

## A Heuristic Algorithm for Constrain Single-Source Problem with Constrained Customers

**S. A. Raisi Dehkordi\***

University of Isfahan

**M. Farahani**

University of Isfahan

**A. Ahmadi**

University of Isfahan

**Abstract.** The Fermat-Weber location problem is to find a point in  $\mathbb{R}^n$  that minimizes the sum of the weighted Euclidean distances from  $m$  given points in  $\mathbb{R}^n$ . In this paper we consider the Fermat-Weber problem of one new facility with respect to  $n$  unknown customers in order to minimizing the sum of transportation costs between this facility and the customers. We assumed that each customer is located in a nonempty convex closed bounded subset of  $\mathbb{R}^n$ .

**AMS Subject Classification:** 90A80; 90C30

**Keywords and Phrases:** Convex, location, allocation, closed, global convergence

### 1. Introduction

In this work we concentrate on presenting a new Weber problem whose customers are unknown and restricted into some specific regions. A popular iterative solution method for the Weber location problem was first introduced by Weiszfeld in 1937. In 1973 Kuhn claimed that if the  $d$

---

Received: March 2011; Accepted: August 2012

\*Corresponding author

given points are not collinear, then for all but a denumerable number of starting points the sequence of iterates generated by Weiszfeld's scheme converges to the unique optimal solution. Let  $a_1, \dots, a_d$  be  $d$  distinct points in  $\mathbb{R}^n$ . Suppose that each point  $a_j, 1 \leq j \leq d$  is associated with a positive weight  $s_j$ . The Fermat-Weber location problem ([17,24]) is to find a point in  $\mathbb{R}^n$  that will minimize the sum of the (weighted) Euclidean distances from the  $a_1, \dots, a_d$ :

$$\min f(x) = \sum_{j=1}^d s_j \|x - a_j\|. \quad (1)$$

It is well known that if the data points are not collinear, the objective is strictly convex, and therefore has a unique optimum. (In the collinear case at least one of the points  $a_1, \dots, a_d$  is optimal and it can be found in linear time by the algorithm introduced in [2].) There are several infinite schemes to solve the Fermat-Weber location problem (see [3-4,17,22,25]). One of the most popular algorithms was discovered by Weiszfeld [25]. It is analysed extensively in [16] and [18]. The algorithm is based on the following mapping of  $\mathbb{R}^n$  into the convex hull of  $a_1, \dots, a_d$ :

$$T(x) = \begin{cases} \frac{\sum_{j=1}^d s_j \|x - a_j\|^{-1} a_j}{\sum_{j=1}^d \|x - a_j\|^{-1}} & \text{if } x \neq a_1, \dots, a_d, \\ a_j & \text{if } x = a_j \text{ for some } j = 1, \dots, d. \end{cases} \quad (2)$$

Weiszfeld's algorithm is defined by the following iterative scheme;

$$x^{r+1} = T(x^r). \quad (3)$$

Recently, the so-called Newton-Bracketing (NB) method for convex minimization was utilized to solve the involved location phase and thus Cooper-NB algorithm was developed in [19]. Due that the gradients of  $\|x - a_j\|$ , ( $j = 1, 2, \dots, n$ ) are used in the iteration, both the Weiszfeld procedure and NB method share the common characteristic that their implementations may terminate unexpectedly when the current iteration

happens to be identical with some location of the customers. i.e. the singular case happens. How to improve the original Weiszfeld procedure and NB method in the singular case for location phase and allocation phase and terminate controllable become the main challenges in this study. The rest of the paper is organized as follows. Section 2 reformulated the involved location phase. Then we present an algorithm include a location phase and an allocation phase. Moreover, we formulated the involved location phase into linear variational inequalities (LVI) and present an effective method for solving these LVIs. In Section 3, convergence results are presented under mild assumptions. In addition, some numerical results are investigated.

## 2. Problem Structure and Solution Method

In this section we consider the generalized constrained single-source Weber problem (GCSWP) whose mathematical model is as follows:

$$\begin{aligned} GCSWP : \min & \sum_{j=1}^n S_j V P_{H_j}(x), \\ & x \in X, \\ & a_j \in H_j, \quad j = 1, 2, \dots, d, \end{aligned} \quad (4)$$

where

- (1)  $H_j \subset \mathbb{R}^n$  is non-empty closed convex subset in  $\mathbb{R}^n$ ,  $j = 1, 2, \dots, d$ ;
- (2)  $x \in \mathbb{R}^n$  is the location of the facility to be determined;
- (3)  $s_j \geq 0$  is the given demand required by the  $j$ th customer;
- (4)  $a_j$  is the location of the  $j$ th customer to be determined,  $j = 1, 2, \dots, d$ ;
- (5)  $\|\cdot\|$  is the norm produced by inner product in  $\mathbb{R}^n$ ;
- (6)  $X$  is non-empty closed convex subset in  $\mathbb{R}^n$ ;
- (7)  $P_{H_j}(x) = \arg \min \{\|x - u\| \mid u \in H_j\}$ ;

$$(8) VP_{H_j}(x) = \min \{\|x - u\| \mid u \in H_j\};$$

$$(9) \text{ let } A = \{H_1, H_2, \dots, H_d\}, \text{ if } H_i, H_j \in A, i \neq j \text{ then } H_i \cap H_j = \emptyset$$

At the  $(k + 1)$ th iteration, the location phase finds the candidate of location of facility by solving generalized single-source Weber problem (GSWP) denoted by

$$\text{GSWP} : x^{k+1} = \arg \min_{x \in X} \left\{ C(x) := \sum_{j=1}^d s_j \|x - a_j^{k+1}\| \right\}, \quad (5)$$

where  $a_j^{k+1}$  is produced by allocation phase. Then the allocation phase involves an allocation, which depends on the  $x$  generated by solving (5). More specifically, if  $x$  is the nearest facility for each region of  $x^{k+1}$ , then  $x$  is the desirable location of facility. Therefore, it is reasonable to allocate the regions from the facility  $x^{k+1}$  in order to minimize the total sum of transportation costs. The overall solution method for (*GCSWP*) can be outlined as follows:

**Input:** Starting locations  $\{a_1^1, a_2^1, \dots, a_n^1\}$ .

*Step 0.* Set  $t = 0$  ( $t$  is the number of reassignment).

*Step 1.* Location phase:

Solving the involved GSWP (5) and find  $x^{k+1}$ .

*Step 2.* Allocation phase:

For  $j = 1, \dots, n$  do:

if  $x^k \notin H_j$ ,

then  $a_j^{k+1} = P_{H_j}(x^k)$ .

if  $x^k \in H_j$ ,

then  $a_j^{k+1} = x^k + \varepsilon \frac{x^k - x^{k-1}}{\|x^k - x^{k-1}\|}$ ,

for  $\varepsilon$  sufficiently small enough such that  $0 < \varepsilon < \|x^k - x^{k-1}\|$ .

## 2.1 LVI Reformulation of GSWP

At solving the involved GSWP (5) in the location phase by a linear variational inequality (LVI) approach. There are many methods for LVI,

see for example [5,8-14,20-21]. Among these methods, the projection and contraction methods are attractive for their simplicity and efficiently, (see [7]).

**Lemma 2.1.1.** *Let  $a_j^{k+1}$  be the solution of allocation phase in iteration  $k$ . Then  $a_j^{k+1}$  is unique and an element of  $H_j$ .*

**Proof.** In case  $a_j^{k+1} = P_{H_j}(x^k)$ , since  $H_j$  is a compact set and  $\|\cdot\|$  is a continues function, it follows that  $a_j^{k+1} \in H_j$ .

On the other hand,  $H_j$  is a convex set and the distant function is convex function, hence  $a_j^{k+1}$  is unique. In semisingular case, by the [15,Theorem1]  $x^k$  is unique. Hence  $a_j^{k+1}$  is unique.  $\square$

It is easy to verify that in each iterate

$$VP_{H_j}(x^k) = \min \left\{ \|x^k - u\| \mid u \in H_j \right\} = \|x^k - a_j^{k+1}\|. \quad (6)$$

For convenience, we use  $a_k^T = (a_1^{kT}, a_2^{kT}, \dots, a_d^{kT})$ , and  $H = H_1 \times H_2 \times \dots \times H_d$ .

More specifically, if  $x^{k+1}$  is the nearest facility for each regions, then  $x^{k+1}$  is the desirable location of facility and  $a_j^{k+1}$ ,  $j = 1, \dots, n$  are the desirable location of customers.

We define

$$B_w = \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq w\}, \text{ for } w > 0 \quad (7)$$

Note that for any  $r, \xi \in \mathbb{R}^n$ , we get  $r^T \xi \leq \|r\| \|\xi\|$ , and

$$\max_{\xi \in B_w} r^T \xi \leq \max_{\xi \in B_w} \|r\| \|\xi\| = w \|r\|. \quad (8)$$

Since  $\frac{wr}{\|r\|} \in B_w$ , it follows that

$$r^T \frac{wr}{\|r\|} \leq \max_{\xi \in B_w} r^T \xi \Rightarrow w \|r\| \leq \max_{\xi \in B_w} r^T \xi. \quad (9)$$

From (8), (9) we deduce that

$$\max_{\xi \in B_w} r^T \xi = w \|r\|. \quad (10)$$

According to (10), GSWP is equivalent to the following min-max problem:

$$\min_{x \in X} \max_{z_i \in B_{s_i}} \sum_{i=1}^d z_i(x - a_i^{k+1}), \quad (11)$$

where each  $z_i$ , ( $i = 2, \dots, d$ ) is a vector in  $B_{s_i} = \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq s_i\}$ . A compact form of (11) is

$$\min_{x \in \mathbb{R}^n} \max_{z_i \in B_{s_i}} z^T(Ax - b), \quad (12)$$

where

$$\begin{aligned} z^T &= (z_1^T, z_2^T, \dots, z_d^T) \quad , \quad \bar{B} = B_{s_1} \times B_{s_2} \times \dots \times B_{s_d} \\ A &= (I_n, I_n, \dots, I_n)^T \quad , \quad b^T = (a_1^{k+1^T}, a_2^{k+1^T}, \dots, a_d^{k+1^T}). \end{aligned}$$

Let  $(x^*, z^*) \in X \times \bar{B}$  be any solution of (12). Then it follows that

$$z^T(Ax^* - b) \leq z^{*T}(Ax^* - b) \leq z^{*T}(Ax - b), \quad \forall x \in X, z \in \bar{B}.$$

Thus  $(x^*, z^*)$  is a solution of the following LVI:

$$\text{Find } x^* \in X, z^* \in \bar{B} \text{ such that } \begin{cases} (x - x^*)(A^T z^*) \geq 0 & \forall x \in X \\ (z - z^*)(-Ax^* + b) \geq 0 & \forall z \in \bar{B} \end{cases}$$

A compact form of the above LVI is

$$\text{LVI} : u^* \in \Omega, (u - u^*)(Mu^* + q) \geq 0, \quad \forall u \in \Omega, \quad (13)$$

where

$$u = \begin{pmatrix} x \\ z \end{pmatrix}, \quad M = \begin{pmatrix} 0 & A^T \\ -A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad \Omega = X \times \bar{B}. \quad (14)$$

Therefore, the GSWP (8) is reformulated into the LVI (13)- (14). It is well known, see, e.g. ([19]), that for any  $\beta > 0$ ,  $u^*$  is a solution of the LVI (13), (14) if and only if

$$e(u^*, \beta) := u^* - P_\Omega[u^* - \beta(Mu^* + q)] = 0, \quad (15)$$

The LVI is solved in [16] and the solution is unique. The solution method for (GSWP) is as follows:

**Location phase Algorithm.** Given a tolerance  $\varepsilon' > 0$  and a initial iterate  $u^0 = (x^0, z^0) \in \Omega$ . For  $k = 0, 1, \dots$ , if  $\|e(u^k)\| > \varepsilon'$  then do:

$$\begin{aligned} z^{k+1} &= P_{\bar{B}} \left[ z^k + (Ax^k - b) \right], \\ x^{k+1} &= P_X \left[ \frac{1}{d} (dx^k - A^T z^{k+1} - A^T (z^{k+1} - z^k)) \right]. \end{aligned}$$

### 3. Convergence

This section analyzes the convergence of this method. The global convergence of it is proved under mild assumptions. Suppose that the map  $B : X \times H \rightarrow X \times H$  is given by

$$B \begin{pmatrix} x^k \\ a^k \end{pmatrix} = \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix}.$$

If  $x^k \notin H_j$ , then

$$a_j^{k+1} = P_{H_j}(x^k)$$

, and if  $x^k \in H_j$ , then

$$a_j^{k+1} = x^k + \varepsilon \frac{x^k - x^{k-1}}{\|x^k - x^{k-1}\|},$$

for  $\varepsilon$  sufficiently small enough such that  $0 < \varepsilon < \|x^k - x^{k-1}\|$ .

We also define the map  $C : X \times H \rightarrow X \times H$  by

$$C \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix} = \begin{pmatrix} x^{k+1} \\ a^{k+1} \end{pmatrix},$$

where  $x^{k+1} = \arg \min_{x \in X} \sum_{j=1}^d s_j \|x - a_j^{k+1}\|$ .

We set  $\alpha \begin{pmatrix} x \\ a \end{pmatrix} = \sum_{j=1}^d S_j \|x - a_j\|$ , where  $a^T = (a_1^T, a_2^T, \dots, a_n^T)$ .

**Lemma 3.2.** *Let  $x^k$  be the sequence produced by the location phase and  $a^{k+1}$  be that by the allocation phase. And also  $a^k \neq a^{k+1}$ , then for each  $k$ :*

$$\alpha \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix} < \alpha \begin{pmatrix} x^k \\ a^k \end{pmatrix}.$$

**Proof.** In the case of  $x^k \notin H_j$ , since  $a_j^k \in H_j$  and  $a_j^{k+1} = P_{H_j}(x^k) = \arg \min \{\|x^k - u\| \mid u \in H_j\}$  it follows that,

$$VP_{H_j}(x^k) \leq \|x^k - a_j^k\|.$$

Since  $a^k \neq a^{k+1}$  and  $a^k$  is unique in each iterate, therefore

$$\|x^k - a_r^{k+1}\| < \|x^k - a_r^k\| \text{ for some } r \in \{1, 2, \dots, d\}.$$

It follows that:

$$\alpha \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix} = \sum_{j=1}^d S_j \|x^k - a_j^{k+1}\| < \sum_{j=1}^d S_j \|x^k - a_j^k\| = \alpha \begin{pmatrix} x^k \\ a^k \end{pmatrix}.$$

Hence  $a_j^k = a_j^{k-1}$  for  $j = 1, 2, \dots, n, j \neq r$ .

In the semisingular case, we have  $VP_{H_r}(x^k) = x^k$ . Hence

$$a_r^{k+1} = x^k + \varepsilon \frac{x^k - x^{k-1}}{\|x^k - x^{k-1}\|}.$$

Since  $x^k \neq x^{k-1}$ , it follows that

$$\exists \varepsilon > 0 \text{ s.t. } \varepsilon < \|x^k - x^{k-1}\|, s_r \|x^k - a_r^{k-1}\| = s_r \varepsilon < s_r \|x^k - a_r^k\|.$$

Thus

$$\alpha \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix} < \alpha \begin{pmatrix} x^k \\ a^k \end{pmatrix}. \quad \square$$

**Lemma 3.3.** *Let  $C \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix} = \begin{pmatrix} x^{k+1} \\ a^{k+1} \end{pmatrix}$ , then the following inequality holds true for each  $k$ :*

$$\alpha \begin{pmatrix} x^{k+1} \\ a^{k+1} \end{pmatrix} \leq \alpha \begin{pmatrix} x^k \\ a^{k+1} \end{pmatrix}.$$

**Proof.** Since in each iterate  $x^{k+1} = \arg \min_{x \in X} \sum_{j=1}^d s_j \|x - a_j^{k+1}\|$  and since  $x^k \in X$ , it follows that

$$\sum_{j=1}^d s_j \|x^{k+1} - a_j^{k+1}\| \leq \sum_{j=1}^d s_j \|x^k - a_j^{k+1}\|,$$

Which implies that  $\alpha \left( \begin{array}{c} x^{k+1} \\ a^{k+1} \end{array} \right) \leq \alpha \left( \begin{array}{c} x^k \\ a^{k+1} \end{array} \right)$ .  $\square$

**Lemma 3.4.** *Let  $\Omega \subseteq X \times H$  be the nonempty solution set of problem GMWP, and  $B : X \times H \rightarrow X \times H$ . Then  $B : X \times H \rightarrow X \times H$  and  $C : X \times H \rightarrow X \times H$  are closed maps over the complement of  $\Omega$ .*

**Proof.** Since  $P_{H_j}$  is continuous where  $H_j$ 's is convex and closed set and

$\arg \min_{x \in X} \sum_{j=1}^d s_j \|x - a_j^{k+1}\|$  is continuous where  $X$  is convex and closed set.

Hence  $B$  and  $C$  are closed over the complement of  $\Omega$ .  $\square$

**Theorem 3.5.** *Let  $X \times H$  be a nonempty closed set in  $\mathbb{R}^n$ , and  $w = \begin{pmatrix} x \\ a \end{pmatrix}$ , and let  $\Omega \subseteq X \times H$  be a nonempty solution set. Let  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function, and consider the point-to-point map  $C : X \times H \rightarrow X \times H$  satisfying the following property: Given  $w \in X \times H$ , then  $\alpha(y) \leq \alpha(w)$  for  $y = C(w)$ . Let  $B : X \times H \rightarrow X \times H$  be a point-to-point map that is closed over the complement of  $\Omega$  and that satisfies  $\alpha(y) < \alpha(w)$  for each if  $w \notin \Omega$ . Now consider the algorithm defined by composite map  $A = CB$ . Given  $w_1 \in X$ , suppose that the sequence  $\{w_k\}$  is generated as follows: If  $w_k \in \Omega$ , then stop; otherwise, let  $w_{k+1} = A(w_k)$ , replace  $k$  by  $k + 1$ , and repeat. Suppose that the set  $\Lambda = \{w; \alpha(w) \leq \alpha(w_1)\}$  is compact. Then either the algorithm stops in a finite number of steps with a point in  $\Omega$  or all accumulation points of  $\{w_k\}$  belong to  $\Omega$ .*

**Proof.** The proof follows from lemma 1, 2, 3 and 4 and Theorem 7.3.4

in [1].  $\square$

## 4. Numerical Example

In this Section, we present an example for proposed method.

**Example.** Consider  $x \in \mathbb{R}^2$  and  $s_j = 1$ ,  $j = 1, 2, 3, 4$ ,

$$\begin{aligned} H_1 &= \sqrt{y_2^2 + (y_1 - 2)^2}, & H_2 &= \sqrt{(y_2 + 4)^2 + (y_1 + 4)^2}, \\ H_3 &= \sqrt{(y_2 - 6)^2 + (y_1 + 5)^2}, & H_4 &= \sqrt{(y_2 + 2)^2 + (y_1 + 7)^2}, \end{aligned}$$

$$a_1 \in H_1, a_2 \in H_2, a_3 \in H_3, a_4 \in H_4, \bigcap_{j=1}^4 H_j = \emptyset.$$

Clearly  $H_j$  for  $j = 1, 2, 3, 4$  are closed bounded convex sets in  $\mathbb{R}^2$ .

Since in each iterate doesn't lie in any regions, it flows that

$a_j^{k+1} = P_{H_j}(x^k)$  for  $j = 1, 2, \dots, 4$ . A summary of computations is presented in the following table.

iter	$x^k{}^T$	$a_1^{(k+1)T}$	$a_2^{(k+1)T}$	$a_3^{(k+1)T}$	$a_4^{(k+1)T}$
$k = 0$	(-0.6878, 2.0253)	(-0.6658, 2.4130)	(-4.2955, 3.4733)	(1.7988, -3.6335)	(3.5840, -3.0017)
$k = 1$	(0.5125, -0.4776)	(1.0479, -0.3057)	(-3.2117, -3.3847)	(-4.3519, 5.2384)	(-6.0199, -1.8014)
$k = 2$	(-3.4918, -1.2664)	(1.0256, -0.2247)	(-3.8172, -3.0168)	(-4.7968, 5.0209)	(-6.0212, -1.7953)
$k = 3$	(-4.0204, -1.3494)	(1.0242, -0.2187)	(-4.0077, -3.0000)	(-4.8679, 5.0088)	(-6.0230, -1.7867)
$k = 4$	(-4.1820, -1.3771)	(1.0239, -0.2174)	(-4.0692, -3.0024)	(-4.8898, 5.0061)	(-6.0236, -1.7842)
$k = 5$	(-4.2348, -1.3865)	(1.0238, -0.2171)	(-4.0895, -3.0040)	(-4.8970, 5.0053)	(-6.0237, -1.7834)
$k = 6$	(-4.2522, -1.3897)	(1.0238, -0.2170)	(-4.0962, -3.0046)	(-4.8993, 5.0051)	(-6.0238, -1.7832)
$k = 7$	(-4.2580, -1.3908)	(1.0238, -0.2169)	(-4.0984, -3.0049)	(-4.9001, 5.0050)	(-6.0238, -1.7831)
$k = 8$	(-4.2599, -1.3911)	(1.0238, -0.2169)	(-4.0991, -3.0049)	(-4.9004, 5.0050)	(-6.0238, -1.7831)
$k = 9$	(-4.2606, -1.3912)	(1.0238, -0.2169)	(-4.0994, -3.0050)	(-4.9005, 5.0050)	(-6.0238, -1.7831)

## References

- [1] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming Theory and Algorithms*, Wiley, 1993, 3th Edition.

- [2] M. Blum, R. W. Floyd, V. R. Pratt, R. L. Rivest, and R. E. Tarjan, Time bounds for selection, *Journal of Computer and System Sciences*, 7 (1972), 448-461.
- [3] P. H. Calamai and A. R. Conn, A second-order method for solving the continuous multifacility location problem, G. A. Watson, ed., Numerical analysis: Proceedings of the Ninth Biennial Conference, Dundee, Scotland, *Lecture Notes in Mathematics*, Vol. 912 (Springer, Berlin, Heidelberg and New York, (1982), 1-25.
- [4] J. A. Chatelon, D. W. Hearn, and T. J. Lowe, A subgradient algorithm for certain minimax and minisum problems, *Mathematical Programming*, 15 (1978), 130-145.
- [5] R. W. Cottle, F. Giannessi, and J. L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, Wiley, New York, 1980.
- [6] J. W. Eyster, J. A. White, and W. W. Wierwille, On solving multifacility location problems using a hyperboloid approximation procedure, *Transactions*, 5 (1973), 1-6.
- [7] D. R. Han. Solvig linear varaitional inequality problems by a self adaptive projection method, *Applied Mathematics and Computation*, 182 (2006), 1765-1771.
- [8] P. T. Harker and J. S. Pang, Finite dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Mathematical Programming*, 48 (1990), 161-220.
- [9] B. S. He, A projection and contraction method for a class of linear complementarity problem and its application in convex quadratic programming, *Applied Mathematics and Optimization*, 25 (1992), 247-262.
- [10] B. S. He and J. Stoer, Solution of projection problems over polytopes, *Numerische Mathematik*, 61 (1992), 73-90.
- [11] B. S. He, A new method for a class of linear variational inequalities, *Mathematical Programming*, 66 (1994), 137-144.
- [12] B. S. He, Solving a class of linear projection equations, *Numerische Mathematik*, 68 (1994), 71-80.

- [13] B. S. He, A modified projection and contraction method for a class of linear complementarity problems, *Journal of Computational Mathematics*, 14 (1996), 54-63.
- [14] B. S. He, A class of projection and contraction methods for monotone variational inequalities, *Applied Mathematics and Optimization*, 35 (1997), 69-76.
- [15] J. L. Jiang and X. M. Yuan, A heuristic algorithm for constrained multi-source Weber problem-The variational inequality approach, *European Journal of Operational Research*, 187 (2008), 357-370.
- [16] I. N. Katz, Local convergence in Fermats problem, *Mathematical Programming*, 6 (1974), 89-104.
- [17] H. W. Kuhn, On a pair of dual nonlinear programs, *J. Abadie, ed., Non-linear Programming* (North-Holland, Amsterdam, (1967), 38-54.
- [18] H. W. Kuhn, A note on Fermats problem, *Mathematical Programming*, 4 (1973), 98-107.
- [19] Y. Levin and A. Ben-Israel, A heuristic method for large-scale multi-facility location problems, *Computers & Operations Research*, 31 (2004), 257-272.
- [20] L. Z. Liao and S. L. Wang, A self-adaptive projection and contraction method for monotone symmetric linear variational inequalities, *Computers and Mathematics with Applications*, 43 (2002), 41-48.
- [21] R. F. Love, J. G. Morris, and G. O. Wesolowsky, *Facilities Location: Models & Methods*, North-Holland Publishing Co., New York, 1988.
- [22] M. L. Overton, A quadratically convergence method for minimizing a sum of Euclidean norms, *Mathematical Programming*, 27 (1983), 34-63.
- [23] J. S. Pang and D. Chan, Iterative methods for variational and complementarity problems, *Mathematical Programming*, 24 (1982), 284-313.
- [24] A. Weber, *Theory of the Location of Industries*, translated by Carl J. Friedrich (The University of Chicago Press, Chicago, 1937.
- [25] E. Weiszfeld, Sur le point par lequel la somme des distances de n points donnees est minimum, *Tohoku Mathematics Journal*, 43 (1937), 355-386.

**Sayyed Ahmadsreza Raisi Dehkordi**

Department of Mathematics  
Ph.D student of Applied Mathematics  
University of Isfahan  
Isfahan, Iran  
E-mail: ahmad.raisi76@yahoo.com

**Masood Farahani**

Department of Mathematics  
M.Sc Student of Applied Mathematics  
University of Isfahan  
Isfahan, Iran  
E-mail: ma farahani@sci.ui.ac.ir

**Abuzar Ahmadi**

Department of Mathematics  
M.Sc Student of Applied Mathematics  
University of Isfahan  
Isfahan, Iran  
E-mail: abozar\_ahmadi@sci.ui.ac.ir