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Original Research Paper

σ-C^*-Dynamics of $\mathcal{K}(H)$

M. Mosadeq*
Larestan Branch, Islamic Azad University

Abstract. Let σ be a linear $*$-endomorphism on a C^*-algebra A so that $\sigma(A)$ acts on a Hilbert space H which including $\mathcal{K}(H)$ and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-C^*-dynamical system on A with the generator δ. In this paper, we demonstrate some conditions under which $\{\alpha_t\}_{t\in\mathbb{R}}$ is implemented by a C_0-groups of unitaries on H. In particular, we prove that for a rank-one projection $p \in A$, which is invariant under α_t, there is a C_0-group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries in $\mathbf{B}(H)$ such that $\alpha_t(a) = u_t\sigma(a)u_t^*$. Furthermore, introducing the concepts of σ-inner endomorphism and σ-bijective map, we prove that each σ-bijective linear endomorphism on A is a σ-inner endomorphism, where σ ia idempotent. Finally, as an application, we characterize each so-called σ-C^*-dynamical system on the concrete C^*-algebra $A := \mathbf{B}(H) \times \mathbf{B}(H)$, where H is a separable Hilbert space and σ is the linear $*$-endomorphism $\sigma(S,T) = (0,T)$ on A.

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1 Introduction

General theory of (semi) groups of linear operators which is the paradigm for modelling and studying phenomena in mathematical physics is estab-

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*Corresponding Author

lished by acting an abstract (semi) group on an arbitrary Banach space. In particular, the action of the additive group $\mathbb{R}^+ := [0, \infty)$ on a Banach space is called "one parameter semigroup". The theory of semigroups can be applied to solve a large class of problems commonly known as evolution equations. They are described by an initial value problem for a differential equation.

Let A be a Banach space and $\sigma : A \to A$ be a bounded linear operator. A one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ (resp. $\{\alpha_t\}_{t\geq 0}$) of bounded linear operators on A is called a *σ-one parameter (semi)group*, if

(i) $\alpha_0 = \sigma$;

(ii) $\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$ (resp. $t, s \geq 0$).

The σ-one parameter (semi) group $\{\alpha_t\}_{t\in\mathbb{R}}$ (resp. $\{\alpha_t\}_{t\geq 0}$) is said to be

(i) *uniformly continuous* if $\lim\limits_{t\to 0} \|\alpha_t - \sigma\| = 0$ (resp. $\lim\limits_{t\to 0^+} \|\alpha_t - \sigma\| = 0$).

(ii) *strongly continuous* if $\lim\limits_{t\to 0} \alpha_t(a) = \sigma(a)$ (resp. $\lim\limits_{t\to 0^+} \alpha_t(a) = \sigma(a)$) for each $a \in A$.

We define the *infinitesimal generator* δ of the σ-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq A \to A$ such that $\delta(a) = \lim\limits_{t\to 0} \dfrac{\alpha_t(a) - \sigma(a)}{t}$ where $D(\delta) = \{a \in A \textit{ such that } \lim\limits_{t\to 0} \dfrac{\alpha_t(a) - \sigma(a)}{t} \textit{exists}\}$.

If $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-one parameter group with the generator δ, then one can easily see that

(i) $\sigma^2 = \sigma$ and $\sigma\alpha_t = \alpha_t\sigma = \alpha_t$ for each $t \in \mathbb{R}$.

(ii) $\alpha_t(A) = \sigma(A)$ and $ker(\alpha_t) = ker(\sigma)$ for each $t \in \mathbb{R}$.

(iii) $\sigma\delta(a) = \delta\sigma(a) = \delta(a)$ for each $a \in D(\delta)$.

(iv) $\sigma(A)$ is a closed subspace of A.

As an example of σ-one parameter group, let M be a closed subspace of Hilbert space H, let $M^\perp$ be the set $\{x \in H : < x, m >=$

$0\ for\ every\ m \in M\}$, and let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a one parameter group on H. If σ is the first projection operator on M, then for $x = y+z \in M\oplus M^\perp = H$, the one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ defined by $\alpha_t(x) = \varphi_t(y)$ is a σ-one parameter group on H with the same continuity of $\{\psi_t\}_{t\in\mathbb{R}}$.

In the case that $\sigma = I_A$ (the identity operator on A), the concept of σ-one parameter (semi) group is nothing than a one parameter (semi) group in the usual sense (see [23, p. 8]). This notion was introduced by Janfada in 2008. We refer the reader to [10] for more details.

One parameter groups of bounded linear operators and their extensions are of more considerable magnitude because of their applications in the theory of dynamical systems. Such groups are applied widely to describe the dynamical systems appearing in quantum field theory and statistical mechanics [4, 5, 7, 22, 24, 25] . The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on C^*-algebras. On the other hand, the infinitesimal generator d of a C^*-dynamical system is a closed densely defined $*$-derivation, that is d is a $*$-linear map and it satisfies the Liebniz rule $d(ab) = d(a)b+ad(b)$ for all $a, b \in D(d)$. Therefore, the theory of C^*-dynamical systems concerns the theory of derivations in C^*-algebras.

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. As an idea, let σ be a linear homomorphism on an algebra A and $d : A \to A$ be a derivation. Then, the mapping $\delta : A \to A$ defined by $\delta(a) := d(\sigma(a))$ satisfies the equation $\delta(ab) = \delta(a)\sigma(b)+\sigma(a)\delta(b)$ for all $a, b \in A$. This motivates us to consider the following definition.

Let A be a $*$-Banach algebra and σ be a $*$-linear operator on A. A $*$-linear map δ from a $*$-subalgebra $D(\delta)$ of A into A is called a *σ-derivations* if $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in D(\delta)$. For instance, let σ be a linear $*$-endomorphism and h be an arbitrary self-adjoint element of A. Then, the mapping $\delta : A \to A$ defined by $\delta(a) = i[h, \sigma(a)]$, where $[h, \sigma(a)]$ is the commutator $h\sigma(a) - \sigma(a)h$, is a σ-derivation which is called *inner*. Moreover, when σ is an automorphism and $\delta : A \to A$ be a σ-derivation, we can consider $d := \delta\sigma^{-1}$ and find out that d is an ordinary derivation. Automatic continuity, innerness, approximately innerness and closability are some of important subjects which are investigated in the theory of σ-derivations (see

[8, 9, 12, 13, 18, 20] and references therein).

In each case of generalization of derivation, a noted point drawing the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamical system. Such dynamical system is usually provided by adjoining a suitable property to an extension of a uniformly (strongly) continuous one parameter groups of bounded linear operators. Some approaches to preparing new dynamical systems and their applications have been explained in [1, 14, 15, 16, 17, 19] and references therein.

In order to construct an extension of a C^*-dynamical system associated with σ-derivation, as its infinitesimal generator, note that each $*$-endomorphism on a C^*-algebra is norm decreasing. This specific property, provides the possibility that σ is considered to be a linear $*$-endomorphism and the desired extension is based on a class of σ-one parameter groups.

Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a strongly continuous σ-one parameter group of linear $*$-endomorphisms on the C^*-algebra A. An immediate consequence of the features $\alpha_t(A) = \sigma(A)$ and $ker(\alpha_t) = ker(\sigma)$ $(t \in \mathbb{R})$ is that by substituting $\sigma = I$, we obtain a classical C^*-dynamical system. In 2013, the author introduced the mentioned extension of C^*-dynamical systems and called it a σ-C^*-*dynamics*. So, this notion covers the classical C^*-dynamical systems and is compatible with the terminology of σ-derivations.

It has been proved in [14] that, the infinitesimal generator δ of the σ-C^*-dynamics $\{\alpha_t\}_{t\in\mathbb{R}}$ is a $*$-σ-derivation such that $\overline{\sigma\big(D(\delta)\big)} = \sigma(A)$.

Assume that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-C^*-dynamical system on A with the infinitesimal generator δ. Then, the one parameter family $\{\widetilde{\alpha}_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\sigma(A)$ defined by $\widetilde{\alpha}_t(\sigma(a)) = \alpha_t(a)$ is a C^*-dynamical system and the mapping $\widetilde{\delta} : \sigma\big(D(\delta)\big) \subseteq \sigma(A) \to \sigma(A)$ defined by $\widetilde{\delta}(\sigma(a)) = \delta(a)$ is its generator (see [14]).

Let σ be a $*$-linear endomorphism on the C^*-algebra A. By a σ-*inner endomorphism*, we mean a linear endomorphism $\alpha : A \to A$ such that $\alpha(a) = u\sigma(a)u^*$ for every $a \in A$ and some unitary element $u \in A$. In order to establish a σ-inner endomorphism, let h be a self-adjoint element of the C^*-algebra A. Then, the mapping $\alpha : A \to A$ given by

$\alpha(a) = e^{ih}\sigma(a)e^{-ih}$ is a σ-inner endomorphism.

Suppose that h is a self-adjoint element in A, $\sigma : A \to A$ is an idempotent linear $*$-endomorphism such that $\sigma(h) = h$. Then, it follows from [14, Theorem 3.7] that, the inner $*$-σ-derivation $\delta : A \to A$ defined by $\delta(a) = i[h, \sigma(a)]$ induces the σ-C^*-dynamical system $\varphi_t(a) = e^{ith}\sigma(a)e^{-ith}$ of $*$-σ-inner endomorphisms.

In functional analysis, an "operator algebras" is an algebra of bounded linear operators on a topological vector space X with the multiplication given by the composition of mappings. In particular, the term operator algebra is usually used in reference to algebras of bounded operators on a Banach space or, even more specially in reference to algebras of operators on a Hilbert space, endowed with the operator norm topology. Let H be a Hilbert space. It is known that the algebra $\mathbf{B}(H)$ with respect to the operator norm and the natural involution given by the Hilbert adjoint operation is a unital C^*-algebra. On the other hand, due to the Gelgand-Naimark-Segal representation, each non-commutative C^*-algebra can be regarded as a C^*-subalgebra of $\mathbf{B}(H)$, for some Hilbert space H. So, the study of C^*-dynamical systems on $\mathbf{B}(H)$ and its C^*-subalgebras has an important role to survey of C^*-dynamical systems in general. Moreover, it is one of the key ideas of quantum mechanics to use C_0-one parameter groups of unitary operators on a Hilbert space H to implement new dynamical systems on the operator algebra $\mathbf{B}(H)$ and its C^*-subalgebras (see [2], [4],[7] and [24]).

Let H be a Hilbert space. A linear map $T : H_1 \to H_2$ between Hilbert spaces H_1 and H_2 is called compact if $T\left(S(H_1)\right)$ is relatively compact in H_2 (i.e., $\overline{T\left(S(H_1)\right)}$, the norm closure of $T\left(S(H_1)\right)$, is a compact subset of H_2), where $S(H_1)$ is the closed unit ball of H_1. It is notable that for a Hilbert space H, the set $\mathcal{K}(H)$, of all compact operators on H, is a closed two sided ideal of $\mathbf{B}(H)$ which is also self-adjoint (see [21, Theorem 2.4.3]). Thus, $\mathcal{K}(H)$ is a C^*-subalgebra of $\mathbf{B}(H)$ which contains $\mathcal{F}(H)$, the set of all finite rank operators on H. Especially, $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$ (see [21, Theorem 2.4.5]).

The above considerations motivate us to investigate σ-C^*-dynamical systems on $\mathbf{B}(H)$ and its C^*-subalgebras.

Let σ be a linear $*$-endomorphism on a C^*-algebra A so that $\sigma(A)$ acts on a Hilbert space H which including $\mathcal{K}(H)$ and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-

C^*-dynamical system on A with the generator δ. In this paper, we prove that for a rank-one projection $p \in A$, which is invariant under α_t, there is a C_0-group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries in $\mathbf{B}(H)$ such that $\alpha_t(a) = u_t\sigma(a)u_t^*$. We also, demonstrate a version of perturbation theorem in the setting of σ-C^*-dynamical systems and applying it, we prove that for a rank-one projection $p \in D(\delta) \cap \sigma(A)$, there exist a self-adjoint operator h^p on H and a bounded $*$-σ-derivation δ^p such that $(\delta + \delta^p)(p) = 0$ and it generates the σ-C^*-dynamical system $\alpha_{t,p}(a) = e^{ith^p}\sigma(a)e^{-ith^p}$ on A. Furthermore, if $h^p \in A$ and $\sigma(h^p) = h^p$, then there is a self-adjoint operator $h : D(h) \subseteq H \to H$ such that for each $a \in D(\delta)\cap\mathcal{K}(H)$, $\delta(a) = i[h, \sigma(a)]$ and $\alpha_t(a) = e^{ith}\sigma(a)e^{-ith}$ on $\mathcal{K}(H)$. Introducing the concept of σ-bijective maps, we prove that each σ-bijective linear endomorphism on A is a σ-inner endomorphism, where σ ia idempotent. Finally, as an application, we characterize each so-called σ-C^*-dynamical system on the concrete C^*-algebra $A := \mathbf{B}(H) \times \mathbf{B}(H)$, where H is a separable Hilbert spaces and σ is the linear $*$-endomorphism $\sigma(S, T) = (0, T)$ on A.

The reader is referred to [3, 6] and [21] for details on Banach algebras and to [4, 25] for more information on dynamical systems.

2 σ-C^*-Dynamics on Some Special Classes of Operator Algebras

In the following theorem, we show that every C_0-group of unitary operators on a Hilbert space H can define a σ-C^*-dynamical system on $\mathcal{K}(H)$.

Theorem 2.1. *Let $\{u_t\}_{t\in\mathbb{R}}$ be a C_0-group of unitary operators on a Hilbert space H and suppose that $\sigma : \mathbf{B}(H) \to \mathbf{B}(H)$ is an idempotent linear $*$-endomorphism satisfying $\sigma(u_t) = u_t$. Then, $\alpha_t(a) = u_t\sigma(a)u_t^*$ is a σ-C^*-dynamical system on $\mathcal{K}(H)$.*

Proof. It is trivial that for each $t \in \mathbb{R}$, α_t is a homomorphism on $\mathcal{K}(H)$ such that $\alpha_0 = \sigma$. Moreover, it follows from the hypotheses $\sigma^2 = \sigma$ and $\sigma(u_t) = u_t$, that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-one parameter group. It is sufficient to prove that $\{\alpha_t\}_{t\in\mathbb{R}}$ is strongly continuous. For this aim, let $x, y \in H$, and define $x \otimes y : \mathcal{H} \longrightarrow \mathcal{H}$ by $(x \otimes y)(z) =< z, y > x$. Then, $x \otimes y \in \mathcal{K}(H)$ and

$$\begin{aligned}\|\alpha_t(x \otimes y) - \sigma(x \otimes y)\| = \|u_t\sigma(x \otimes y)u_t^* - \sigma(x \otimes y)\| \\ =&\|u_t(\sigma(x) \otimes y)u_t^* - \sigma(x) \otimes y\| \\ =&\|u_t(\sigma(x)) \otimes u_t(y) - \sigma(x) \otimes y\| \\ \leq&\|u_t(\sigma(x)) \otimes u_t(y) - \sigma(x) \otimes u_t(y)\| \\ &+ \|\sigma(x) \otimes u_t(y) - \sigma(x) \otimes y\| \\ \leq&\|(u_t(\sigma(x)) - \sigma(x)) \otimes u_t(y)\| \\ &+ \|\sigma(x) \otimes (u_t(y) - y)\| \\ \leq&\|(u_t(\sigma(x)) - \sigma(x))\|\|u_t(y)\| \\ &+ \|\sigma(x)\|\|u_t(y) - y\|.\end{aligned}$$

Since $\{u_t\}_{t\in\mathbb{R}}$ is strongly continuous, $\lim\limits_{t\to 0} \|\alpha_t(x \otimes y) - \sigma(x \otimes y)\| = 0$. This shows that $\lim\limits_{t\to 0} \|\alpha_t(a) - \sigma(a)\| = 0$ for each $a \in \mathcal{F}(H)$. Also, $\mathcal{F}(H)$ is dense in $\mathcal{K}(H)$. Hence, $\lim\limits_{t\to 0} \|\alpha_t(a) - \sigma(a)\| = 0$ for each $a \in \mathcal{K}(H)$, and therefore, $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-C^*-dynamical system on $\mathcal{K}(H)$. $\square$

We are going to establish some conditions making the converse of the above theorem be held. More precisely, we like to investigate some restrictions under which a σ-C^*-dynamical system on A can be characterized with respect to a C_0-group of unitaries on H.

Theorem 2.2. *Let σ be a linear $*$-endomorphism on a C^*-algebra A so that $\sigma(A)$ acts on a Hilbert space H which including $\mathcal{K}(H)$ and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-C^*-dynamical system on A with the generator δ. If there exists a rank one projection $p \in A$ which is invariant under α_t (i.e., $\alpha_t(p) = p$ for each $t \in \mathbb{R}$), then there is a C_0-group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries in $\mathbf{B}(H)$ such that $\alpha_t(a) = u_t\sigma(a)u_t^*$.*

Proof. Since p is invariant under α_t, $\sigma(p) = \alpha_0(p) = p$. So, $p \in \sigma(A)$. However, p is a rank-one projection. Hence, there is a unit vector $x \in H$ such that $p = x \otimes x$ (see [21, p. 55]). Using some ideas of the proof of [2, Theorem 4.1], we define, $u_t(\sigma(a)x) = \alpha_t(a)x$ for each $a \in A$. We have

$$\begin{aligned}\|\sigma(a)x\| &= \| < x, x > \sigma(a)x\| \\ &= \|(\sigma(a)x \otimes x)x\| \\ &= \|\sigma(a)(x \otimes x)x\| \\ &\leq \|\sigma(a)(x \otimes x)\|.\|x\|.\end{aligned}$$

Thus, $\|\sigma(a)x\| \leq \|\sigma(a)(x \otimes x)\|$. On the other hand,

$$\begin{aligned}\|\sigma(a)(x \otimes x)\| &= \|\sigma(a)x \otimes x\| \\ &\leq \|\sigma(a)x\|.\|x\| \\ &= \|\sigma(a)x\|.\end{aligned}$$

Hence, $\|\sigma(a)(x \otimes x)\| = \|\sigma(a)x\|$.

Considering the associated C^*-dynamics $\widetilde{\alpha}_t(\sigma(a)) = \alpha_t(a)$ on $\sigma(A)$, we obtain $\|\alpha_t(a)\| = \|\widetilde{\alpha}_t(\sigma(a))\| = \|\sigma(a)\|$. Further, applying the facts that $\alpha_t(A) = \sigma(A)$ and $ker(\alpha_t) = ker(\sigma)$, one can conclude that for each $t \in \mathbb{R}$ and $a \in A$, there exists $b \in A$ such that $\alpha_t(a) = \sigma(b)$ and $\|\alpha_t(a)x\| = \|\sigma(b)x\| = \|\sigma(b)(x \otimes x)\| = \|\alpha_t(a)(x \otimes x)\|$. Therefore,

$$\begin{aligned}\|u_t(\sigma(a)x)\| &= \|\alpha_t(a)x\| \\ &= \|\alpha_t(a)(x \otimes x)\| \\ &= \|\alpha_t(a)\alpha_t(x \otimes x)\| \\ &= \|\alpha_t(a.x \otimes x)\| \\ &= \|\sigma(a.x \otimes x)\| \\ &= \|\sigma(a).\sigma(x \otimes x)\| \\ &= \|\sigma(a).(x \otimes x)\| \\ &= \|\sigma(a).x\|.\end{aligned}$$

So, u_t is well-defined isometry on $\sigma(A)x$. Since $\sigma(A)$ includes $\mathcal{K}(H)$, we have $z \otimes x \in \sigma(A)$ for every $z \in H$. But, $z = (z \otimes x)x$. This means, that $z \in \sigma(A)x$ and hence, $\overline{[\sigma(A)x]} = H$. Consequently, u_t can be extended to a unitary on H. Let $b_0 \in A$ and $t \in \mathbb{R}$. Using the properties $\alpha_t(A) = \sigma(A)$ and $ker(\alpha_t) = ker(\sigma)$ once more, it follows that there exists $a_0 \in A$ such that $\sigma(b_0) = \alpha_t(a_0)$ and $\sigma(b_0)x = \alpha_t(a_0)x = u_t(\sigma(a_0)x)$. So,

$$u_t^*(\sigma(b_0)x) = u_t^{-1}(\sigma(b_0)x) = \sigma(a_0)x = \alpha_{-t}(\sigma(b_0))x = \alpha_{-t}(b_0)x.$$

To justify group properties of $\{u_t\}_{t\in\mathbb{R}}$, note that

$$u_0(\sigma(a)x) = \alpha_0(a)x = \sigma(a)x$$

and for each $a, b \in A$

$$\begin{aligned}
< u_s u_t(\sigma(a)x), \sigma(b)x > &=< u_t(\sigma(a)x), u_s^*(\sigma(b)x) > \\
&=< \alpha_t(a)x, \alpha_{-s}(b)x > \\
&=< \alpha_{-s}(b^*)\alpha_t(a)x, x > \\
&=< \alpha_{-s}(b^*\alpha_{t+s}(a))x, x > \\
&=< u_{-s}(\sigma(b^*\alpha_{t+s}(a)))x, x > \\
&=< u_{-s}(\sigma(b)^*\alpha_{t+s}(a))x, x > \\
&=< \sigma(b)^*\alpha_{t+s}(a)x, u_{-s}^*((x\otimes x)x) > \\
&=< \sigma(b)^*\alpha_{t+s}(a)x, u_{-s}^*(\sigma(x\otimes x)x) > \\
&=< \sigma(b)^*\alpha_{t+s}(a)x, \alpha_s(x\otimes x)x > \\
&=< \sigma(b)^*\alpha_{t+s}(a)x, (x\otimes x)x > \\
&=< \sigma(b)^*\alpha_{t+s}(a)x, x > \\
&=< \alpha_{s+t}(a)x, \sigma(b)x > \\
&=< u_{s+t}(\sigma(a)x), \sigma(b)x > .
\end{aligned}$$

Since $\overline{[\sigma(A)x]} = H$, then $u_{s+t} = u_s u_t$. Strong continuity of $\{u_t\}_{t\in\mathbb{R}}$ follows by

$$\|u_t(\sigma(a)x) - \sigma(a)x\| = \|\alpha_t(a)x - \sigma(a)x\| \quad \leq \|\alpha_t(a) - \sigma(a)\|\|x\|.$$

Therefore, $\{u_t\}_{t\in\mathbb{R}}$ is a C_0-group of unitaries on H.

Finally, for each $a, b \in A$, we have

$$\begin{aligned}
u_t\sigma(a)u_t^*(\sigma(b)x) &= u_t\sigma(a)\alpha_{-t}(b)x \\
&= u_t\sigma(a.\alpha_{-t}(b))x \\
&= \alpha_t(a.\alpha_{-t}(b))x \\
&= \alpha_t(a)(\sigma(b)x),
\end{aligned}$$

which implies that $\alpha_t(a) = u_t\sigma(a)u_t^*$. $\square$

The following result gives us a version of perturbation theorem in the setting of σ-C^*-dynamical systems.

Theorem 2.3. *Let δ_1 be the generator of a σ-C^*-dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ on A and let δ_2 be a bounded $*$-σ-derivation on A such that $\delta_2\sigma = \delta_2 = \sigma\delta_2$. Then, $\delta_1+\delta_2$ generates a σ-C^*-dynamical system on A.*

Proof. Define the operator $\widetilde{\delta_1} : \sigma\big(D(\delta_1)\big) \subseteq \sigma(A) \to \sigma(A)$ by $\widetilde{\delta_1}(\sigma(a)) = \delta_1(a)$. Then, $\widetilde{\delta_1}$ is the generator of the associated C^*-dynamical system $\{\widetilde{\alpha}_t\}_{t\in\mathbb{R}}$ on $\sigma(A)$ defined by $\widetilde{\alpha}_t(\sigma(a)) = \alpha_t(a)$. Also, the operator $\widetilde{\delta_2}$: $\sigma\big(D(\delta_2)\big) \subseteq \sigma(A) \to \sigma(A)$ defined by $\widetilde{\delta_2}(\sigma(a)) = \delta_2(a)$ is a bounded $*$-derivation on $\sigma(A)$. By perturbation theorem ([23, Theorem 3.1.1]) $\widetilde{\delta_1} + \widetilde{\delta_2}$ (resp. $-(\widetilde{\delta_1} + \widetilde{\delta_2})$) is the generator of a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ (resp. $\{S_t\}_{t\geq 0}$) on $\sigma(A)$. Then, $\widetilde{\delta_1}+\widetilde{\delta_2}$ is the generator of the strongly continuous one parameter group $\{\widetilde{\beta}_t\}_{t\in\mathbb{R}}$ defined by $\widetilde{\beta}_t := \{ \begin{matrix} T_t & t \geq 0 \\ S_{-t} & t \leq 0 \end{matrix}$ on $\sigma(A)$.

Consider the one parameter family $\{\beta_t\}_{t\in\mathbb{R}}$ defined by $\beta_t(a) = \widetilde{\beta}_t(\sigma(a))$. It is easy to check that $\{\beta_t\}_{t\in\mathbb{R}}$ is a strongly continuous σ-one parameter group on A and the $*$-σ-derivation $\delta_1+\delta_2$ is its generator. Denote $\delta_1+\delta_2$ by δ and let $a, b \in D(\delta)$. Note that β_t is actually a $*$-homomorphism.

Because for each $t \in \mathbb{R}$, one calculates that

$$\begin{aligned}
\frac{d}{dt}\beta_{-t}\Big(\beta_t(a).\beta_t(b)\Big) &= \lim_{h\to 0}\frac{\beta_{-t-h}\Big(\beta_{t+h}(a).\beta_{t+h}(b)\Big) - \beta_{-t}\Big(\beta_t(a).\beta_t(b)\Big)}{h} \\
&= \lim_{h\to 0}\frac{\beta_{-t}}{h}\{\beta_{-h}\Big(\beta_{t+h}(a).\beta_{t+h}(b)\Big) - \beta_t(a).\beta_t(b)\} \\
&= \lim_{h\to 0}\frac{\beta_{-t}}{h}\{\beta_{-h}\Big(\beta_{t+h}(a).\beta_{t+h}(b) - \beta_t(a).\beta_{t+h}(b)\Big)\} \\
&\quad + \lim_{h\to 0}\frac{\beta_{-t}}{h}\{\beta_{-h}\Big(\beta_t(a).\beta_{t+h}(b) - \beta_t(a).\beta_t(b)\Big)\} \\
&\quad + \lim_{h\to 0}\frac{\beta_{-t}}{h}\{\beta_{-h}\Big(\beta_t(a).\beta_t(b)\Big) - \beta_t(a).\beta_t(b)\} \\
&=\beta_{-t}\lim_{h\to 0}\beta_{-h}\Big(\frac{\beta_{t+h}(a) - \beta_t(a)}{h}.\beta_{t+h}(b)\Big) \\
&\quad + \beta_{-t}\lim_{h\to 0}\beta_{-h}\Big(\beta_t(a).\frac{\beta_{t+h}(b) - \beta_t(b)}{h}\Big) \\
&\quad + \beta_{-t}\lim_{h\to 0}\frac{\beta_{-h}\Big(\beta_t(a).\beta_t(b)\Big) - \beta_t(a).\beta_t(b)}{h} \\
&=\beta_{-t}\sigma[\delta(\beta_t(a)).\sigma(\beta_t(b)) + \sigma(\beta_t(a)).\delta(\beta_t(b))] \\
&\quad - \beta_{-t}\delta\Big(\beta_t(a).\beta_t(b)\Big)
\end{aligned}$$

$$\begin{aligned}
&=\beta_{-t}[\delta\Big(\beta_t(a).\beta_t(b)\Big) - \delta\Big(\beta_t(a).\beta_t(b)\Big)] \\
&=\beta_{-t}(0) \\
&=0.
\end{aligned}$$

Thus, $\beta_{-t}[\beta_t(a).\beta_t(b)] = \beta_0[\beta_0(a).\beta_0(b)] = \sigma(ab)$ and consequently, $\beta_t(a).\beta_t(b) = \beta_t(a.b)$.

Applying boundedness of β_t $(t \in \mathbb{R})$ together with the facts that $D(\delta) = D(\delta_1)$ and $\overline{\sigma\big(D(\delta_1)\big)} = \sigma(A)$, it follows that β_t $(t \in \mathbb{R})$ is a homomorphism. Further, δ and σ are $*$-linear operators and the conjugation operation is norm continuous. So, a simple calculation indicates that

$$\frac{d}{dt}\beta_{-t}\Big(\beta_t(a)^*\Big) = \beta_{-t}[\sigma\delta\Big(\beta_t(a)\Big)^* - \delta\Big(\beta_t(a)^*\Big)] = 0.$$

Thus,

$$\beta_{-t}\Big(\beta_t(a)^*\Big) = \beta_0\Big(\beta_0(a)^*\Big) = \sigma(\sigma(a)^*) = \sigma^2(a^*) = \sigma(a^*)$$

and therefore, $\beta_t(a)^* = \beta_t\Big(\sigma(a^*)\Big) = \beta_t(a^*)$.
□

Theorem 2.4. *Let σ be a linear $*$-endomorphism on a C^*-algebra A so that $\sigma(A)$ acts on a Hilbert space H which including $\mathcal{K}(H)$ and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-C^*-dynamical system on A with the generator δ. If there is a rank-one projection $p \in D(\delta) \cap \sigma(A)$, then there exists a bounded $*$-σ-derivation δ^p on A such that $(\delta + \delta^p)(p) = 0$ and $\delta + \delta^p$ generates a σ-C^*-dynamical system on A.*

Proof. First, note that $p \in \sigma(A)$. Thus, $p = \sigma(q)$ for some $q \in A$. But, from the group property of $\{\alpha_t\}_{t\in\mathbb{R}}$ it follows that $\sigma^2 = \sigma$ and therefore, $\sigma(p) = \sigma^2(q) = \sigma(q) = p$. Also, p is projection. So, $p = p^2$ and $\delta(p) = \delta(p^2) = \delta(p).p + p.\delta(p)$. Thus,

$$\begin{aligned} p.\delta(p).p &= p.\delta(p^2).p \\ &= p.\delta(p).p^2 + p^2.\delta(p).p \\ &= 2p.\delta(p).p \end{aligned}$$

and consequently, $p.\delta(p).p = 0$.

Using some ideas of [4, p. 246], we define $h_p = i[\delta(p), p]$ and $\delta^p(a) = i[h_p, \sigma(a)]$. Trivially, h_p is self-adjoint and δ^p is a bounded $*$-σ-derivation on A satisfying $\delta^p\sigma = \delta^p = \sigma\delta^p$. Moreover, $\delta^p(p) = i[h_p, p]$ since $\sigma(p) = p$. Hence,

$$\begin{aligned} (\delta + \delta^p)(p) &= \delta(p) + i[h_p, p] \\ &= \delta(p) + i\Big(h_p.p - p.h_p\Big) \\ &= \delta(p) - \Big(\delta(p).p^2 + p^2.\delta(p)\Big) + 2p.\delta(p).p \\ &= \delta(p) - \delta(p^2) + 2p.\delta(p).p \\ &= 0. \end{aligned}$$

The final part is obtained from the previous theorem. □

The next result manifests a uniqueness theorem for σ-C^*-dynamical systems.

Theorem 2.5. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{\beta_t\}_{t\in\mathbb{R}}$ be two σ-C^*-dynamical systems on a C^*-algebra A, with the same generator δ. Then, $\alpha_t = \beta_t$ $(t \in \mathbb{R})$.*

Proof. Consider the associated the one parameter families $\{\widetilde{\alpha}_t\}_{t\in\mathbb{R}}$ and $\{\widetilde{\beta}_t\}_{t\in\mathbb{R}}$ on $\sigma(A)$ defined by $\widetilde{\alpha}_t(\sigma(a)) = \alpha_t(a)$ and $\widetilde{\beta}_t(\sigma(a)) = \beta_t(a)$. As we mentioned in the introduction, $\{\widetilde{\alpha}_t\}_{t\in\mathbb{R}}$ and $\{\widetilde{\beta}_t\}_{t\in\mathbb{R}}$ are C^*-dynamical systems. But, δ is the generator of $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{\beta_t\}_{t\in\mathbb{R}}$. Hence, the mapping $\widetilde{\delta} : \sigma\big(D(\delta)\big) \subseteq \sigma(A) \to \sigma(A)$ defined by $\widetilde{\delta}(\sigma(a)) = \delta(a)$ is the generator of $\{\widetilde{\alpha}_t\}_{t\in\mathbb{R}}$ and $\{\widetilde{\beta}_t\}_{t\in\mathbb{R}}$ and by uniqueness ([23, Theorem 1.1.3]), we conclude that $\widetilde{\alpha}_t = \widetilde{\beta}_t$. Consequently,

$$\begin{aligned}\alpha_t(a) &= \widetilde{\alpha}_t(\sigma(a))\\ &= \widetilde{\beta}_t(\sigma(a))\\ &= \beta_t(a).\end{aligned}$$

□ Before we state the next theorem, we need the following useful proposition which can be found in [10].

Proposition 2.6. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a strongly continuous σ-one parameter group with the generator δ. Then, for each $a \in D(\delta)$,*

(i) $\alpha_t(a) \in D(\delta)$ *and* $\delta\left(\alpha_t(a)\right) = \alpha_t\left(\delta(a)\right) = \frac{d}{dt}\alpha_t(a)$;

(ii) $\alpha_t(a) - \alpha_s(a) = \displaystyle\int_s^t \alpha_\tau\Big(\delta(a)\Big)d\tau.$

Theorem 2.7. *Let σ be a linear $*$-endomorphism on a C^*-algebra A so that $\sigma(A)$ acts on a Hilbert space H which including $\mathcal{K}(H)$ and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-C^*-dynamical system on A with the generator δ. Then, for a rank one projection $p \in D(\delta) \cap \sigma(A)$, there exist a σ-C^*-dynamical system $\{\alpha_{t,p}\}_{t\in\mathbb{R}}$ on A and a self-adjoint operator $h^p : D(h^p) \subseteq H \to H$ such that $\alpha_{t,p}(a) = e^{ith^p}\sigma(a)e^{-ith^p}$. Furthermore, if $h^p \in A$ and $\sigma(h^p) = h^p$, then there is a self-adjoint operator $h : D(h) \subseteq H \to H$ such that for each $a \in D(\delta) \cap \mathcal{K}(H)$, $\delta(a) = i[h, \sigma(a)]$ and $\alpha_t(a) = e^{ith}\sigma(a)e^{-ith}$ on $\mathcal{K}(H)$.*

Proof. By Theorem 2.4, there exists a bounded $*$-σ-derivation δ^p such that $(\delta + \delta^p)(p) = 0$ and $\delta + \delta^p$ generates a perturbed σ-C^*-dynamical system on A, namely $\{\alpha_{t,p}\}_{t\in\mathbb{R}}$ which satisfies $\alpha_{t,p}(p) = p$. Since, applying the previous proposition, we have $\frac{d}{ds}\alpha_{s,p}(p) = \alpha_{s,p}\Big((\delta+\delta^p)(p)\Big) = 0$. Therefore, $\alpha_{t,p}(p)-\sigma(p) = \displaystyle\int_0^t \alpha_{s,p}\Big((\delta+\delta^p)(p)\Big)ds = 0$ and consequently, $\alpha_{t,p}(p) = \sigma(p)$. From the comment as stated at the beginning of the proof of Theorem 2.4, $\sigma(p) = p$. Hence, $\alpha_{t,p}(p) = p$ $(t \in \mathbb{R})$ which means that p is invariant under $\{\alpha_{t,p}\}_{t\in\mathbb{R}}$ and it follows from Theorem 2.2 that, $\{\alpha_{t,p}\}_{t\in\mathbb{R}}$ is implemented by a C_0-group $\{u_{t,p}\}_{t\in\mathbb{R}}$ of unitaries in $\mathbf{B}(H)$ (i.e., $\alpha_{t,p}(a) = u_{t,p}\sigma(a)u^*_{t,p}$). Assume that $\widehat{h^p}$ to be the generator of $\{u_{t,p}\}_{t\in\mathbb{R}}$ and take $h^p := -i\widehat{h^p}$. By Stone's theorem ([21, Theorem 1.10.8]), h^p is a self-adjoint operator in H and $u_{t,p} = e^{ith^p}$.

Now, take $h := h^p - h_p$, where $h_p := i[\delta(p), p]$ (as we defined in the proof of Theorem 2.4). Since, h_p is bounded, $D(h) = D(h^p)$ and h is self-adjoint. Using Stone's theorem once more, we reach the conclusion that ih is the generator of the C_0-group $\{e^{ith}\}_{t\in\mathbb{R}}$ of unitaries. Also, the assumption $\sigma(h^p) = h^p$ together with continuity of σ, imply that $\sigma(e^{ith}) = e^{ith}$. By Theorem 2.1, $\beta_t(a) = e^{ith}\sigma(a)e^{-ith}$ is a σ-C^*-dynamical system on $\mathcal{K}(H)$ whose generator is $\delta_h(a) = i[h, \sigma(a)]$.

But, for each $a \in D(\delta) \cap \mathcal{K}(H)$ we have

$$\begin{aligned}
\delta_h(a) &= i[h, \sigma(a)] \\
&= i[h^p, \sigma(a)] - i[h_p, \sigma(a)] \\
&= (\delta + \delta^p)(a) - \delta^p(a) \\
&= \delta(a).
\end{aligned}$$

It follows that $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{\beta_t\}_{t\in\mathbb{R}}$ have identical generators on $\mathcal{K}(H)$ and by Theorem 2.5, $\alpha_t(a) = e^{ith}\sigma(a)e^{-ith}$ which completes the proof. $\square$

Definition 2.8. *Let σ be a linear mapping on a vector space A. A linear map α on A is called σ-bijective if $\sigma(A) \subseteq \alpha(A)$ and $ker(\alpha) \subseteq ker(\sigma)$.*

Theorem 2.9. *Let σ be an idempotent linear $*$-endomorphism on a C^*-algebra A so that $\sigma(A)$ acts on a Hilbert space H which including $\mathcal{K}(H)$. Then, each σ-bijective linear endomorphism on A is a σ-inner endomorphism.*

Proof. Let α be an arbitrary σ-bijective linear endomorphism on A. Consider the associated map β on $\sigma(A)$ defined by $\beta(\sigma(a)) := \alpha(a)$. Since α is σ-bijective, β is a linear automorphism. Let p be an arbitrary rank one projection in $\mathcal{K}(H)$. So, there is a unit vector $x \in H$ such that $p = x \otimes x$. Also, by the assumption $\mathcal{K}(H) \subseteq \sigma(A)$, it follows that p is contained in $\sigma(A)$ and there exists an element $a_p \in A$ such that $\sigma(a_p) = p$. However, $\sigma^2 = \sigma$. So, $\sigma(p) = \sigma^2(a_p) = \sigma(a_p) = p$. On the other hand, following as stated in the proof of [21, Theorem 2.4.8], $\beta(p)$ is also a rank-one projection, namely q. Thus, $\alpha(p) = \beta(\sigma(p)) = \beta(p) = q$ and $q = y \otimes y$, for some unit vector $y \in H$. Define for each $a \in A$, $u(\sigma(a)x) = \alpha(a)y$. Then, by the same reasoning as in the proof of Theorem 2.2, we have

$$\begin{aligned}
\|u(\sigma(a)x)\| &= \|\alpha(a)y\| \\
&= \|\alpha(a)(y \otimes y)\| \\
&= \|\alpha(a)\alpha(x \otimes x)\| \\
&= \|\alpha(a.x \otimes x)\| \\
&= \|\sigma(a.x \otimes x)\| \\
&= \|\sigma(a).\sigma(x \otimes x)\| \\
&= \|\sigma(a).(x \otimes x)\| \\
&= \|\sigma(a)(x \otimes x)x\| \\
&= \|\sigma(a).x\|.
\end{aligned}$$

So, u is well-defined isometry on $\sigma(A)x$. Also, u is onto since for each $z \in H$, $z = (z \otimes y)(y)$, and by the assumption $\mathcal{K}(H) \subseteq \sigma(A)$ we obtain $z \otimes y \in \sigma(A)$. However, α is σ-bijective so, there is an element $a \in A$ such that $\alpha(a) = z \otimes y$. Therefore, $z = \alpha(a)(y) = u(\sigma(a)x)$.

Moreover, applying the relation $z = (z \otimes x)x$ together with the the assumption $\mathcal{K}(H) \subseteq \sigma(A)$ one concludes that $z \in \sigma(A)x$ and $\overline{[\sigma(A)x]} = H$. Similarly, $\overline{[\sigma(A)y]} = H$. Consequently, u can be extended to a unitary on H.

Let $b \in A$. Since β is an automorphism, there exists $c \in A$ such that $\sigma(b) = \beta(\sigma(c)) = \alpha(c)$ and $\sigma(b)y = \alpha(c)y = \beta(\sigma(c))\, y = u(\sigma(c)x)$. So,

$$u^*(\sigma(b)y) = u^{-1}(\sigma(b)y) = u^{-1}\left(\beta(\sigma(c))y\right)\right) = \sigma(c)x.$$

Consequently, for each $a, b \in A$

$$\begin{aligned} u\sigma(a)u^*(\sigma(b)y) &= u\sigma(a)\sigma(c)x \\ &= u\sigma(a.c)x \\ &= \alpha(a.c)y \\ &= \alpha(a)\alpha(c)y \\ &= \alpha(a)(\sigma(b)y), \end{aligned}$$

which implies that $\alpha(a) = u\sigma(a)u^*$ on $\sigma(A)y$ and the density of $\sigma(A)y$ in H completes the proof. $\square$

Substituting $\sigma := I_A$ (the identity operator on A), we have the following corollary.

Corollary 2.10. *Let A be a C^*-subalgebra of $\mathbf{B}(H)$ including $\mathcal{K}(H)$. Then, each linear automorphism on A is inner.*

Now, let's return to $\mathbf{B}(H)$ and its C^*-dynamical systems. We are going to show that each C^*-dynamical system on $\mathbf{B}(H)$, where H is a separable Hilbert space is implemented by a C_0-group of unitaries on H. First, we recall that a strongly closed $*$-subalgebra of $\mathbf{B}(H)$, where H is a Hilbert space is called a *von Neumann* algebra. Obviously, $\mathbf{B}(H)$ is a von Neumann algebra (see [21, p.116]). Now, we need the following useful theorem which can be found in [11].

Theorem 2.11. *Let A be a C^*-algebra acting on a separable Hilbert space H, M be a von Neumann algebra generated by A, and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one parameter group of $*$-automorphisms on A such that $t \to \alpha_t(a)x$ is continuous for all $a \in A$ and $x \in H$. Suppose that for each $t \in \mathbb{R}$, α_t extends to be an inner automorphism on M. Then, there exists a C_0-group of unitary operators in M such that $\alpha_t(a) = u_t a u_t^*$.*

Before we state the next remark, it is necessary to recall that for a Hilbert space H, a net $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq H$ converges weakly to an element x in H if and only if $< x_\lambda, y > \to < x, y >$ for each $y \in H$.

Remark 2.12. Let A be a C^*-algebra acting on a separable Hilbert space H, and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a C^*-dynamical system on A. Then, as stated in [2, p.91], due to the Cauchy- Schwarz inequality we have

$$\begin{aligned} | < \alpha_t(a)x, y > - < \alpha_s(a)x, y > | &= | < (\alpha_t(a) - \alpha_s(a))x, y > | \\ &\leq \|\alpha_t(a) - \alpha_s(a)\| \, \|x\| \, \|y\| \end{aligned}$$

for each $s, t \in \mathbb{R}$ and $x, y \in H$ and $a \in A$.

By applying strong continuity of $\{\alpha_t\}_{t\in\mathbb{R}}$, one can deduce that

$$\lim_{t\to s} < \alpha_t(a)x, y >=< \alpha_s(a)x, y > \qquad (a \in A, x, y \in H).$$

That is, $t \to \alpha_t(a)x$ is weakly continuous for each $x \in H$.

The following theorem is an immediate consequence of Theorem 2.11 and the previous remark.

Theorem 2.13. *Let A be a C^*-algebra acting on a separable Hilbert space H, M be a von Neumann algebra generated by A, and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a C^*-dynamical system on A. Suppose that for each $t \in \mathbb{R}$, there is a unitary operator v_t in M such that $\alpha_t(a) = v_t a v_t^*$ for each $a \in A$. Then, there exists a C_0-group $\{u_t\}_{t\in\mathbb{R}}$ of unitary operators in M such that $\alpha_t(a) = u_t a u_t^*$ ($a \in A$).*

Applying corollary 2.10 and substituting $M = A = \mathbf{B}(H)$ in the above theorem, we have the following main result.

Theorem 2.14. *Let H be a separable Hilbert space and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a C^*-dynamical system on $\mathbf{B}(H)$. Then, there exists a C_0-group $\{u_t\}_{t\in\mathbb{R}}$ of unitary operators in $\mathbf{B}(H)$ such that $\alpha_t(a) = u_t a u_t^*$ ($a \in A$).*

3 Comments and an Application

In this section, we characterize each σ-C^*-dynamical system on the concrete C^*-algebra $A := \mathbf{B}(H) \times \mathbf{B}(H)$, where H is a separable Hilbert spaces and σ is the linear $*$-endomorphism $\sigma(S, T) = (0, T)$ on A.

For this aim, suppose that A_j ($j = 1, 2$) is a C^*-algebra. It is easy to observe that, $A := A_1 \times A_2$ is also a C^*-algebra by regarding the following algebraic structure

(i) $(a, b) + (c, d) = (a + c, b + d)$,

(ii) $\lambda(a,b) = (\lambda a, \lambda b)$

(iii) $(a,b).(c,d) = (ac, bd)$, $(a,b)^* = (a^*, b^*)$

(iv) $\| (a,b) \| = \max\{\| a \|, \| b \|\}$.

Now, consider A_j $(j = 1,2)$ as the concrete C^*-algebra $\mathbf{B}(H)$, where H is a separable Hilbert space and define $\sigma : A \to A$ by $\sigma(S,T) := (0,T)$. Evidently, σ is an idempotent norm decreasing linear $*$-endomorphism on A. The following theorem characterizes each so-called σ-C^*-dynamical system on A.

Theorem 3.1. *Let $\mathcal{H}$ be a separable Hilbert space. The following assertions are equivalent.*

(i) $\{\alpha_t\}_{t\in\mathbb{R}}$ *is a σ-C^*-dynamical system on A.*

(ii) *There exists a C_0-group $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators in A satisfying $\sigma(U_t) = U_t$ and $\alpha_t(S,T) = U_t\sigma(S,T)U_t^*$ $(S,T \in \mathbf{B}(H))$.*

Proof. Suppose that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-C^*-dynamical system on A. So, for each $t \in \mathbb{R}$ and $(S,T) \in A$, there is a unique pair $(S',T') \in A$ such that $\alpha_t(S,T) = (S',T')$. However, $\sigma(\alpha_t(S,T)) = \alpha_t(S,T)$ and therefore, $S' = 0$. On the other hand, $\alpha_t(S,T) = \alpha_t(\sigma(S,T))$. That is, $\alpha_t(S,T) = \alpha_t(0,T)$. Define $\beta_t : \mathbf{B}(H) \to \mathbf{B}(H)$ by $\beta_t(T) := T'$. Hence, for each $t \in \mathbb{R}$ and $S,T \in \mathbf{B}(H)$ we have

$$\begin{aligned}(0,\beta_t(TS)) &= \alpha_t(0,TS)\\ &= \alpha_t(0,T).\alpha_t(0,S)\\ &= (0,\beta_t(T)).(0,\beta_t(S))\\ &= (0,\beta_t(T)\beta_s(T)).\end{aligned}$$

So, for each $t \in \mathbb{R}$, $\beta_t(TS) = \beta_t(T).\beta_t(S)$ and β_t is a homomorphism. Similarly, one can show that β_t $(t \in \mathbb{R})$ is $*$-linear.

Let $t \in \mathbb{R}$ and suppose that $T \in ker\beta_t$. Thus, $\alpha_t(0,T) = (0,\beta_t(T)) = (0,0)$ and so, $\sigma(0,T) = \alpha_{-t}(\alpha_t(0,T)) = (0,0)$. By the definition of σ, we get $T = 0$ and therefore β_t is injective. Also, for each $S \in \mathbf{B}(H)$, $\sigma(0,S) = (0,S)$ and hence, $(0,S) \in \sigma(A)$. From the comment as stated at the beginning of the introduction, $\sigma(A) = \alpha_t(A)$ $(t \in \mathbb{R})$. This feature implies that $(0, S) = \alpha_t(T', T)$ for some $T', T \in \mathbf{B}(H)$. But, $\alpha_t(T', T) = \alpha_t(0, T)$ and

$$\begin{aligned}(0, \beta_t(T)) &= \alpha_t(0, T) \\ &= \alpha_t(T', T) \\ &= (0, S).\end{aligned}$$

Consequently, $\beta_t(T) = S$ for each $t \in \mathbb{R}$. So for each $t \in \mathbb{R}$, β_t is a bounded linear $*-$automorphism on A.

To justify group property, note that for each $T \in \mathbf{B}(H)$, we have $(0, T) = \alpha_0(0, T) = (0, \beta_0(T))$ and thus, $\beta_0 = I$. Moreover, for each $s, t \in \mathbb{R}$ and $T \in \mathbf{B}(H)$ we have

$$\begin{aligned}(0, \beta_{t+s}(T)) &= \alpha_{t+s}(0, T) \\ &= \alpha_t\left(\alpha_s(0, T)\right) \\ &= \alpha_t\left(0, \beta_s(T)\right) \\ &= \left(0, \beta_t\left(\beta_s(T)\right)\right)\end{aligned}$$

Furthermore, applying the relation $(0, \beta_t(T) - T) = \alpha_t(0, T) - \sigma(0, T)$ together with strong continuity of $\{\alpha_t\}_{t\in\mathbb{R}}$, we observe that $\{\beta_t\}_{t\in\mathbb{R}}$ is also strongly continuous. Consequently, $\{\beta_t\}_{t\in\mathbb{R}}$ is a C^*-dynamics on $\mathbf{B}(H)$.

By Theorem 2.14, there exists a C_0-group $\{V_t\}_{t\in\mathbb{R}}$ of unitary operators in $\mathbf{B}(H)$ such that $\beta_t(T) = V_t T V_t^*$. Hence, for each $T \in \mathbf{B}(H)$

$$\begin{aligned}\alpha_t(0, T) &= (0, \beta_t(T)) \\ &= (0, V_t T V_t^*) \\ &= (0, V_t)\sigma(0, T)(0, V_t^*).\end{aligned}$$

By taking $U_t := (0, V_t)$, it follows that the family $\{U_t\}_{t\in\mathbb{R}}$ is a C_0-group of unitaries in A satisfying $\sigma(U_t) = U_t$ $(t \in \mathbb{R})$ and the σ-C^*-dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ is implemented by the C_0-group $\{U_t\}_{t\in\mathbb{R}}$ of unitaries in A.

Conversely, let $\{U_t\}_{t\in\mathbb{R}}$ be a C_0-group of unitary operators in A satisfying $\sigma(U_t) = U_t$ and $\alpha_t(S, T) = U_t\sigma(S, T)U_t^*$ $(S, T \in \mathbf{B}(H))$. Trivially, for each $t \in \mathbb{R}$, α_t is a homomorphism on A such that $\alpha_0 = \sigma$. Moreover, it follows from the fact that $\sigma^2 = \sigma$ and $\sigma(U_t) = U_t$, that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a

σ-one parameter group. It is sufficient to prove that $\{\alpha_t\}_{t\in\mathbb{R}}$ is strongly continuous. For each $S, T \in \mathbf{B}(H)$, we have

$$\begin{aligned}
\|\alpha_t(S,T) - \sigma(S,T)\| &= \|U_t\sigma(S,T)U_t^* - \sigma(S,T)\| \\
&= \|(U_t\sigma(S,T) - \sigma(S,T)U_t)U_t^*\| \\
&\leq \|U_t\sigma(S,T) - \sigma(S,T)U_t\| \\
&\leq \|U_t\sigma(S,T) - \sigma(S,T)\| + \|\sigma(S,T) - U_t\sigma(S,T)\| \\
&\leq 2\|U_t\sigma(S,T) - \sigma(S,T)\|.
\end{aligned}$$

Since $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous, $\lim\limits_{t\to 0}\|\alpha_t(S,T) - \sigma(S,T)\| = 0$ and therefore, $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-C^*-dynamical system on A. $\square$

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Maysam Mosadeq
Department of Mathematics
Assistant Professor of Mathematics
Larestan Branch, Islamic Azad University
Larestan, Iran
E-mail: mosadeq@mshdiau.ac.ir,
maysammosaddegh@yahoo.com