ 0 & c\end{bmatrix}$ and $B=\begin{bmatrix} a' & b' \\ 0 & c'\end{bmatrix}\in R$ such that $AB\in nil(R)$. Then $aa'=0=cc'$, since F is a filed. Hence $a'=c'=0$ or $a=c'=0$ or $a=c=0$ or $a'=c=0$. Let $a'=c=0$. Then $A\alpha(B)=\begin{bmatrix} 0 & -ab'+bc' \\ 0 & 0\end{bmatrix}\in nil(R)$. In each other cases, by a similar

argument one can show that $A\alpha(B) \in nil(R)$.
Now assume that $A\alpha(B) \in nil(R)$. Then by a similar argument as above one can show that $AB \in nil(R)$. Thus $nil(R)$ is an α-compatible ideal of R, and hence by Proposition 2.1, R is nil α-Armendariz.

Example 2.10. Let $\mathbb{Z}$ be the set of all integers. Consider the ring $R = \left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} | a - b \equiv c \equiv 0 \bmod (2) \text{ and } a, b, c \in \mathbb{Z} \right\}$. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha(\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}) = \begin{bmatrix} a & -c \\ 0 & b \end{bmatrix}$. Then R is not α-Armendariz. For, $p = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x$ and $q = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x \in R[x;\alpha]$, we have $pq = 0$, but $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \neq 0$. Since $nil(R) = \left\{ \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} | c \in 2\mathbb{Z} \right\}$ is an α-compatible ideal of R, hence by Proposition 2.1, R is nil α-Armendariz.
Example 2.11. shows that there exists a nil α-Armendariz ring R such that $\alpha(e) \neq e$ for some $e^2 = e \in R$. For example $e = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ is an idempotent of R and $\alpha(e) \neq e$. Recall that a ring R is called *abelian*, if each idempotent of R is central.

Proposition 2.11. *Let R be an abelian ring with $\alpha(e) = e$ for any $e = e^2 \in R$. Then the following statements are equivalent:*

1. *R is nil α-Armendariz;*
2. *eR and $(1-e)R$ are nil α-Armendariz for any $e = e^2 \in R$;*
3. *eR and $(1-e)R$ are nil α-Armendariz for some $e = e^2 \in R$.*

Proof. It is enough to show (3) $\Rightarrow$ (1). Let $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ with $pq \in nil(R)[x]$. Then $(ep)(eq) \in nil(eR)[x]$ and $((1-e)p)((1-e)q) \in nil((1-e)R)[x]$ for some $e = e^2 \in R$ by hypothesis. Since eR and $(1-e)R$ are nil α-Armendariz, we have $ea_ib_j \in nil(eR)$ and $(1-e)a_ib_j \in nil(1-e)R$, for all $0 \leqslant i \leqslant m$ and $0 \leqslant$

$j \leqslant n$. Let $k \geqslant 1$, such that $(ea_ib_j)^k = 0 = ((1-e)a_ib_j)^k$. Then $(a_ib_j)^k = ((ea_ib_j) + (1-e)a_ib_j)^k = (ea_ib_j)^k + ((1-e)a_ib_j)^k = 0$, since $(ea_ib_j)((1-e)a_ib_j) = 0 = ((1-e)a_ib_j)(ea_ib_j)$. Therefore R is nil α-Armendariz. $\square$

Lemma 2.12. *If R is a nil α-Armendariz ring and $\alpha^t = I_R$, for some $t \geqslant 1$, then $nil(R[x;\alpha]) \subseteq nil(R)[x]$.*

Proof. Suppose that $f(x) \in nil(R[x;\alpha])$ and $f(x)^m = 0$ for some $m \geqslant 1$. By Lemma 2.3, we have $a_1 \cdots a_m \in nil(R)$ where $a_i \in coef(f(x))$ for $i = 1, \cdots, m$. In particular, for every $a \in coef(f(x))$, a^m is nilpotent. Therefore $a \in nil(R)$ for all $a_i \in coef(f(x))$ and hence $f(x) \in nil(R)[x]$. $\square$

Proposition 2.13. *Let R be a nil ring. Then R is nil α-Armendariz for each endomorphism α over R.*

Proof. Since $nil(R) = R$, hence $a\alpha(b) \in nil(R)$, for each $a, b \in R$.

Smoktunowicz [18] proved that for each countable filed K there is a nil algebra R over K (generated by three elements), such that polynomial algebra $R[x]$ over R is not nil. In Lemma 2.13 we have seen the other inclusion for α-Armendariz rings which $\alpha^t = I_R$, hence we have proved: $\square$

Corollary 2.14. *If R is an α-Armendariz ring and $\alpha^t = I_R$, for some $t \geqslant 1$, then $nil(R[x;\alpha]) = nil(R)[x]$.*

Corollary 2.15. [3, Corollary 5.2] *If R is an Armendariz ring, then $nil(R[x]) = nil(R)[x]$.*

Theorem 2.16. *Let R be a nil α-Armendariz ring and $\alpha^t = I_R$, for some $t \geqslant 1$. Then $R[x;\alpha]$ is nil-Armendariz if and only if $nil(R[x;\alpha]) = nil(R)[x]$.*

Proof. If $R[x;\alpha]$ is nil-Armendariz, by Theorem 2.5, we have that $nil(R[x;\alpha])$ is a subrng of $R[x;\alpha]$. Let $a \in nil(R)$. Since $nil(R)$ is an α-compatible subrng of R, we have that $a\alpha(a) \cdots \alpha^{t-1}(a) \in nil(R)$. If $(a\alpha(a) \cdots \alpha^{t-1}(a))^s = 0$, then since $\alpha^t = I_R$, we have $(ax)^{st} =$

$(a\alpha(a)\cdots\alpha^{t-1}(a))^{st}x^{st}=0$. By a similar argument one can show that ax^r is nilpotent for any $r\geqslant 2$. Hence $nil(R)[x]\subseteq nil(R[x;\alpha])$. Now, since R is nil α-Armendariz, by Lemma 2.12, we have the other inclusion. Hence $nil(R[x;\alpha])=nil(R)[x]$.

Now suppose that $nil(R[x;\alpha])=nil(R)[x]$. Let $f(y),g(y)\in R[x;\alpha][y]$ such that $f(y)g(y)\in nil(R[x;\alpha])[y]$. Also, let $f(y)=f_0(x)+f_1(x)y+\cdots+f_m(x)y^m$ where $f_i(x)=\sum_{k=0}^{s_i} f_{i_k}x^k$ and $g(y)=g_0(x)+g_1(x)y+\cdots+g_n(x)y^n$ where $g_j(x)=\sum_{\ell=0}^{t_j} g_{j_\ell}x^\ell$, and $M>max\{deg(f_i(x)),deg(g_j(x))\}$ for any $0\leqslant i\leqslant m$ and $0\leqslant j\leqslant n$, where the degree is as polynomials in $R[x]$ and the degree of zero polynomial is taken to be 0. Let $f(x^{tM})=f_0(x)+f_1(x)x^{tM}+\cdots+f_m(x)x^{tmM}$, and $g(x^{tM})=g_0(x)+g_1(x)x^{tM}+\cdots+g_n(x)x^{tnM}$ in $R[x;\alpha]$. Then the set of coefficients of $f_i(x)$'s (resp., $g_j(x)$'s) equals the set of coefficients of $f(x^{tM})$ (resp., $g(x^{tM})$). Since $f(y)g(y)\in nil(R[x;\alpha])[y]$, x^{tM} commutes with elements of R in $R[x;\alpha]$, and $nil(R[x;\alpha])=nil(R)[x]$ is a subrng of $R[x;\alpha]$, we have $f(x^{tM})g(x^{tM})\in nil(R[x;\alpha])=nil(R)[x]$. Since R is nil α-Armendariz, $f_{i_k}g_{j_\ell}\in nil(R)$ for all i,j,k,ℓ. Now since $nil(R)$ is an α-compatible subrng of R, we have $f_i(x)g_j(x)\in nil(R)[x]$. Finally, since $nil(R[x;\alpha])=nil(R)[x]$, $f_i(x)g_j(x)$ is nilpotent. □

Corollary 2.17. [3, Theorem 5.3]*Let R be a nil-Armendariz ring. Then $R[x]$ is nil-Armendariz if and only if $nil(R[x])=nil(R)[x]$.*

Proof. It follows from Theorem 2.17, wheneve $\alpha=id_R$. Recall that if α is an endomorphism of a ring R, then the map α can be extended to an endomorphism of the polynomial ring $R[x]$ defined by $\sum_{i=0}^m a_ix^i\mapsto\sum_{i=0}^m\alpha(a_i)x^i$. We shall also denote the extended map $R[x]\to R[x]$ by α and the image of $f\in R[x]$ by $\alpha(f)$. □

Theorem 2.18. *Let α be an endomorphism of a ring R and $\alpha^t=I_R$ for some positive integer t. If R is a nil α-Armendariz ring and $nil(R[x][y;\alpha])=nil(R[x])[y]$, then $R[x]$ is nil α-Armendariz.*

Proof. Let $f(y),g(y)\in R[x][y;\alpha]$ such that $f(y)g(y)\in nil(R[x])[y]$. Let $f(y)=f_0(x)+f_1(x)y+\cdots+f_m(x)y^m$ where $f_i(x)=\sum_{k=0}^{s_i} f_{i_k}x^k$ and $g(y)=g_0(x)+g_1(x)y+\cdots+g_n(x)y^n$ where $g_j=\sum_{\ell=0}^{t_j} g_{j_\ell}x^\ell$. Then

$h_0(x)=f_0(x)g_0(x)\in nil(R[x])$,

$h_1(x) = f_0(x)g_1(x) + f_1(x)\alpha(g_0(x)) \in nil(R[x])$,
$h_2(x) = f_0(x)g_2(x) + f_1(x)\alpha(g_1(x)) + f_2(x)\alpha^2(g_0(x)) \in nil(R[x])$,

$$\vdots$$

$h_{m+n}(x) = f_m(x)\alpha^m(g_n(x)) \in nil(R[x])$.

Hence

$h_0(x^t) = f_0(x^t)g_0(x^t) \in nil(R[x])$,
$h_1(x^t) = f_0(x^t)g_1(x^t) + f_1(x^t)\alpha(g_0(x^t)) \in nil(R[x])$,
$h_2(x^t) = f_0(x^t)g_2(x^t)+f_1(x^t)\alpha(g_1(x^t))+f_2(x^t)\alpha^2(g_0(x^t)) \in nil(R[x])$,

$$\vdots$$

$h_{m+n}(x^t) = f_m(x^t)\alpha^m(g_n(x^t)) \in nil(R[x])$.

Thus

$$(f_0(x^t) + f_1(x^t)y + f_2(x^t)y^2 + \cdots + f_m(x^t)y^m)(g_0(x^t) + g_1(x^t)y + g_2(x^t)y^2 + \cdots + g_n(x^t)y^n) \in nil(R[x])[y].$$

Let $M > max\{ts_i, tt_j\}_{i,j}$, $f(x^{Mt+1}) = f_0(x^t) + f_1(x^t)x^{Mt+1} + \cdots + f_m(x^t)x^{(Mt+1)m}$ and $g(x^{Mt+1}) = g_0(x^t)+g_1(x^t)x^{Mt+1}+\cdots+g_n(x^t)x^{(Mt+1)n}$ in $R[x]$. Then the set of coefficients of the f_i's (resp., g_j's) equals the set of coefficients of $f(x^{Mt+1})$ (resp., $g(x^{Mt+1})$). Since $\alpha^t = I_R$, the set of coefficients of the h_i's equals the set of coefficients of $f(x^{Mt+1})g(x^{Mt+1})$ in $R[x;\alpha]$. Also, since $nil(R[x;\alpha]) = nil(R)[x]$, $f(x^{Mt+1})g(x^{Mt+1}) \in nil(R)[x]$. Since R is nil α-Armendariz, $f_{i_k}g_{j_\ell} \in nil(R)$. Now, since $nil(R)$ is a subring of R, $\alpha^t = I_R$ and $nil(R[x;\alpha]) = nil(R)[x]$, we have that $f_i(x^t)g_j(x^t) \in nil(R[x;\alpha])$ and so $f_i(x^t)g_j(x^t)$ is nilpotent, for each i, j. Therefore in $R[x]$, $f_i(x)g_j(x)$ is nilpotent, for each i, j. □

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