Integral Closure of a Filtration Relative to an Injective Module

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Abstract. In this paper we will introduce the integral closure of a filtration relative to an injective module.

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1 Introduction

Throughout this paper $R$ denotes a commutative Noetherian ring with identity. Further $\mathbb{N}$ and $\mathbb{N}_0$ will denote the set of natural integers and non-negative integers respectively. Also $\mathbb{Z}$ will denote the set of integer numbers. Further $E$ is an injective $R$–module.

The ideas of reduction and integral closure of an ideal in a commutative Noetherian ring $R$ (with identity) were introduced by Northcott and Rees in [3]. It is appropriate for us to recall these definitions.

Let $I$ and $J$ be ideals of a commutative Noetherian ring $R$. The ideal $I$ is a reduction of the ideal $J$ if $I \subseteq J$ and there exists an integer $n \in \mathbb{N}$ such that $IJ^n = J^{n+1}$. Also an element $x$ of $R$ is said to be integrally

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dependent on $I$ if there exist a positive integer $n$ and elements $c_k \in I^k$, $k = 1, ..., n$, such that

$$x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n = 0.$$ 

We know from [3], $x \in R$ is integrally dependent on $I$ if and only if $I$ is a reduction of the ideal $I + Rx$. Further, we know that the set of all elements of $R$ which are integrally dependent on $I$ is an ideal of $R$. This ideal is called the integral closure of $I$ and is denoted by $I^\ast$.


Let $I$ and $J$ be ideals of $R$. The ideal $I$ is said to be a reduction of the ideal $J$ relative to $E$, if $I \subseteq J$ and there exists an integer $n \in \mathbb{N}$ such that $(0:_{E}IJ^m) = (0:_{E}J^{n+1})$. Also an element $x$ of $R$ is said to be integrally dependent on $I$ relative to an injective $R$-module $E$, if there exists a positive integer $n$ such that

$$(0:_{E}\sum_{i=1}^{n}x^{n-i}I^n) \subseteq (0:_{E}x^n).$$

We know from [2], an element $x$ of $R$ is integrally dependent on $I$ relative to an injective $R$-module $E$, if and only if $I$ is a reduction of the ideal $I + Rx$ relative to $E$. Moreover in [2], it is shown that the set of all elements of $R$ which are integrally dependent on $I$ relative to $E$ is an ideal of $R$. This is denoted by $I^\ast(E)$ and is called the integral closure of $I$ relative to $E$.

Here, we give some definitions and notations which will be helpful for us in the rest of the paper.

A filtration $F = \{I_n\}_{n \geq 0}$ on $R$ is a descending sequence of ideals $I_n$ of $R$ such that $I_0 = R$ and $I_nI_m \subseteq I_{n+m}$ for all $n, m \in \mathbb{N}_0$. Let $F = \{I_n\}_{n \geq 0}$ and $G = \{J_n\}_{n \geq 0}$ be two filtrations. We say $F \subseteq G$ if $I_n \subseteq J_n$ for every $n$. Also the filtration $\{I_nJ_n\}_{n \geq 0}$ is denoted by $FG$.

The integral closure of a filtration $F = \{I_n\}_{n \geq 0}$ is defined in [4]. For every $n \geq 0$, let $J_n$ be the set of all $x \in R$ such that $x$ satisfies an equation

$$x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k = 0$$
for a positive integer \( k \) and \( a_i \in I_{n_i} \). Then \( \mathcal{F}^{-} = \{ J_n \}_{n \geq 0} \) is a filtration such that \( \mathcal{F} \subseteq \mathcal{F}^{-} \). The filtration \( \mathcal{F}^{-} = \{ J_n \}_{n \geq 0} \) is called the integral closure of the filtration \( \mathcal{F} = \{ I_n \}_{n \geq 0} \).

In this paper we will introduce the integral closure of a filtration relative to an injective module and study some related topics.

2 Auxiliary results

In this section we define the concepts of reduction and integral closure of a filtration relative to injective modules and prove some of their properties. We begin to remind some definitions.

**Definition 2.1.** (See [5, 2.1.3].) Let \( \mathcal{F} = \{ I_n \}_{n \geq 0} \) and \( \mathcal{G} = \{ J_n \}_{n \geq 0} \) be filtrations on \( R \). \( \mathcal{F} \) is said to be a reduction of \( \mathcal{G} \) if \( \mathcal{F} \subseteq \mathcal{G} \) and there exists a positive integer \( d \) such that

\[
J_n = \sum_{i=0}^{d} I_{n-i}J_i \quad \text{for every } n \geq 1.
\]

Here, and throughout this paper, \( I_i = R \) if \( i \leq 0 \).

**Definition 2.2.** (See [5, 2.1.4].) Let \( R \) be a Noetherian ring. A filtration \( \mathcal{F} = \{ I_n \}_{n \geq 0} \) on \( R \) is Noetherian in case there exists a positive integer \( d \) such that

\[
I_n = \sum_{i=0}^{d} I_{n-i}I_i \quad \text{for every } n \geq 1.
\]

**Definition 2.3.** Let \( \mathcal{F} = \{ I_n \}_{n \geq 0} \) and \( \mathcal{G} = \{ J_n \}_{n \geq 0} \) be filtrations on \( R \). Then \( \mathcal{F} \) is said to be a reduction of \( \mathcal{G} \) relative to an injective \( R \)-module \( E \) if \( \mathcal{F} \subseteq \mathcal{G} \) and there exists a positive integer \( d \) such that

\[
(0 :_E J_n) = (0 :_E \sum_{i=0}^{d} I_{n-i}J_i) \quad \text{for every } n \geq 1.
\]

**Remark 2.4.** Let \( \mathcal{F} = \{ I_n \}_{n \geq 0} \) and \( \mathcal{G} = \{ J_n \}_{n \geq 0} \) be filtrations on \( R \). Let \( \mathcal{F} \) be a reduction of \( \mathcal{G} \) relative to an injective \( R \)-module \( E \). Then
there exists a positive integer $d$ such that

$$(0 :_E J_n) = (0 :_E \sum_{i=0}^{d} I_{n-i}J_i) \text{ for every } n \geq 1.$$ 

Let $d < d'$. Since $\sum_{i=d+1}^{d'} I_{n-i}J_i \subseteq J_n$, we have

$$(0 :_E J_n) = (0 :_E \sum_{i=0}^{d} I_{n-i}J_i) \cap (0 :_E \sum_{i=d+1}^{d'} I_{n-i}J_i) = (0 :_E \sum_{i=0}^{d'} I_{n-i}J_i).$$

**Theorem 2.5.** (See [2, 1.3]). Let $\mathcal{F} = \{I_n\}_{n \geq 0}, \mathcal{G} = \{J_n\}_{n \geq 0}, \mathcal{H} = \{H_n\}_{n \geq 0},$ and $\mathcal{K} = \{K_n\}_{n \geq 0}$ be filtrations on $R$ and let $E$ be an injective $R$–module.

(a) If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to $E$ then $\mathcal{G}$ is a reduction of $\mathcal{H}$ relative to $E$.

(b) If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to $E$ and $\mathcal{G}$ is a reduction of $\mathcal{H}$ relative to $E$ then $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to $E$.

(c) If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to $E$ and $\mathcal{H}$ is a reduction of $\mathcal{K}$ relative to $E$ then $\mathcal{F}\mathcal{H}$ is a reduction of $\mathcal{G}\mathcal{K}$ relative to $E$.

**Proof.** (a) and (b) are clear.

(c) Since $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to $E$ and $\mathcal{H}$ is a reduction of $\mathcal{K}$ relative to $E$ then there are two positive integers $d,d'$ such that for every $n \geq 1$,

$$(0 :_E J_n) = (0 :_E \sum_{i=0}^{d} I_{n-i}J_i)$$

and

$$(0 :_E K_n) = (0 :_E \sum_{i=0}^{d'} H_{n-i}K_i).$$

By Remark 2.4, we can assume $d = d'$. Then for every $n \geq 1$, we have

$$(0 :_E J_nK_n) = ((0 :_E J_n) :_E K_n) = ((0 :_E \sum_{i=0}^{d} I_{n-i}J_i) :_E K_n).$$
\[(0 : E K_n) : E \sum_{i=0}^{d} I_{n-i}J_i) \]

\[= ((0 : E \left( \sum_{i=0}^{d} I_{n-i}J_i \right)) \left( \sum_{t=0}^{d} H_{n-t}K_t \right)).\]

It is easy to see that \(\left( \sum_{i=0}^{d} I_{n-i}J_i \right) \left( \sum_{t=0}^{d} H_{n-t}K_t \right) = \sum_{i=0}^{d} I_{n-i}H_{n-i}J_iK_i.\) Thus we have

\[(0 : E J_nK_n) = (0 : E \sum_{i=0}^{d} I_{n-i}H_{n-i}J_iK_i) \quad \text{for every } n \geq 1\]

and so \( \mathcal{KH} \) is a reduction of \( \mathcal{GK} \) relative to \( E. \) \( \square \)

Now we mention a useful notation from [2]. Let \( I \) be an ideal of \( R. \) For a subset \( P \) of \( \text{Spec}(R), \) the notation \( I(P) \) denotes \( I \) if \( I = R \) and, if \( I \) is proper, the intersection of those primary terms in a minimal primary decomposition of \( I \) which are contained in at least one member of \( P. \) We know \( I(P) = \bigcap_{P \in P} I(\{P\}) \) we shall abbreviate \( I(\{P\}) \) (for \( P \in \text{Spec}(R) \)) by \( I(P). \) Note that \( I(P) \) is just the contraction back to \( R \) of the extension of \( I \) to \( R_P \) under the natural ring homomorphism.

**Remark 2.6.** (See [2, 1.6].) Let \( P \in \text{Spec}(R), I \) and \( J \) be ideals of \( R. \) Let \( E = E(R/P). \) Then the following statements are equivalent:

(a) \( (0 : E I) \subseteq (0 : E J); \)

(b) \( IR_P \subseteq JR_P; \)

(c) \( I(P) \subseteq J(P). \)

Let \( \mathcal{F} = \{I_n\}_{n \geq 0} \) be a filtration on \( R. \) For every prime ideal \( P \) of \( R, \) \( \{I_nR_P\}_{n \geq 0} \) is a filtration on \( R_P. \) We will denote this filtration on \( R_P \) by \( \mathcal{F}_P. \)

**Lemma 2.7.** Let \( \mathcal{F} = \{I_n\}_{n \geq 0} \) be a Noetherian filtration on \( R. \) Then for every prime ideal \( P \) of \( R, \)

\[(\mathcal{F}_P)^- = (\mathcal{F}^-)_P. \]
Proof. Let $\mathcal{F}^- = \{U_n\}_{n \geq 0}$ and $(\mathcal{F}_P)^- = \{H_n\}_{n \geq 0}$ where $\mathcal{F}^-$ and $(\mathcal{F}^-)_P$ are filtrations on $R$ and $R_P$ respectively. Let $\bar{x} \in U_n R_P$. Then there exist $u \in U_n$ and $t \in R \setminus P$ such that $\bar{x} = \frac{u}{t}$. Thus there exists $s \in R \setminus P$ such that $su = stx$. Since $u \in U_n$ we have $u^k \in \sum_{i=1}^{k} u^{k-i} I_{ni}$ and so 

$$(su)^k \in \sum_{i=1}^{k} (su)^{k-i} I_{ni}$$

for a positive integer $k$. But $su = stx$ implies that 

$$(stx)^k \in \sum_{i=1}^{k} (stx)^{k-i} I_{ni} R_P$$. Now by Remark 2.6 and $\sum_{i=1}^{k} (stx)^{k-i} I_{ni} \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni}$, we have 

$$(0 : E(R/P) \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 : E(R/P) \sum_{i=1}^{k} (stx)^{k-i} I_{ni}) \subseteq (0 : E(R/P) (stx)^k)$$. 

Now since $s, t \in R \setminus P$ we can see that 

$$(0 : E(R/P) \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 : E(R/P) x^k)$$. So by Remark 2.6, $(\bar{x})^k \in \sum_{i=1}^{k} (\bar{x})^{k-i} I_{ni} R_P$. In other words $\bar{x} \in H_n$ and so $(\mathcal{F}^-)_P \subseteq (\mathcal{F}_P)^-$. 

Conversely, let $\bar{x} = \frac{x}{t} \in H_n$. Then $(\bar{x})^k \in \sum_{i=1}^{k} (\bar{x})^{k-i} I_{ni} R_P$ for a positive integer $k$. Then there are $a_1 \in I_{n1}, \ldots, a_k \in I_{nk}$ and $s_1, \ldots, s_k \in R \setminus P$ such that 

$$(\frac{x}{1})^k + \frac{a_1}{s_1} (\frac{x}{1})^{k-1} + \cdots + \frac{a_{k-1}}{s_{k-1}} (\frac{x}{1})^1 + \frac{a_k}{s_k} = 0.$$ 

Let $s = s_1 \ldots s_k$. Then there exists $t \in R \setminus P$ such that 

$$(tsx)^k \in \sum_{i=1}^{k} (tsx)^{k-i} I_{ni}.$$ 

This shows $tsx \in U_n$. But $\bar{x} = \frac{tsx}{ts} \in U_n R_P$ and so $(\mathcal{F}_P)^- \subseteq (\mathcal{F}^-)_P$ and this completes the proof. □
By well-known work of Matlis and Gabriel, we know for every injective $R$-module $E$, there is a family $\{P_\lambda : \lambda \in \Lambda\}$ of prime ideals of $R$ such that $E = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ (we use $E(L)$ to denote the injective envelope of an $R$-module $L$). Further we know the set $\{P_\lambda : \lambda \in \Lambda\}$ is the set of all associated prime ideals of $R$ which is denoted by $Ass_R(E)$.

**Remark 2.8.** Let $E = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ be an injective $R$-module. Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a filtration on $R$. Let $U_n$ be the set of all $x \in R$ such that

$$ (0 : E \sum_{i=1}^k x^{k-i}I_{ni}) \subseteq (0 : E x^k) $$

for a positive integer $k$. Since $R$ is a Noetherian ring, $Ass_R(E)$ is a finite set. We know

$$ (0 : E \sum_{i=1}^k x^{k-i}I_{ni}) \subseteq (0 : E x^k) $$

for the positive integer $k$ if and only if for every $P \in Ass_R(E)$,

$$ (0 : E(R/P) \sum_{i=1}^k x^{k-i}I_{ni}) \subseteq (0 : E(R/P) x^k). $$

But by Remark 2.6, for every $P \in Ass_R(E)$,

$$ (0 : E(R/P) \sum_{i=1}^k x^{k-i}I_{ni}) \subseteq (0 : E(R/P) x^k) $$

for the positive integer $k$ if and only if

$$ (\frac{x}{1})^k \in \sum_{i=1}^k (\frac{x}{1})^{k-i}I_{ni}R_P. $$

By Lemma 2.7, we have $(\mathcal{F}_P)^- = (\mathcal{F}^-)_P$. Let $\mathcal{F}^- = \{J_n\}_{n \geq 0}$. Then we see $x \in U_n$ if and only if $\frac{x}{1} \in J_nR_P$ for every $P \in Ass_R(E)$. Since $(\mathcal{F}_P)^- = \{J_nR_P\}_{n \geq 0}$ is a filtration of ideals on $R_P$, it is easy to see that $\{U_n\}_{n \geq 0}$ is a filtration of ideals on $R$. 
**Definition 2.9.** Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a filtration on $R$ and let $E$ be an injective $R$–module. For every $n \geq 0$, we assume that $U_n$ contains all $x \in R$ such that

$$(0 : E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 : E x^k)$$

for a positive integer $k$. By Remark 2.8, we know $\{U_n\}_{n \geq 0}$ is a filtration on $R$. This filtration is denoted by $\mathcal{F}^*(E)$ and is called the integral closure of a filtration $\mathcal{F} = \{I_n\}_{n \geq 0}$ relative to an injective $R$–module $E$. By Remark 2.8, we can see $(\mathcal{F}^*(E))_P = (\mathcal{F}^*)_P$ for every $P \in \text{Ass}_R(E)$.

**Theorem 2.10.** Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a filtration on $R$ and let $E$ be an injective $R$–module. Let $\mathcal{F}^*(E) = \{U_n\}_{n \geq 0}$. Further for a non negative integer $n$ and $x \in R$, let $L_k = Rx^k + x^{k-1} I_{n1} + x^{k-2} I_{n2} + \cdots + x I_{n(k-1)} + I_{nk}$ and $H_k = I_{nk}$. Then $x \in U_n$ if and only if the filtration $\{H_k\}_{k \geq 0}$ is a reduction of filtration $\{L_k\}_{k \geq 0}$ relative to $E$.

**Proof.** ($\Rightarrow$) Let $x \in U_n$. Then there exists a positive integer $k$ such that

$$(0 : E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 : E x^k).$$

Since $x^{k-i} I_{ni} \subseteq H_{k-(k-i)} L_{k-i}$ for every $1 \leq i \leq k$,

$$(0 : E \sum_{i=0}^{k} H_{k-i} L_i) = (0 : E \sum_{i=0}^{k} H_{k-(k-i)} L_{k-i}) \subseteq (0 : E \sum_{i=1}^{k} x^{k-i} I_{ni}).$$

But $(0 : E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 : E x^k)$ and so

$$(0 : E \sum_{i=0}^{k} H_{k-i} L_i) \subseteq (0 : E \sum_{i=0}^{k} x^{k-i} I_{ni})) = (0 : E L_k).$$
Also we know \( \sum_{i=0}^{k} H_{k-i}L_i \subseteq L_k \). Then
\[
(0 : E L_k) = (0 : E \sum_{i=0}^{k} H_{k-i}L_i).
\]

Now, we will show that
\[
(0 : E L_t) = (0 : E \sum_{i=0}^{k} H_{t-i}L_i) \quad \text{for every } t \geq 1.
\]
First let \( t < k \). Since \( t < k \),
\[
(0 : E \sum_{i=0}^{k} H_{t-i}L_i) \subseteq (0 : E H_0L_t) = (0 : E L_t).
\]
Also we know \( \sum_{i=0}^{k} H_{t-i}L_i \subseteq L_t \). Thus we have
\[
(0 : E L_t) = (0 : E \sum_{i=0}^{k} H_{t-i}L_i) \quad \text{for every } t < k.
\]
Now let \( t > k \). This is clear that \( \sum_{i=0}^{k} H_{t-i}L_i = \sum_{i=0}^{k} x^i I_{n(t-i)} \) and so
\[
(0 : E \sum_{i=0}^{k} H_{t-i}L_i) = (0 : E \sum_{i=0}^{k} x^i I_{n(t-i)})
\]
Since \( (0 : E \sum_{i=1}^{k} x^{k-i} I_{n_i}) \subseteq (0 : E x^k) \), we can see that
\[
(0 : E \sum_{i=1}^{k} x^{k-i} I_{n(r+i)}) \subseteq (0 : E x^{k+r}).
\]
But by
\[
(0 : E x^{k+r} I_{n(t-(k+r))}) \supseteq (0 : E \sum_{i=1}^{k} x^{k-i} I_{n(r+i)} I_{n(t-(k+r))})
\]
we have

\[ (0 : E L_t) = (0 : E \sum_{i=0}^{k} x^i I_{n(t-i)}) \cap (0 : E \sum_{i=0}^{k} x^i I_{n(t-i)}) \]

Then

\[ (0 : E L_t) = (0 : E \sum_{i=0}^{k} H_{t-i} L_i) \quad \text{for every } t \geq 1. \]

(⇐) Let \( \{H_k\}_{k \geq 0} \) be a reduction of filtration \( \{L_k\}_{k \geq 0} \) relative to \( M \). Then there exists a positive integer \( d \) such that

\[ (0 : E L_k) = (0 : E \sum_{i=0}^{d} H_{k-i} L_i) \quad \text{for every } k \geq 1. \]

Particularly, we have \( (0 : E L_{d+1}) = (0 : E \sum_{i=0}^{d} H_{d+1-i} L_i) \). But by

\[ \sum_{i=0}^{d} H_{d+1-i} L_i = \sum_{i=0}^{d} x^i I_{n(d+1-i)} \subseteq \sum_{i=0}^{d} x^i I_{n(d-i)} \]

we have

\[ (0 : E x^{d+1}) \supseteq (0 : E L_{d+1}) = (0 : E \sum_{i=0}^{d} H_{d+1-i} L_i) \supseteq (0 : E \sum_{i=0}^{d} x^i I_{n(d-i)}). \]

Hence \( x \in U_n. \) \( \square \)

The following theorem shows that \( \mathcal{F} \to \mathcal{F}^*(E) \), is a semi-prime operation.

**Theorem 2.11.** (See [4, 2.4].) Let \( \mathcal{F} = \{I_n\}_{n \geq 0} \) and \( \mathcal{G} = \{J_n\}_{n \geq 0} \) be filtrations on \( R \). Then for every injective \( R \)-module \( E \), we have
Thus there are $P$ for every $x$ such that for $t$

Proof. (a) and (b) are clear.

(c) By (a) and (b), we have $F^{*}(E) \subseteq (F^{*}(E))^{*}(E)$. Let $F = \{I_n\}_{n \geq 0}$, $F^{*}(E) = \{U_n\}_{n \geq 0}$, $F^{-} = \{J_n\}_{n \geq 0}$ and $(F^{*}(E))^{*}(E) = \{K_n\}_{n \geq 0}$. Let $x \in K_n$. By Lemma 2.7 and Remark 2.8, we have

$((F^{*}(E))^{*}(E))_P = (((F^{*}(E))^{-})_P = (((F^{*}(E)))_P^{-}

= ((F^{-})_P^{-} = ((F^{-})_P^{-})$

for every $P \in \text{Ass}_R(E)$. Also we know from [4, 2.4.3], $(F^{-})^{-} = F^{-}$. Thus there are $a \in J_n$ and $s \in R \setminus P$ such that $\frac{x}{t} = \frac{\alpha}{\beta}$. Hence $t x y = t a$ for $t \in R \setminus P$. Since $t x y = t a \in J_n$, there exists a positive integer $k$ such that $(t x y)^k \in \bigcup_{i=1}^{k} (t x y)^{k-i} I_{ni}$. Now since $t s y \in R \setminus P$, it is easy to see that

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} x^k)$$

for every $P \in \text{Ass}_R(E)$. Then $x \in U_n$ and so $(F^{*}(E))^{*}(E) \subseteq F^{*}(E)$. This follows (c).

(d) Let $F^{*}(E) = \{U_n\}_{n \geq 0}$ and $G^{*}(E) = \{V_n\}_{n \geq 0}$. Let $x \in U_n$ and $y \in V_n$. Further let $P \in \text{Ass}_R(E)$. We have

$$(F^{*}(E))_P(G^{*}(E))_P = (F^{-})_P(G^{-})_P = (F^{-}G^{-})_P.$$

But by [4, 2.4.4], we know $F^{-}G^{-} \subseteq (F G)^{-}$. This shows if $(F G)^{-} = \{H_n\}_{n \geq 0}$ then $\frac{t}{\alpha} y \frac{u}{t} \in H_n R P$. Thus there are $a \in H_n$ and $s, t \in R \setminus P$ such that $t x y = t a$. Since $t a \in H_n$, there is a positive integer $k$ such that $(s t x y)^k \in \bigcup_{i=1}^{k} (s t x y)^{k-i} I_{ni} J_{ni}$. This implies that

$$(0 :_{E(R/P)} \sum_{i=1}^{k} (xy)^{k-i} I_{ni} J_{ni}) \subseteq (0 :_{E(R/P)} \sum_{i=1}^{k} (ts xy)^{k-i} I_{ni} J_{ni})$$
Since $s, t \in R \setminus P$, for every $P \in \text{Ass}_R(E)$ we have

$$(0 :_{E(R/P)} (tsxy)^k) \subseteq (0 :_{E(R/P)} (xy)^k).$$

Then $$(0 :_{E} \sum_{i=1}^{k} (xy)^{k-i} I_{ni}J_{ni}) \subseteq (0 :_{E} (xy)^k)$$ and so if $(FG)^{\ast(E)} = \{W_n\}_{n \geq 0}$ then $xy \in W_n$. □

3 Main results

Let $I$ be an ideal of $R$ and $E$ be an injective $R$–module. In [2], it is shown that $I^{\ast(E)} = I^{-}(\text{Ass}_R(E))$. In this section we will prove a similar theorem for the integral closure of a filtration $F$ relative to an injective $R$–module $E$. First we introduce the following notation.

Let $F = \{I_n\}_{n \geq 0}$ be a filtration on $R$ and let $P$ be a subset of $\text{Spec}(R)$. For every $P \in P$, we have

$I_n(P)I_m(P) \subseteq I_n(P)I_m(P) = (I_nR_P)^c(I_mR_P)^c \subseteq (I_nI_mR_P)^c$

\[\subseteq (I_{n+m}R_P)^c = I_{n+m}(P).\]

Then

$I_n(P)I_m(P) \subseteq \bigcap_{P \in P} I_{n+m}(P) = I_{n+m}(P)$.

This shows that $\{I_n(P)\}_{n \geq 0}$ is a filtration on $R$. We denote this filtration by $F(P)$.

Now we are ready to prove the main proposition of this section.

**Theorem 3.1.** (See [2, 2.6].) Let $F = \{I_n\}_{n \geq 0}$ be a filtration on $R$ and let $E$ be an injective $R$–module. Then $F^{\ast(E)} = F^{-}(\text{Ass}_R(E))$.

**Proof.** Let $F^{\ast(E)} = \{U_n\}_{n \geq 0}$ and $F^{-} = \{J_n\}_{n \geq 0}$. We will show $U_n = J_n(\text{Ass}_R(E))$ for every $n$. 

Let $x \in U_n$. Then for every $P \in \text{Ass}_R(E)$, $\bar{x} \in J_n R_P$. Let $J_n = Q_1 \cap \cdots \cap Q_k$ be a minimal primary decomposition of $J_n$. Let

$$\sqrt{Q_i} \cap P = \emptyset \text{ for every } 1 \leq i \leq l$$

and

$$\sqrt{Q_i} \cap P \neq \emptyset \text{ for every } l + 1 \leq i \leq k.$$  

Then $\bar{x} \in J_n R_P = (Q_1 \cap \cdots \cap Q_l) R_P$ and so there are $y \in Q_1 \cap \cdots \cap Q_l$ and $s, t \in R \setminus P$ such that $stx = sy \in Q_1 \cap \cdots \cap Q_l$. Since $st \notin \sqrt{Q_i}$ for every $1 \leq i \leq l$, $x \in Q_1 \cap \cdots \cap Q_l$ and so $x \in J_n(P)$ for every $P \in \text{Ass}_R(E)$. Hence $x \in J_n(\text{Ass}_R(E))$ and so $U_n \subseteq J_n(\text{Ass}_R(E))$. For converse inclusion, let $x \in J_n(P)$ for every $P \in \text{Ass}_R(E)$. Then $\bar{x} \in J_n R_P$ for every $P \in \text{Ass}_R(E)$. Hence there are $y \in J_n$ and $s, t \in R \setminus P$ such that $stx = sy \in J_n$. Then there is a positive integer $k$ such that $(stx)^k \in \sum_{i=1}^k (tsx)^{k-i} I_{ni}$. By $s, t \in R \setminus P$ we can see

$$(0 :_{E(R/P)} \sum_{i=1}^k x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} \sum_{i=1}^k (tsx)^{k-i} I_{ni})$$

$$\subseteq (0 :_{E(R/P)} (stx)^n)$$

$$\subseteq (0 :_{E(R/P)} x^n).$$

Thus we have

$$(0 :_{E(R/P)} \sum_{i=1}^k x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} x^n)$$

for every $P \in \text{Ass}_R(E)$. Thus $x \in U_n$ and so $J_n(\text{Ass}_R(E)) \subseteq U_n$. This follows $U_n = J_n(\text{Ass}_R(E))$. \hfill \Box

**Definition 3.2.** (See [4, 3.1(2)].) Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a filtration on $R$ and $\mathcal{F}^- = \{J_n\}_{n \geq 0}$. Members of

$$A^-(\mathcal{F}) = \{P : P \in \text{Ass}(R/J_n) \text{ for some } n \geq 1\}$$

are called the asymptotic prime divisors of $\mathcal{F}$. 

Let $E$ be an injective $R$–module. We know from [1, 2.2], for each ideal $I$ of $R$, the module $(0 :_E I)$ has a secondary representation, and so we can form the finite set of prime ideals $\text{Att}_R(0 :_E I)$. In fact,

$$\text{Att}_R(0 :_E I) = \{ P' \in \text{ass}(I) : P' \subseteq P \text{ for some } P \in \text{Ass}_R(E) \}.$$  

**Definition 3.3.** Let $\mathcal{F} = \{ I_n \}_{n \geq 0}$ be a filtration on $R$ and $E$ be an injective $R$–module. Let $\mathcal{F}^*(E) = \{ U_n \}_{n \geq 0}$. We will show the set

$$\{ P : P \in \text{Att}(0 :_E U_n) \text{ for some } n \geq 1 \}$$

by $\text{At}^*(\mathcal{F}, E)$.

**Theorem 3.4.** (See [2, 3.2].) Let $\mathcal{F} = \{ I_n \}_{n \geq 0}$ be a Noetherian filtration on $R$. Let $E$ be an injective $R$–module. Then $\text{At}^*(\mathcal{F}, E)$ is a finite set.

**Proof.** Let $\mathcal{F}^*(E) = \{ U_n \}_{n \geq 0}$ and $\mathcal{F}^- = \{ J_n \}_{n \geq 0}$. By Note 3, we know

$$\text{At}^*(\mathcal{F}, E) = \{ P' \in \text{ass}(U_n) : P' \subseteq P \text{ for some } P \in \text{Ass}_R(E) \}.$$  

But we know from Theorem 3.1, $\mathcal{F}^*(E) = \mathcal{F}^-(\text{Ass}_R(E))$. Then

$$\text{At}^*(\mathcal{F}, E) = \{ P' \in \text{ass}(J_n(\text{Ass}_R(E))) : P' \subseteq P \text{ for some } P \in \text{Ass}_R(E) \}$$

$$= \{ P' \in \text{A}^- (\mathcal{F}) : P' \subseteq P \text{ for some } P \in \text{Ass}_R(E) \}.$$  

Now the proof is completed by [4, 3.3]. \qed

**References**


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