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General Decay of Energy for a Class of Integro-Differential Equation with Nonlinear Damping

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Abstract. In this paper, a class of non-linear Integro-differential equations is considered in a bounded domain Ω with a smooth boundary $\partial\Omega$ as follows:

$$u_{tt} + M(\|D^m u\|_2^2)(-\Delta)^m u(t) + \int_0^t g(t-s)(-\Delta)^m u(s)ds + |u_t|^{\alpha-1}u_t = |u|^{p-1}u.$$

The asymptotic behavior of solutions is discussed by some conditions on g. Decay estimates of the energy function of solutions are also given.

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1. Introduction

Consider the initial boundary value problem for a higher-order integro-differential equation:

$$\begin{cases} u_{tt} + M(\|D^m u\|_2^2)(-\Delta)^m u(t) + \int_0^t g(t-s)(-\Delta)^m u(s)ds & \\ +|u_t|^{\alpha-1}u_t = |u|^{p-1}u, & t>0 \\ \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,2,...,m-1, & x \in \partial\Omega, t \geqslant 0 \\ u(x,0) = u_0, u_t(x,0) = u_1 & x \in \Omega, \end{cases} \tag{1}$$

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where $p > 1, m \geqslant 1, \alpha \geqslant 1$, Ω is a bounded domain of $\mathbb{R}^n, n \geqslant 1$, with the smooth boundary $\partial\Omega$, so that, divergence theorem can be applied, ν is unit outward normal on $\partial\Omega$, and $\frac{\partial^i u}{\partial \nu^i}$ denotes the $i-$order normal derivation of u, and D denotes the gradient operator, that is $Du = (u_{x_1}, u_{x_2}, ..., u_{x_n})$, and:

$$D^m = \overbrace{\nabla.\nabla.\cdots.\nabla}^{m}.$$

Before further progress, without the viscoelastic term, that is $g = 0$, for the case that m=1 and M being not a constant function, equation(1) is Kirchhoff equation which has been introduced in order to describe the nonlinear vibrations of an elastic string. Kirchhoff ([6]) was the first one to study the oscillations a stretched string and plates. In this case, the existence and nonexistence of solutions were discussed by many authors ([3,9,13]).

With $g \neq 0$, in the case of $M = 1$, equation (1) becomes a semilinear viscoelastic equation. Cavalcanti el.al ([2]) treated equation (1) with damping term $a(x)u_t$; here $a(x)$ may be null on apart of boundary. By assuming the kernel g in the memory term decays exponentially, they obtained an exponentially decay rate. On the other hand, Jiang and Rivera ([4]) proved, in the framework of nonlinear viscoelasticity, the exponential decay of the energy provided that the kernel g decays exponentially without imposing damping term. In the case M is not a constant function, equation (1) is a model in which describe the motion of deformable solids as hereditary effect is incorporated. This equation was first studied by Torrejon and Young ([12]) who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera ([6]) showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Recently, Wu and Tsai ([11]) discussed the global as well as energy decay of equation(1). In that paper, the following assumption on the negative kernel $g'(t) \leqslant 0$, for all $t \geqslant 0$ for some $r > o$, which motivated the present researcher to consider the problem of how to obtain the energy decay of the solutions when the above assumption is replaced by $g'(t) \leqslant 0$, for all $t \geqslant 0$.

In this paper, the global solution and the energy decays exponentially and polynomially under some conditions on g were established. The content of this paper is organized as follows: In Section 2, some important Lemmas and assumptions which will be used later and the state the local existence Theorem 2.1. are given. In Section 3, the results of global existence and decay property of the solutions of equation(1) are given by Theorem 3.1.

2. Preliminary Notes

In this section, the material needed for proving the main result is introduced. The standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ are used with their usual scalar products and norms. Meanwhile,

$$H_0^m(\Omega) = \{u \in H^m(\Omega) : \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0, 1, 2, ..., m-1\}$$

is defined and the following abbreviations are introduced:
$\|.\|_{H^m} = \|.\|_{H^m(\Omega)}$, $\|.\|_{H_0^m} = \|.\|_{H_0^m(\Omega)}$, $\|.\|_2 = \|.\|_{L^2(\Omega)}$, $\|.\|_p = \|.\|_{L^p(\Omega)}$ for any real number $p > 1$.
It is assumed that:

(A1) $M(s)$ is positive C^1-function for $s \geqslant 0$ and $M(s) = m_0 + s^q$ for $m_0 > 0, q \geqslant 1$ and $s \geqslant 0$.

(A2) $g \in C^1([0, \infty))$ is a bounded function satisfying:

$$m_0 - \int_0^t g(s) = l > 0, \quad \forall t > 0,$$

and there exist positive constants ξ_1 and ξ_2 such that:

$$-\xi_1 g(t) \leqslant g'(t) \leqslant -\xi_2 g(t). \tag{2}$$

(A3) $1 < p < \infty$ for $n \leqslant 2m$, $1 < p < \frac{2m}{n-2m}$ for $n > 2m$.

It is necessary to state that the local existence theorem for equation(1) will be established by combining the arguments of [3] and [12].

Theorem 2.1. *Assume that $M(s)$,$g(x)$ and p satisfy $(A1)$, $(A2)$ and $(A3)$ respectively. Then for any given $(u_0, u_1) \in (H_0^m(\Omega) \cap H^{2m}(\Omega)) \times H_0^m(\Omega)$, the problem (1) has a unique local solution satisfying:*

$$u \in C([0,T]; H_0^m(\Omega)), \quad u_t \in C([0,T]; L^2(\Omega)) \cap L^2(\Omega \times (0,T)),$$
$$u_{tt} \in L^\infty((0,T); L^2(\Omega)).$$

Lemma 2.2.(Sobolev-Poincare inequality [1]). *If p satisfies (A_3) for all $u \in H_0^m(\Omega)$, then $H_0^m(\Omega) \longrightarrow L^p(\Omega)$, $\|u\|_{p+1} \leqslant B\|D^m u\|_2$, where B is the optimal constant of the Sobolev embeding.*

Lemma 2.3. ([7]) *Let $\phi(t)$ be a nonincreasing and nonnegative function defined on $[0,T]$, $\ T > 1$, satisfying:*

$$\phi^{1+r}(t) \leqslant k_0(\phi(t) - \phi(t+1)),$$

for $t \in [0,T]$, $\ k_0 > 1$ and $r \geqslant 0$. Then we have for each $t \in [0,T]$,

$$\begin{cases} \phi(t) \leqslant \phi(0)e^{k(t-1)^+}, & r = 0 \\ \phi(t) \leqslant (\phi^{-r}(0) + k_0 r^{-1}(t-1)^+)^{\frac{-1}{r}}, & r > 0 \end{cases}$$

where $(t-1)^+ = \max\{t-1, 0\}$ and $k = \ln(\frac{k_0}{k_0-1})$. Furthermore, the energy function $E(t)$ of the problem (1) is defined by:

$$\begin{aligned} E(t) &= \tfrac{1}{2}(m_0 - \textstyle\int_0^t g(s)ds)\|D^m u(t)\|_2^2 + \tfrac{1}{2}(goD^m u)(t) \\ &+ \tfrac{1}{2(q+1)}\|D^m u(t)\|_2^{2(q+1)} + \tfrac{1}{2}\|u_t\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1}, \end{aligned} \tag{3}$$

where $(goD^m u)(t) = \int_0^t g(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds$.

Lemma 2.4. *Assume that $(A1)$,$(A2)$ and $(A3)$ hold and let u be the solution of problem (1). Then $E(t)$ decreases, in other words:*

$$E'(t) = \frac{1}{2}(g' o D^m u)(t) - \frac{1}{2}g(t)\|D^m u(t)\|_2^2 - \|u_t\|_{\alpha+1}^{\alpha+1} \leqslant 0,$$

furthermore, for all $t \geqslant 0$,

$$E(t) \leqslant E(0). \tag{4}$$

Proof. By multiplying equation(1) by u_t and integrating the result over Ω, the following result is obtained:

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\|u_t\|_2^2 + M(\|D^m u(t))\|_2^2 \int_\Omega (-\Delta)^m u(t) u_t dx + \|u_t\|_{\alpha+1}^{\alpha+1}\\ &+ \int_0^t g(t-s)\int_\Omega (-\Delta)^m u(s) u_t dx ds = \tfrac{d}{dt}\tfrac{1}{p+1}\|u\|_{p+1}^{p+1},\end{aligned} \tag{5}$$

for any regular solution, this result remains valid for weak solutions by a simple density argument. After being integrated m times by parts for the second term on the left side of (4) and noting $\frac{\partial^i u}{\partial \nu^i} = 0$, the following identity will be obtained:

$$\begin{aligned}\int_\Omega [(-\Delta)^m u(t)] u_t dx = (-1)^m \int_\Omega D^{2m} u u_t dx &= \int_\Omega D^m u(t).D^m u_t(t) dx\\ &= \tfrac{1}{2}\tfrac{d}{dt}\|D^m u(t)\|_2^2.\end{aligned} \tag{6}$$

Inserting (6) in (5) and applying $(A1)$, result in:

$$\begin{aligned}&\tfrac{d}{dt}\Big\{\tfrac{1}{2(q+1)}\|D^m u(t))\|_2^{2(q+1)} + \tfrac{m_0}{2}\|D^m u(t))\|_2^2 + \tfrac{1}{2}\|u_t\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1}\Big\}\\ &= \int_0^t g(t-s)\int_\Omega D^m u_t.D^m u(s) dx ds - \|u_t\|_{\alpha+1}^{\alpha+1},\end{aligned} \tag{7}$$

Also:

$$\begin{aligned}&\int_0^t g(t-s)\int_\Omega D^m u_t.D^m u(s) dx ds\\ &= \int_0^t g(t-s)\int_\Omega D^m u_t.[D^m u(s) - D^m u(t)] dx ds\\ &+ \int_0^t g(t-s) ds \int_\Omega D^m u_t.D^m u(t) dx\\ &= -\tfrac{1}{2}\int_0^t g(t-s)\tfrac{d}{dt}\int_\Omega |D^m u(s) - D^m u(t)|^2 dx ds\\ &+ \int_0^t g(s) ds \tfrac{d}{dt}\tfrac{1}{2}(\int_\Omega |D^m u(t)|^2 dx) ds.\end{aligned}$$

But,

$$= -\tfrac{1}{2}\tfrac{d}{dt}\Big[\int_0^t g(t-s)\int_\Omega |D^m u(s) - D^m u(t)|^2 dxds\Big]$$

$$+\tfrac{d}{dt}\tfrac{1}{2}\Big[\int_0^t g(s)\int_\Omega |D^m u(t)|^2 dxds\Big] \tag{8}$$

$$+\tfrac{1}{2}\int_0^t g'(t-s)\int_\Omega |D^m u(s) - D^m u(t)|^2 dxds - \tfrac{1}{2}g(t)\int_\Omega |D^m u(t)|^2 dx.$$

Then, (8) is inserted in (7) to get:

$$\tfrac{d}{dt}\Big\{\tfrac{1}{2}(m_0 - \int_0^t g(s)ds)\|D^m u(t))\|_2^2 + \tfrac{1}{2(q+1)}\|D^m u(t))\|_2^{2(q+1)}$$

$$+\tfrac{1}{2}(goD^m)u(t) + \tfrac{1}{2}\|u_t\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1}\Big\} \tag{9}$$

$$= -\|u_t\|_{\alpha+1}^{\alpha+1} + \tfrac{1}{2}\int_0^t g'(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds - \tfrac{1}{2}\|D^m u(t)\|_2^2.$$

Using the definition of $E(t)$, the proof is completed. □

3. The Main Result

In this section, the main result is proved.

Theorem 3.1. (Global existence and energy decay) *Let the assumptions of Theorem 2.1. hold and $1 \leqslant \alpha \leqslant \frac{n+2}{n-2}$. If the initial datum satisfies,*

$$\|u_0\|_{p+1} < \lambda_0 = l^{\frac{1}{p-1}} B^{\frac{-2}{p-1}}, \qquad E(0) < E_0 = \frac{p-1}{2(p+1)}\lambda_0^{p+1}, \tag{10}$$

where B is the optimal constant of the Sobolev embedding(Sobolev-Poincare inequality). Then the cauchy problem (1) has a unique global solution. Moreovere,

$$E(t) \leqslant E(0)e^{-k(t-1)^+}, \qquad t \geqslant 0, \qquad \alpha = 1, \tag{11}$$

$$E(t) \leqslant (E^{\frac{\alpha-1}{2}}(0) + \frac{\alpha-1}{2}c_{12}^{-1}(t-1)^+)^{\frac{-2}{\alpha-1}} \qquad t \geqslant 0, \qquad \alpha > 1, \tag{12}$$

where $k = \ln(\frac{3c_{10}}{3c_{10}-1})$ and c_{10} and c_{12} are given in (41)and (44) respectively.

Proof. By decreasing of energy $E(t)$, one obtains:

$$E(t) \leqslant E(0) < E_0 = \frac{p-1}{2(p+1)}\lambda_0^{p+1}. \tag{13}$$

Therefore the following inequality is claimed:

$$\|u(.,t)\|_{p+1} < \lambda_0, \qquad \forall t \geqslant 0. \tag{14}$$

Suppose (14) is not true, by continuity of $\|u(.,t)\|_{p+1}$-norm; then there exist a t_0 such that $\|u(.,t_0)\|_{p+1} = \lambda_0$. Using Sobolev-Poincare inequality the following relation can be presented:

$$E(t) \geqslant \frac{1}{2} l B^{-2}\|u(t)\|_{p+1}^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} \qquad \forall t \geqslant 0. \tag{15}$$

Then,

$$\begin{aligned} E(t_0) &\geqslant \tfrac{1}{2} l B^{-2}\|u(t_0)\|_{p+1}^2 - \tfrac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1} \\ &= \tfrac{p-1}{2(p+1)}\lambda_0^{p+1} = E_0, \end{aligned} \tag{16}$$

in which (16) contradicts with (13).
On the other hand for all $t \geqslant 0$,

$$\begin{aligned} l\|D^m u(t)\|_2^2 &= 2E(t) - \|u_t\|_2^2 - \tfrac{1}{q+1}\|D^m u(t)\|_2^{2(q+1)} \\ &+ \tfrac{2}{p+1}\|u(t)\|_{p+1}^{p+1} + (goD^m u)(t) \\ &\leqslant \tfrac{p-1}{p+1} l^{\frac{p+1}{p-1}} B^{\frac{-2(p+1)}{p-1}} + \tfrac{2}{p+1} l^{\frac{p+1}{p-1}} B^{\frac{-2(p+1)}{p-1}} \\ &= \lambda_0^{p+1}. \end{aligned} \tag{17}$$

By continuation argument and (17), the local solution constructed by Theorem 2.1. will be exist globally. Furthermore, the large time behavior of equation(1) is considered.
According to (17), the initial condition and Sobolev-Poincare inequality, the following relation can be concluded:

$$\begin{aligned} l\|D^m u(t)\|_2^2 &< 2E(t) + \tfrac{2}{p+1} B^{p+1}\|D^m u(t)\|_2^{p+1} \\ &< 2E(t) + \tfrac{2}{p+1} l\|D^m u(t)\|_2^2, \end{aligned} \tag{18}$$

and consequently:

$$\|D^m u(t)\|_2^2 < (\frac{2(p+1)}{l(p-1)}E(0))^{\frac{1}{2}}. \tag{19}$$

The parameter β is defined as follows:

$$0 \leqslant \beta = \frac{B^{p+1}}{l}(\frac{2(p+1)}{l(p-1)}E(0))^{\frac{p-1}{2}} < 1. \tag{20}$$

From (19), (20) and Sobolev-Poincar inequality, the following can be received:

$$\begin{aligned} \|u(t)\|_{p+1}^{p+1} &< \beta l\|D^m u(t)\|_2^2 \\ &< l\|D^m u(t)\|_2^2. \end{aligned} \tag{21}$$

Therefore, if $I(t)$ is defined as follows:

$$I(t) = l\|D^m u(t)\|_2^2 + \|D^m u(t)\|_2^{2(q+1)} + (goD^m u)(t) - \|u(t)\|_{p+1}^{p+1}, \tag{22}$$

then, by considering (21), the following can be presented:

$$I(t) > l(1-\beta)\|D^m u(t)\|_2^2 > 0. \tag{23}$$

Now, $F(t)$ is set as follows:

$$\begin{aligned} F^{\alpha+1}(t) &= -\frac{1}{2}\int_t^{t+1}\int_0^t g'(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds dt \\ &+ \int_t^{t+1}\|u_t(t)\|_{\alpha+1}^{\alpha+1}dt + \frac{1}{2}\int_t^{t+1} g(t)\|D^m u(t)\|_2^2 dt. \end{aligned} \tag{24}$$

Thanks to mean value Theorem and Holder inequality,

$$\begin{aligned} \frac{1}{4}\|u_t(t_1)\|_2^2 + \frac{1}{4}\|u_t(t_2)\|_2^2 &\leqslant \int_t^{t+1}\|u_t(t)\|_2^2 dt \\ & |\Omega|^{\frac{\alpha-1}{\alpha+1}}(\int_t^{t+1}\|u_t(t)\|_{\alpha+1}^{\alpha+1}dt)^{\frac{2}{\alpha+1}}, \end{aligned} \tag{25}$$

holds for some $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$.
Hence, by (24), the following is presented:

$$\|u_t(t_i)\|_2^2 \leqslant cF^2(t), \qquad i = 1, 2, \tag{26}$$

where $c = 4|\Omega|^{\frac{2(\alpha-1)}{(\alpha+1)^2}}$.
Afterwards, multiplying equation(1) by u and integrating it over $\Omega \times [t_1, t_2]$ the following identity is presented:

$$\begin{aligned}
&\int_{t_1}^{t_2} [l\|D^m u(t)\|_2^2 + \|D^m u(t)\|_2^{2(q+1)} - \|u(t)\|_{p+1}^{p+1}]dt \\
&= -\int_{t_1}^{t_2}\int_\Omega u(t)u_{tt}(t)dxdt - \int_{t_1}^{t_2}\int_\Omega u(t)|u_t(t)|^{\alpha-1}u_t(t)dxdt \qquad (27)\\
&+ \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)D^m u(t).[D^m u(s) - D^m u(t)]dsdxdt.
\end{aligned}$$

Then, using (22), the following is obtained:

$$\begin{aligned}
&\int_{t_1}^{t_2} I(t)dt \\
&= -\int_{t_1}^{t_2}\int_\Omega u(t)u_{tt}(t)dxdt - \int_{t_1}^{t_2}\int_\Omega u(t)|u_t(t)|^{\alpha-1}u_t(t)dxdt \\
&+ \int_{t_1}^{t_2}(goD^m)u(t)dt \qquad (28)\\
&+ \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)D^m u(t).[D^m u(s) - D^m u(t)]dsdxdt.
\end{aligned}$$

Note that by integrating by parts and Holder inequality, the following inequality is achieved:

$$-\int_{t_1}^{t_2}\int_\Omega u(t)u_{tt}(t)dxdt \leqslant \sum_{i=1}^{2}\|u_t(t_i)\|_2^2 + \int_{t_1}^{t_2}\int_\Omega u_t^2(t)dxdt. \qquad (29)$$

Also the following relation is obtained by considering Young inequality:

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)D^m u(t).[D^m u(s) - D^m u(t)]dsdxdt \\
&\leqslant \delta\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^m u(t)\|_2^2 dsdt + \tfrac{1}{4\delta}\int_{t_1}^{t_2}(goD^m)u(t)dt, \qquad (30)
\end{aligned}$$

where δ is some positive constant to be chosen later.
By using (18) and Sobolev-Poincare inequality, the following result is concluded:

$$\|u(t_i)\|_{p+1} \leqslant c_1 \sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s), \qquad (31)$$

where $c_1=(\frac{2(p+1)}{l(p-1)})^{\frac{1}{2}}$.
Then, by (26) and (29)-(31), the following relation is deduced:

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt &\leqslant c_2F(t)\sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s)+cF^2(t)\\ &+\int_{t_1}^{t_2}\int_\Omega |u(t)||u_t(t)|^\alpha dxdt+(\tfrac{1}{4\delta}+1)\int_{t_1}^{t_2}(goD^mu)(t)dt\\ &+\delta\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^mu(t)\|_2^2dsdt,\end{aligned}\tag{32}$$

where $c_2=\sqrt{2}c_1c$.
On the other hand, the following inequality is obtained from (2) and (24):

$$\int_{t_1}^{t_2}(goD^mu)(t)dt\leqslant -\frac{1}{\xi_2}\int_{t_1}^{t_2}(g'oD^mu)(t)dt\leqslant \frac{2}{\xi_2}F^{\alpha+1}(t).\tag{33}$$

The following relation is achieved by considering (2) and (23):

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^mu(t)\|_2^2dsdt &\leqslant \tfrac{1}{\xi_1}\int_{t_1}^{t_2}\int_0^t g'(t-s)\|D^mu(t)\|_2^2dsdt\\ &\leqslant \tfrac{1}{\xi_1}\int_{t_1}^{t_2} g(0)\|D^mu(t)\|_2^2dt\\ &\leqslant \tfrac{g(0)}{(1-\beta)l\xi_1}\int_{t_1}^{t_2} I(t)dt.\end{aligned}\tag{34}$$

Hence, by choosing δ such that $\frac{\delta g(0)}{(1-\beta)l\xi_1}=\frac{1}{2}$ and by using (32)-(34), the following is obtained:

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt &\leqslant 2c_2F(t)\sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s)+2cF^2(t)\\ &c_3F^{\alpha+1}(t)+\int_{t_1}^{t_2}\int_\Omega |u(t)||u_t(t)|^\alpha dxdt,\end{aligned}\tag{35}$$

where $c_3=(1+\frac{g(0)}{(1-\beta)l\xi_1})\frac{1}{\xi_2}$.
By using Holder inequality and Sobolev-Poincare inequality, the following is resulted:

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |u(t)||u_t(t)|^\alpha dxdt &\leqslant B\int_{t_1}^{t_2}\|u_t(t)\|_{\alpha+1}^\alpha\|D^mu(t)\|_2dt\\ &\leqslant c_1\sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s)F^\alpha(t).\end{aligned}\tag{36}$$

By putting (36) into (35) and due to decreasing of energy $E(t)$, it can be concluded that:

$$\int_{t_1}^{t_2} I(t)dt \leqslant c_4(F(t)E^{\frac{1}{2}}(t) + F^{\alpha}(t)E^{\frac{1}{2}}(t) + F^2(t) + F^{\alpha+1}(t)), \quad (37)$$

where $c_4 = \max\{c, c_1, c_2, c_3\}$.
Moreover, from (3), (22) and (23), it is seen that:

$$E(t) \leqslant \frac{1}{2}\|u_t\|_2^2 + c_5 l\|D^m u(t)\|_2^2 + c_5(goD^m u)(t) + c_6 I(t), \quad (38)$$

where $c_5 = \frac{1}{2} - \frac{1}{p+1}$ and $c_6 = (\frac{1}{p+1} + \frac{1}{2(q+1)})$.
By integrating (38) over (t_1, t_2) also using (23), (26) and (33), the following is achieved:

$$\int_{t_1}^{t_2} E(t)dt \leqslant \frac{c}{2}F^2(t) + c_7\int_{t_1}^{t_2} I(t)dt + c_8 F^{\alpha+1}(t), \quad (39)$$

where $c_7 = c_6 + \frac{c_5}{1-\beta}$ and $c_8 = \frac{2c_5}{\xi_2}$.
On the other hand, from the nonincreasing of $E(t)$ one obtain:

$$\int_{t_1}^{t_2} E(t)dt \geqslant \frac{1}{2}E(t_2).$$

Therefore, from (39),

$$\begin{aligned} E(t) &= E(t_2) - \tfrac{1}{2}\textstyle\int_t^{t_2}\int_0^t g'(t-s)\|D^m u(s) - D^m u(t)\|_2^2 dsdt \\ &\textstyle\int_t^{t_2}\|u_t(t)\|_{\alpha+1}^{\alpha+1}dt + \tfrac{1}{2}\int_t^{t_2} g(t)\|D^m u(t)\|_2^2 dt \\ &\leqslant 2\textstyle\int_{t_1}^{t_2} E(t)dt + F^{\alpha+1}(t) \\ &\leqslant c_9(F(t)E^{\frac{1}{2}}(t) + F^{\alpha}(t)E^{\frac{1}{2}}(t) + F^2(t) + F^{\alpha+1}(t)), \end{aligned} \quad (40)$$

where $c_9 = \max\{c + 2c_7c_4, 2c_7c_4, 1 + 2c_8 + c_7c_4\}$.
After that, by Young inequality, the following inequality is achieved:

$$E(t) \leqslant c_{10}(F^2(t) + F^{2\alpha}(t) + F^{\alpha+1}(t)), \quad (41)$$

where $c_{10} = |\frac{2c_9}{1-c_9}|$.
If $\alpha = 1$ in (41), then

$$E(t) \leqslant 3c_{10}(F^2(t)) = 3c_{10}(E(t) - E(t+1)), \tag{42}$$

therefore (11) follows from (42) and Lemma 2.3.
If $\alpha > 1$, since:

$$F^{\alpha+1}(t) = E(t) - E(t+1) \leqslant E(0),$$

then, the following relation is obtained:

$$\begin{aligned} E(t) &\leqslant c_{10}(1 + F^{2(\alpha-1)}(t) + F^{\alpha-1}(t))F^2(t) \\ &\leqslant c_{10}(1 + (E(0))^{\frac{2(\alpha-1)}{\alpha+1}}(t) + (E(0))^{\frac{\alpha-1}{\alpha+1}}(t))F^2(t) \\ &\leqslant c_{11}F^2(t), \end{aligned} \tag{43}$$

where $c_{11} = (1 + (E_0)^{\frac{2(\alpha-1)}{\alpha+1}}(t) + (E_0)^{\frac{\alpha-1}{\alpha+1}}(t))$.
Thus, (43) implies (44) as follows:

$$E^{\frac{\alpha+1}{2}}(t) \leqslant c_{11}^{\frac{\alpha+1}{2}} F^{\alpha+1}(t) = c_{12}(E(t) - E(t+1)), \tag{44}$$

where $c_{12} = c_{11}^{\frac{\alpha+1}{2}}$.
Hence,(12)follows from (44) and Lemma 2.3. This finishes the proof. □

4. Conclusion

In this paper, a difference inequality [Lemma 2.3.] for a class of Integro-Diferential equations with nonlinear damping has been applied. The main goal of this work is estimating general decay energy of these equations. The mentioned target is satisfied by the propose method. Also, the asymptotic behavior of solutions is discussed.

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