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Malmquist Productivity Index on Efficiency Layers

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Abstract. Data Envelopment Analysis (DEA), a popular linear programming technique is useful to rate comparatively operational effiency of decision Making Unit (DMU) based on the their deterministic input-output data. The Malmquist productivity index in DEA, calculable with the distance function, for measurement the productivity change among two variant time period or two variant group in the same time. This index is based on two factor of efficiency change index and a technological change index. In this paper, we operate on the collective Malmquist productivity index, which performs clustering operation DMUs with classification into different levels of efficient frontier, and then we discuss on the relation between Malmquist index on the efficiency layers and their attractiveness and progress.

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1. Introduction

Data Envelopment Analysis is a non-parametric analytical method by the mathematical programming procedure, calculated the relative efficiency, that concerning the production frontier for multiple inputs and multiple outputs. DEA first proposed by Charnes, Cooper and Rhodes

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for measuring the relative efficiency of DMUs ([4]) and then has extended with Banker et al. ([1]). DEA provided the benchmark information, that use for improvement efficiency of the evaluated DMU. If Decision making unit (DMU) lies on production frontiers, DMU is efficient, otherwise DMU is inefficient.
Fare ([5]) extend the Malmquist productivity index which proposed at the first by Malmquist. Fare presented for each unit the Malmq-uist index with combine the Farrell views for measurement of efficiency.
Malmquist productivity index is divisible on two factors of efficiency change index which indicator of increase or decrease of efficiency and a technology change index that is measure the value of frontier movement. Among achievement in this domain, the researchs Fenz (1994) and RD (1997) that formulated by non-parametric programming models, the approximate value of malmquist index. The difference between variables and that estimations, cause distinction between fore going methods.
One unit, for example is possibility have not the attractiveness against the activity domain or include the less than of attractiveness value versus other unit ([6]). In this article we shows a comparison between groups Malmquist index according to groups selection of Efficiency Layers. the attractive and progress index have the close relation with Malmquist index.
The reminder of this paper is organized as follows. Section 2 introduces the primary concepts DEA . Section 3 presents the attractiveness and progress relative, in Section 4 and 4.1 Malmquist productivity index, the Malmquist productivity index for different groups and in Section 4.2 Malmquist productivity index on the efficiency layers are presented. In Section 5 the relation between Malmquist index with attractiveness and progress is discussed. An illustrative application is in Section 6 and concluding remarks are given in Section 7.

2. The Background of DEA

Data Envelopment Analysis is a mathematical programming technique for evaluating the relative efficiency of homogeneous DMUs set. The variouse models have presented for evaluation the efficiency. One of

this methods calculate efficiency a DMU according to its projection on efficient frontier.
Suppose we have a set of units, $DMU_j (j = 1, ..., n)$. Each DMU uses m inputs $x_{ij}(i = 1, ..., m)$ to produces the s outputs $y_{rj}(r = 1, ..., s)$. Suppose that $X_j = (x_{1j}, x_{2j}, ..., x_{mj})$ and $Y_j = (y_{1j}, y_{2j}, ..., y_{sj})$ are the vectors of inputs and outputs value of DMU_j, respectively and let $X_j \geqslant 0, Y_j \geqslant 0$ and $X_j \neq 0, Y_j \neq 0$ and x_{io} and y_{ro} are the corresponding inputs and outputs for the DMU_o which should begining evaluated. The CCR model in input oriented with constant return to scale is following:

$$
\begin{aligned}
& min \quad \theta \\
s.t \quad & \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta x_{io} \qquad i = 1, ..., m \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geqslant y_{ro} \qquad r = 1, ..., s \\
& \lambda_j \geqslant 0. \qquad j = 1, ..., n
\end{aligned} \tag{1}
$$

The model (1) obtain efficiency of DMU_o, θ^* , according to decreasing input. θ^* is the optimal value that represented the efficiency value. If $\theta^* = 1$, DMU_o is efficient, otherwise the evaluated DMU is inefficient. The CCR model in output oriented with constant return to scale is following:

$$
\begin{aligned}
& max \quad \varphi \\
s.t \quad & \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leqslant x_{io} \qquad i = 1, ..., m \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geqslant \varphi y_{ro} \qquad r = 1, ..., s \\
& \lambda_j \geqslant 0. \qquad j = 1, ..., n
\end{aligned} \tag{2}
$$

The model 2 calculate efficiency of DMU_o, φ^*, according to increasing output. If $\varphi^* = 1$ DMU_o, is efficient, otherwise is inefficient.

Theorem 2.1. *In CCR model the optimal value in input oriented (the value θ^*) is equal to $\frac{1}{\varphi^*}$, that is the optimal value in output oriented for evaluation of DMU_o [2].*

3. Attractiveness and Progress in DEA

Research in the field of self-selection show that their choice in most cases the choice is influenced by the interval. For example, if there are three categories of goods in three price range, then the customer will always choose the best product to each of the three categories. This case is discussed in Data Envelopment Analysis as attractiveness and progress. Relative attractiveness is obtained when the units get worse as the field assessment is evaluated. Relative progress is obtained when the units with better evaluation assessed as to be considered.
Suppose that $J^1 = \{DMU_j, j = 1, ..., n\}$ be the set of all n DMUs and define $J^{l+1} = J^l - E^l$ where $E^l = \{DMU_k \in J^l | \phi^*(l, k) = 1\}$ and $\phi^*(l, k)$ is the optimal value of model (3).

$$
\begin{aligned}
&\varphi_o^* = max\ \varphi_o \\
&s.t. \quad \textstyle\sum_{j\in J^l} \lambda_j x_{ij} \leqslant x_{io} && i = 1, ..., m \\
&\qquad \textstyle\sum_{j\in J^l} \lambda_j y_{rj} \geqslant \varphi y_{ro} && r = 1, ..., s \\
&\qquad \lambda_j \geqslant 0 && j \in J^l.
\end{aligned} \tag{3}
$$

I and O are the set of inputs and outputs, respectively, x_{ij} and y_{rj} represent the input i and output r at DMU_j. when $l = 1$, model (3) is the output oriented CCR model. Model (3), classify all DMUs in levels of E^l step by step. Model (4) calculate the relative attractiveness of DMU_o at one especial level E^{l_o} with respect to after steps $E^{l+d}, d = 1, ..., L - l_o$.[6]

$$
\begin{aligned}
&\varphi_o^*(d) = max\ \varphi_o(d) && d = 1, ..., L - l_o \\
&s.t. \quad \textstyle\sum_{j\in E^{l_o+d}} \lambda_j x_{ij} \leqslant x_{io} && i = 1, ..., m \\
&\qquad \textstyle\sum_{j\in E^{l_o+d}} \lambda_j y_{rj} \geqslant \varphi(d) y_{ro} && r = 1, ..., s \\
&\qquad \lambda_j \geqslant 0 && j \in E^{l_o+d}.
\end{aligned} \tag{4}
$$

That
1. $\varphi_o^*(d) < 1,\ d = 1, ..., L - l_o$
2. $\varphi_o^*(d+1) < \varphi_o^*(d)$.

Definition 3.1. $A_o^*(d) = \frac{1}{\varphi_o^*(d)}$ *is called the d-degree relative attractiveness of evaluated DMU* (DMU_o) *from a specific level* E^{l_o}.
In model (4), each efficient frontier of E^{l_o+d} *represents a level of efficiency for measuring the relative attractiveness of DMUs in especial level* E^{l_o}. *The larger* $A_o^*(d)$, *means more attractive.*
For calculating the progress measure for a specific $DMU_o \in E^{l_o}, l_o \in \{2, ..., L\}$. *One can be used the following model (5):*

$$
\begin{aligned}
& P_o^*(g) = max\ P_o(g) && g = 1, ..., l_o - 1 \\
& s.t. \quad \textstyle\sum_{j \in E^{l_o - g}} \lambda_j x_{ij} \leqslant x_{io} && i = 1, ..., m \\
& \qquad \textstyle\sum_{j \in E^{l_o - g}} \lambda_j y_{rj} \leqslant P_o(g) y_{ro} && r = 1, ..., s \\
& \qquad \lambda_j \geqslant 0 && j \in E^{l_o - g}
\end{aligned}
\tag{5}
$$

That
1. $P_o^*(g) > 1,\ g = 1, ..., l_o - 1$
2. $P_o^*(g+1) > P_o^*(g)$

Definition 3.2. $P_o^*(g)$, *is called the d-degree relative progress of evaluated DMU* (DMU_o) *from a specific level* E^{l_o}.
For a larger $P_o^*(g)$, *more progress is for* DMU_o.

4. Malmquist Productivity Index

The Malmquist productivity index introduced in DEA efficiency measures is defined as the ratio of the efficiency measure for the same production unit in two various time periods or between two various observation for the same period ([2]). The Malmquist index can be divided into two factors. One shows the relative change in efficiency and other componet

is known as frontier productivity index.
Suppose that n DMU, that input vector $X^t \in R^m$ to produce output vector $Y^t \in R^s$, for introducing the distance function in time period t.The measure value DMU_o can be obtained by using the CCR model[8].

$$D(X^t, Y^t) = min\ \theta_o^t$$

$$s.t \quad \sum_{j=1}^{n} \lambda_j x_{ij}^t \leqslant \theta_o^t x_{io}^t \qquad i = 1, ..., m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^t \geqslant y_{ro}^t \qquad r = 1, ..., s$$

$$\lambda_j \geqslant 0 \qquad j = 1, ..., n. \tag{6}$$

Where x_{ij}^t and y_{rj}^t are the corresponding ith input vector and rth output vector for the DMU_j, respectively, and x_{io}^t is the ith input and y_{ro}^t is the rth output for DMU_o in time period t. The optimal value of model (6), θ_o^t shows decreasing in inputs while the output level still have keeped. If $\theta_o^t = 1$, DMU_o is efficient in time period t, otherwise is the inefficient.
Malmquist index calculation requires two single period and two mixed period measures. It can be replaced x_{io}^t and y_{ro}^t by vectors with x_{io}^{t+1} and y_{ro}^{t+1} vectors, respectively $(i = 1, ..., m)$, $(r = 1, ..., s)$, $o \in \{1, ..., n\}$,then model (6) can evaluate technical efficiency of $\theta_o^t(x_o^{t+1}, y_o^{t+1})$ for DMU_o at the time period t and also calculating the technical efficiency $\theta_o^{t+1}(x_o^{t+1}, y_o^{t+1})$ and $\theta_o^{t+1}(x_o^t, y_o^t)$ in time period t+1, similarly in model (6).
Fare ([5]) defined an input oriented productivity index as the geometric mean of the two Malmquist indices developed by Caves ([3]). referring to the technologies at time periods t and t+1,yielding the following Malmquist-type measure of productivity: ([2])

$$M^{t,t+1} = [\frac{D^t(x^{t+1}, y^{t+1})}{D^t(x^t, y^t)} \times \frac{D^{t+1}(x^{t+1}, y^{t+1})}{D^{t+1}(x^t, y^t)}]^{\frac{1}{2}} \tag{7}$$

Gaves et al. (1982) and Fare et al. (1994) defined that $M^{t,t+1} > 1$ indicates productivity gain, if $M^{t,t+1} < 1$ indicates productivity loss, and $M^{t,t+1} = 1$ means no change in productivity from time t to t+1.
Fare et al. (1994) decomposed the Malmquist productivity index into

two components:

$$M^{t,t+1} = \frac{D^{t+1}(x^{t+1},y^{t+1})}{D^t(x^t,y^t)}[\frac{D^t(x^{t+1},y^{t+1})}{D^{t+1}(x^{t+1},y^{t+1})}\times\frac{D^t(x^t,y^t)}{D^{t+1}(x^t,y^t)}]^{\frac{1}{2}} \quad (8)$$

or

$$M = E.P \quad (9)$$

$E = \frac{D^{t+1}(x^{t+1},y^{t+1}}{D^t(x^t,y^t)}$ represented the change in efficiency and
$P = [\frac{D^t(x^{t+1},y^{t+1})}{D^{t+1}(x^{t+1},y^{t+1})} \times \frac{D^t(x^t,y^t)}{D^{t+1}(x^t,y^t)}]^{\frac{1}{2}}$ represented the measure of technical progress.

4.1 Malmquist Index for the Group of DMUs in Different Conditions

This section present the Malmquist productivity index to calculate the efficiency of group decision-making units under different conditions. Group performance is calculated with this method. Malmquist index evaluates the productivity variation among different groups rather than two differnt time. Note that the choice of different groups is something desired by the manager.
Consider δ_A as cardinal of DMUs in group A, using inputs $X^A \in R^m_+$ to produce outputs $Y^B \in R^s_+$, and δ_B as cardinal of DMUs in group B, using inputs $X^B \in R^m_+$ to produce outputs $Y^B \in R^s_+$. (X^A_j, Y^A_j) is the jth input-output vector for group A and (X^B_j, Y^B_j) is the jth input-output vector for group B. $D^A(X^B_j, Y^B_j)$ represents the input distance function for a DMU in group B with respect to the frontier of group A. The Malmquist index is calculated between two groups (A and B group)as follows ([2]).

$$I^{AB} = [\frac{(\prod_{j=1}^{\delta_A} D^A(X^A_j,Y^A_j))^{\frac{1}{\delta_A}}}{(\prod_{j=1}^{\delta_A} D^A(X^B_j,Y^B_j))^{\frac{1}{\delta_A}}}\times\frac{(\prod_{j=1}^{\delta_B} D^B(X^A_j,Y^A_j))^{\frac{1}{\delta_B}}}{(\prod_{j=1}^{\delta_B} D^B(X^B_j,Y^B_j))^{\frac{1}{\delta_B}}}]^{\frac{1}{2}} \quad (10)$$

4.2 Malmquist Index on the Efficiency Frontier Layers

In the previous section, the Malmquist index were introduced to a group of decision-making units under different conditions. In this section we want evaluate, Malmquist index using the efficiency of boundary layer classification obtained from decision-making units.

consider $J^1 = \{DMU_j,\ j = 1, ..., n\}$ be the set of all n DMUs. Define $J^{l+1} = J^l - E^l$ where $E^l = \{DMU_k \in J^l \mid \varphi^*(l,k) = 1\}$, and $\varphi^*(l,k)$ is the optimal value of model (3):

when l=1, model (3) is the output oriented CCR model and DMUs in set E^l define the first-level efficient frontier. When l=2, give the second-level efficient frontier after the elimination of the first-level efficient DMUs, and continue similarly. With this manner, identified the several levels of efficient frontier. E^l is the lth-level efficient frontier. The following algorithm reflect process computation of efficiency frontiers.

Step 1. Let l=1, calculate for J^l the model (3), for obtain the first-level efficient DMUs (E^l).

Step 2. eliminate the efficient DMUs in last step. $J^{l+1} = J^l - E^l$. (if $J^{l+1} = \varnothing$ then stop.)

Step 3. form the new subset of inefficient DMUs, J^{l+1}, and by model (3) to obtain a new set of efficient DMUs E^{l+1}.

Step 4. let l=l+1 and go to step 2.

now, considered the result efficiency layers instead different groups in computation the Malmquist index groups.

E^{k_1}, E^{k_2} the set of DMUs belong to the two different efficiency frontier, that introduce instead two different group. Consider δ_{k_1} DMUs in group E^{k_1} , using inputs $X^{E^{k_1}} \in R^m_+$ to produce outputs $Y^{E^{k_1}} \in R^s_+$, and $(X_j^{E^{k_1}}, Y_j^{E^{k_1}})$ is the jth input-output vector for group E^{k_1} and same condition hold for E^{k_2} group. $D^{E^{k_1}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}})$ represents the input distance function for a DMU in group E^{k_2} with respect to the frontier of group E^{k_1}. Evaluated the Malmquist index for efficiency frontier by following model:

$$I^{E^{k_1}E^{k_2}} = [\frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_1}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_1}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}} \times \frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_2}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_2}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}}]^{\frac{1}{2}} \quad (12)$$

$D^{E^{k_1}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}})$ represents the input distance function for a DMU in group E^{k_1} with respect to the frontier of group E^{K_1}, that really is the units efficiency measure that located on own frontier and this value is one, therefore is rewriting the model (12) into following:

$$I^{E^{k_1}E^{k_2}} = [\frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_2}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_1}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}}]^{\frac{1}{2}} \quad (13)$$

Theorem 4.2.1. *For each tree different efficiency frontier $E^{k_1}, E^{k_2}, E^{k_3}$ that $k_1 < k_2 < k_3$, hold the following relation:*

$$I^{E^{k_1}E^{k_2}} \times I^{E^{k_2}E^{k_3}} = I^{E^{k_1}E^{k_3}} \quad (14)$$

Proof. Considering each $I^{E^{k_1}E^{k_2}}$ index to decompose to product of efficiency change index $IE^{E^{k_1}E^{k_2}}$ and technical change index $IF^{E^{k_1}E^{k_2}}$, must prove:

$$IE^{E^{k_1}E^{k_2}} \times IE^{E^{k_2}E^{k_3}} = IE^{E^{k_1}E^{k_3}}$$
$$IF^{E^{k_1}E^{k_2}} \times IF^{E^{k_2}E^{k_3}} = I^{E^{k_1}E^{k_3}}$$

With using the $IE^{E^{k_1}E^{k_2}}$ in [2] we have:

$$IE^{E^{k_1}E^{k_2}} \times IE^{E^{k_2}E^{k_3}} = \frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_1}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_2}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}} \times$$
$$\frac{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_2}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}}{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_3}}(X_j^{E^{k_3}}, Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}} = \frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_1}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_3}}(X_j^{E^{k_3}}, Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}} = E^{E^{k_1}E^{k_3}}$$

for calculating IF when we have N groups,the IF_{adj} is introduced in this way ([2]):

$$IF_{adj}^{E^{k_1}E^{k_2}} = [\prod_{i=1}^{n} \frac{\prod_{j=1}^{\delta_i} D^{E^{k_2}}(X_j^i, Y_j^i))^{\frac{1}{\delta_i}}}{\prod_{j=1}^{\delta_i} D^{E^{k_1}}(X_j^i, Y_j^i))^{\frac{1}{\delta_i}}}]^{\frac{1}{N}}$$

Now we having three groups $E^{k_1}, E^{k_2}, E^{k_3}$ we will have:

$$IF_{adj}^{E^{k_1}E^{k_2}} \times IF_{adj}^{E^{k_2}E^{k_3}} =$$

$$\Big(\frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_2}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_1}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}} \times \frac{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_2}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_1}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}} \times$$
$$\frac{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_2}}(X_j^{E^{k_3}},Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}}{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_1}}(X_j^{E^{k_3}},Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}}\Big)^{\frac{1}{3}} \times \Big(\frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_3}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_2}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}} \times$$
$$\frac{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_3}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_2}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}} \times \frac{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_3}}(X_j^{E^{k_3}},Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}}{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_2}}(X_j^{E^{k_3}},Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}}\Big)^{\frac{1}{3}} =$$
$$\Big(\frac{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_3}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}}{(\prod_{j=1}^{\delta_{k_1}} D^{E^{k_1}}(X_j^{E^{k_1}},Y_j^{E^{k_1}}))^{\frac{1}{\delta_{k_1}}}} \times \frac{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_3}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}}{(\prod_{j=1}^{\delta_{k_2}} D^{E^{k_1}}(X_j^{E^{k_2}},Y_j^{E^{k_2}}))^{\frac{1}{\delta_{k_2}}}} \times$$
$$\frac{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_3}}(X_j^{E^{k_3}},Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}}{(\prod_{j=1}^{\delta_{k_3}} D^{E^{k_1}}(X_j^{E^{k_3}},Y_j^{E^{k_3}}))^{\frac{1}{\delta_{k_3}}}}\Big)^{\frac{1}{3}} = IF_{adj}^{E^{k_1}E^{k_3}}$$

5. Layers Malmquist Index and Attractiveness and Progress

So far, we could use layering decision making units and we introduced the different groups to measure the Malmquist index. We will now examine the relationship between the Malmquist index and the attractive and development.

Distance function introduced in the efficiency index $I^{E^{k_1}E^{k_2}}$ on the layer model (13) meaning $D^{E^{k_2}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}})$, and it represents the distance function in input oriented for a DMU in the E^{k_1} group compared with the E^{k_2} group.Distance function introduced in the denominator fraction $I^{E^{k_1}E^{k_2}}$ model (13) meaning $D^{E^{k_1}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}})$ shows the distance function in input oriented for a DMU in the E^{k_2} group compared to the efficiency frontier E^{k_1} group.

On the other hand, model (5) calculates the value of progress with the distance function $P_o^*(g)$. It is also the distance function in output oriented that evaluated relative to later its efficiency frontier.

According to Theorem 1 can be said that the distance function in input oriented in the denominator fraction Malmquist index ($D^{E^{k_2}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}})$), shows an inverse of progress the distance function in output oriented value ($P_o^*(g)$). To calculated the attractive the objective function ($\varphi_o^*(d)$) the model (5) the same amount of space in the output function is computed to its next performance boundaries. According to the definition $A_o^*(d) = \frac{1}{\varphi_o^*(d)}$ the fraction numerator Malmquist index shows the attractive.

Therefore we can say Malmquist index is multiplied to the geometric mean of attractive and progress units on layer performance.

6. Example

We apply our approach to the data set of 37 computer printers [6], 1 input and 6 outputs were employed. Price (I_1) (in US dollars) as the single input. The following measures are chosen as outputs: (1) input buffer (O_1); (2) mean time between failure (MTBF) (O_2); (3) 80-column throughput (O_3); (4) graphics throughput (O_4); (5) sound level (O_5) and (6) print quality (O_6). Table 1 provides the data for example .

By using the CCR model (1), we obtain 4 levels of efficiency frontiers. They are:

$E^1 = \{DMU_j | j = 1, 2, 3, 5, 19, 20, 26\}$
$E^2 = \{DMU_j | j = 4, 7, 10, 11, 12, 15, 31\}$
$E^3 = \{DMU_j | j = 6, 8, , 9, 13, 22, 27, 30\}$
$E^4 = \{DMU_j | j = 14, 16, 17, 18, 21, 23, 24, 25, 28, 29, 32\}$

Table 1.
Data for the 23 *printer*

Printer name	DMU no.	I_1	O_1	O_2	O_3	O_4	O_5	O_6
Epson LQ-500	1	499	8	4000	101	850	72	5
NEC p2200	2	499	8	4000	85	830	72	5
Seikosha SL-80AL	3	549	16	3200	56	451	68	4
Copal WH 6700	4	795	50	4000	102	450	69	3
Epson LQ-850	5	799	38	4000	148	1350	71	7
Printronix P1013	6	895	2	4000	107	683	78	6
Panasonic KX-P1524	7	899	45	4000	107	850	75	7
Brother M-1724L	8	949	32	4000	107	931	72	5
Citizen Tribute 224	9	949	24	5000	122	917	73	6
ALPS AlQ 324	10	995	71	5000	105	562	69	6
Fujitsu DL3400	11	995	24	8000	146	1440	63	7
NEC P7	12	995	50	5000	111	1255	65	6
Sanyo Pr-241	13	999	10	8000	90	955	68	6
Dataproducts 9044	14	1099	32	5000	121	687	72	5
Epson LQ-1050	15	1099	48	6000	147	1367	71	7
Facit B3450	16	1245	16	4000	134	1090	72	5
C. Itoh C-715A	17	1295	32	7200	131	1186	74	7
nissho NP-2405	18	1295	36	6000	139	650	72	7
ALPS P2400C	19	1395	256	6000	146	1000	70	7
Okidata microline 393	20	1399	30	4000	184	2400	67	9
Epson LQ-2500	21	1449	40	6000	128	1459	70	6
Fujitsu DL2600	22	1495	80	6000	146	1588	69	8
NEC P5XL	23	1495	40	7000	132	1421	68	7
Radio Shack DMP-2120	24	1599	64	3000	150	465	68	7
AT T 477	25	1695	80	6000	146	1301	69	7
Hewlett-Packard RW 480	26	1695	36	20000	191	542	69	15
Nissho NP-2410	27	1745	54	6000	169	683	71	12
NEC P9XL	28	1795	48	7000	170	1928	68	8
Mannesmann Tally MT330	29	1799	32	4800	205	1069	63	7
C.Itoh C-815	30	1995	42	7200	182	2823	72	10
Fujitsu dL5600	31	2195	24	8000	236	3176	68	12
Japan Dgtl.Labs lDL-850	32	2495	128	4000	169	497	63	9

Table 2 report the Malmquist index for layers with model 13:

Table 2.
The layers Malmquist index

	$I^{E_1E_2}$	$I^{E_1E_3}$	$I^{E_1E_4}$	$I^{E_2E_3}$	$I^{E_2E_4}$	$I^{E_3E_4}$
Malmquist index	1.4	105	1.8	1.1	1.3	1.2

In this table, the $I^{E^{k_1}E^{k_2}}$ is the Malmquist index for the frontier E^{k_1} and frontier E^{k_2}, and interpretation in this manner the others value in this row.
Table 3 reports the attractive and progress score for the library in E^{k_1} and E^{k_2}.

Table 3.
Attractiveness and progress score for the first and second frontiers

DMU no.	$D^{E^{k_2}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}}) = A_o^*(1)$	$D^{E^{k_3}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}}) = A_o^*(2)$	$D^{E^{k_4}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}}) = A_o^*(3)$
1	1.75205	1.83201	2.31503
2	1.7433	1.81594	2.30121
3	1.44483	1.52703	1.89061
5	1.33577	1.59209	1.81572
19	2.57176	3.42939	1.26133
20	1.18537	1.31248	1.59717
26	1.46755	1.64944	2.12225
	$D^{E^{k_1}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}}) = \frac{1}{P_o^*(1)}$	$D^{E^{k_3}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}}) = A_o^*(1)$	$D^{E^{k_4}}(X_j^{E^{k_2}}, Y_j^{E^{k_2}}) = A_o^*(2)$
4	0.7967	1.43261	1.64116
7	0.8544	1.23076	1.4506
10	0.7893	1.39515	1.58585
11	0.9883	1.3685	1.53108
12	0.8628	1.18414	1.41272
15	0.8333	1.18991	1.38622
31	0.8466	1.09066	1.34711

Each one of columns of Table 3, is the represent a distance function and the value of attractiveness and progress. For example the first value of $D^{E^{k_2}}(X_j^{E^{k_1}}, Y_j^{E^{k_1}})$ column, means 1.75205 is the distance function of DMU one on the first efficiency frontier (E^{k_1}) relative to the DMUs on the second frontier (E^{k_2}). that is the attractiveness of DMU one relative to the second efficiency frontier DMUs.

7. Conclusion

In this paper, using layerring efficiency frontiers, we obtained some categories to calculate the Malmquist index group, so that the grouping of units by one manager has been in previous articles. With this classification, in addition to low computational process, we showed that each of the functions and denominator if the relationship between Malmquist index shows the attractive and progress. In fact, Malmquist index layer is the geometric average of attractive and progress.

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