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Lineability of Space of Quasi-Everywhere Surjective Functions

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Abstract. We want to show the lineability of space of quasi-everywhere surjective functions, i.e. we want to show that the space contains a vector space with infinite dimension.

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1. Introduction

This paper is a contribution to the very new area of research in mathematical analysis, that is to search for large algebraic structures (linear spaces or algebras) in space of functions that are enjoying a special property. It has become a usual notation to call a subset M of a topological vector space X, μ-lineable (respectively, μ-spaceable) if $M \cup \{0\}$ contains a linear vector space (respectively, closed vector space) of dimension μ. If M contains an infinite-dimensional (closed) vector space, then M will be shortly called lineable (spaceable). Lineability, spaceability and algebrability are called pathological properties. These properties was introduced by Gurariy in early 2000's and were studied in [4] for the first time.
The origin of lineability and spaceability is due to Gurariy ([18], [19]) that showed that there exists an infinite dimensional linear space such

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that every non-zero element of which is a continuous nowhere differentiable function on $\mathcal{C}[0;1]$. Many examples of vector spaces of functions on $\mathbb{R}$ or $\mathbb{C}$ enjoying certain special properties have been constructed in the recent years. More recently, many authors got interested in this subject and gave a wide range of examples. For instance, in [4] it was shown that the set of everywhere surjective functions in $\mathbb{R}$ is $2^{\mathfrak{c}}$-lineable (where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$) and that the set of differentiable functions on $\mathbb{R}$ which are nowhere monotone, i. e. there is no non-trivial interval on which the function is monotone, is lineable in $\mathcal{C}(\mathbb{R})$. These behaviors occur, sometimes, in particularly interesting ways. For example, in [20], Hencl showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0;1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Holder, i. e., there is no set with non-empty interior on which the following equation is hold

$$|f(x) - f(y)| \leqslant c|x - y|^{\alpha},$$

where $x, y \in [0,1]$ and c, α are positive real numbers. We refer the interested reader to [2, 3, 7-14, 16, 17] for a wider range of results in this topic of lineability and spaceability.

Of course, one could go further and not just consider linear spaces but, instead, larger or more complex structures. For instance, in [1] the authors showed that there exists an uncountably generated algebra every non-zero element of which is an everywhere surjective function on $\mathbb{C}$ and in [5] it was shown that, if $E \subset \mathbb{T}$, the unit circle, is a set of measure zero, and if $\mathcal{F}(\mathbb{T})$ denotes the subset of $\mathcal{C}(\mathbb{T})$ of continuous functions whose Fourier series expansion diverges at every point of E, then $\mathcal{F}(\mathbb{T})$ contains an infinitely generated and dense subalgebra. One of the newest result in this area ([8]) proves the existence of uncountably generated algebras inside the following sets of special functions: Sierpinski-Zygmund functions, perfectly everywhere surjective functions, and nowhere continuous Darboux functions. That a space contains an infinitely generated algebra is called algebrability. It is clear that algebrability implies lineability but studying the algebrability of a space is sometimes far harder than lineability.

The notation of everywhere surjective functions that was mentioned earlier was first introduced by Lebesgue ([21]) in 1904 by showing the existence of a function $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f((a,b)) = \mathbb{R},$$

for every non-trivial interval (a, b). Space of everywhere surjective functions on $\mathbb{R}$ will be denoted by $\mathcal{ES}$ in this paper. These sort of functions was not taken in to account until recently that Aron and Seoane-Sepulveda [6] has investigated the algebraic structure contained in the space of these functions. This trend of research was continued in [15] that they defined perfectly everywhere surjective and strongly everywhere surjective functions and showed some pathological properties of space of such functions.
Our main concept in this paper is to expand the theory of everywhere surjective functions by defining quasi-everywhere surjective functions and investigating the pathological properties of those spaces.

2. Quasi-Everywhere Surjective Functions

This section is devoted to defining quasi-everywhere surjective functions and studying their pathological properties.

Definition 2.1. *Let X and Y be two topological vector spaces. A function $f : X \to Y$ is called quasi-everywhere surjective if $f(U)$ is dense in Y for every open subset U of X. We will show the collection of all quasi-everywhere surjective functions by $\mathcal{QES}(X, Y)$ and if $X = Y$, it will be shown by $\mathcal{QES}(X)$. We use the notation $\mathcal{QES}$, in a spacial case that $X = Y = \mathbb{R}$.*
Here we provide the reader with an example of a quasi-everywhere surjective function that is not everywhere surjective. The idea of the following example is partially inspired from [6, 15]. By a Cantor like set in the next example we mean a subset of $\mathbb{R}$ that is isomorphic to the ternary Cantor set. These sets are obviously nowhere dense and have cardinality $\mathfrak{c}$.

Example 2.2. A function $f : \mathbb{R} \to \mathbb{R}$ that is quasi-everywhere surjective but not everywhere surjective.

To construct this function let $\{I_n\}$ be the sequence of all open intervals in $\mathbb{R}$ with rational end points. We create a collection of mutually disjoint uncountable nowhere dense sets by induction. Let C_1 be Cantor like subset of I_1. Since C_1 is nowhere dense we can take C_2 in $I_2 \setminus C_1$ to be a Cantor like set. Now we can continue this way to choose C_n to be a Cantor like subset of $I_n \setminus \{C_1 \cup C_2 \cup \ldots \cup C_{n-1}\}$.
Now take h_n to be any bijection between C_n and $\mathbb{R} \setminus \bigcup C_n$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} h_n(x) & if\ x \in C_n, \\ 0 & otherwise. \end{cases}$$

It is clear that for any open subset $U \subset \mathbb{R}$, $f(U)$ is equal to $\mathbb{R} \setminus \bigcup C_n$ which is dense in $\mathbb{R}$ by Bair's Theorem.

Remark 2.3. *We have used the fact that if X is a nowhere dense subset of $\mathbb{R}$ then its complement is uncountable. This can be easily proved by Bair's theorem as well.*
The following lemmas help us creating an infinite dimensional vector space contained in the space of all quasi-everywhere surjective functions that are not everywhere surjective.

Lemma 2.4. *If $f : X \to Y$ is a quasi-everywhere surjective function and $g : Y \to Z$ is a continuous surjective function, then $g \circ f$ is quasi-everywhere surjective.*

Proof. Let U be an open subset of X. By quasi-everywhere surjectiveness of f, $\overline{f(U)} = Y$. Since g is surjective, we have $g(\overline{f(U)}) = Z$. But continuity of g implies that $g(\overline{f(U)}) \subseteq \overline{g(f(U))}$. This completes the proof. □

Lemma 2.5. *There exists a vector space Λ of functions $\mathbb{R} \to \mathbb{R}$ with the following properties.*
(i) Every non-zero element of Λ takes dense sets of $\mathbb{R}$ to dense subsets.
(ii) dim(Λ)= $\aleph_\circ$.

Proof. Let Λ be the span of $\{x^n : n \in \mathbb{N}, n \text{ is odd}\}$. It is clear that each element of Λ is an odd order polynomial. Every such polynomial is surjective and continuous. Take $p \in \Lambda$, since $p(\bar{U}) \subset p(\bar{U})$, p takes dense subsets of $\mathbb{R}$ to dense subsets and (i) holds. That $\dim(\Lambda) = \aleph_\circ$ is clear. $\square$

Theorem 2.6. *Space of all quasi-everywhere surjective functions on* $\mathbb{R}$ *that are not everywhere surjective is lineable, i.e.,* $(\mathcal{QES} \setminus \mathcal{ES}) \bigcup \{0\}$ *contains an infinite dimensional vector space.*

Proof. Let f be the function that was presented in the Example 2.2 and Λ be the vector space in the Lemma 2.5. Since every polynomial p in Λ is a continuous surjective function on $\mathbb{R}$, by Lemma 2.4, $p \circ f$ is a quasi-everywhere surjective function on $\mathbb{R}$. On the other hand, $Ran(f) = \mathbb{R} \setminus \bigcup C_n$. It shows that $Ran(f)$ doesn't contain any open subset of $\mathbb{R}$ since each open subset contains C_n for some n. Now if $p \in \Lambda$, p is injective on an interval $[r, \infty]$ for some large r, so $p \circ f$ is not everywhere surjective. This completes the proof. $\square$

Remark 2.7. *We can improve earlier result and show the c-lineability of* $\mathcal{QES} \setminus \mathcal{ES}$ *by taking* $\Lambda = Span\{\tan(\alpha x); \alpha \in [0, 1]\}$ *with the same proofs. This shows that* $\mathcal{QES} \setminus \mathcal{ES}$ *ia actually c-lineable. The thing that can be studied is to find the largest cardinal number* μ *that* $\mathcal{QES} \setminus \mathcal{ES}$ *is* μ*-lineable. Another trend of research in this area is to study the spaceability and algebrability of this space.*

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