Numerical Range in C*-Algebras

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Abstract. Let $A$ be a C*-algebra with unit 1 and let $S$ be the state space of $A$, i.e., $S = \{ \varphi \in A^* : \varphi \geq 0, \varphi(1) = 1 \}$. For each $a \in A$, the C*-algebra numerical range is defined by

$$V(a) := \{ \varphi(a) : \varphi \in S \}.$$  

We prove that if $V(a)$ is a disc with center at the origin, then $\|a + a^*\| = \|a - a^*\|$.

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1. Introduction

Let $T$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We can write

$$T = A + iB,$$  

where $A$ and $B$ are Hermitian operators. Such a decomposition is unique; we have

$$A = \frac{1}{2}(T + T^*), B = \frac{1}{2i}(T - T^*).$$  

The elements $A, B$ are called the real and imaginary parts of $T$, denoted by $Re(T)$ and $Im(T)$, respectively, and the decomposition (1) is called the Cartesian decomposition of $T$.

The numerical range of $T$ is the set

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\[ W(T) := \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}, \]
in the complex plane, where \( \langle ., . \rangle \) denotes the inner product in \( \mathcal{H} \). In other words, \( W(T) \) is the image of the unit sphere \( \{ x \in \mathcal{H} : \|x\| = 1 \} \) of \( \mathcal{H} \) under the (bounded) quadratic form \( x \mapsto \langle Tx, x \rangle \).

Some properties of the numerical range follow easily from the definition. Recall that, the numerical range is unchanged under the unitary equivalence of operators: \( W(T) = W(U^*TU) \) for any unitary operator \( U \). It also behaves nicely under the operation of taking the adjoint of an operator: \( W(T^*) = \{ \bar{z} : z \in W(T) \} \). One of the most fundamental properties of the numerical range is it’s convexity, stated by the famous Toeplitz-Hausdorff Theorem. Other important property of \( W(T) \) is that its closure contains the spectrum of the operator. Also, \( W(T) \) is a connected set and it is compact in the finite dimensional case.

2. Numerical Range and Norm

Suppose \( E \) is a bounded convex subset of the plane. For \( 0 \leq \theta < 2\pi \) define

\[ p_E(\theta) := \sup \{ \text{Re}(e^{-i\theta}z) : z \in E \}. \quad (3) \]

Note that for \( z \in \mathbb{C} \), the number \( \text{Re}(e^{-i\theta}z) \) is the real dot product of the plane vectors \( e^{i\theta} \) and \( z \), i.e., the signed length of the projection of \( z \) in the direction of \( e^{i\theta} \). Thus the set

\[ \Pi_\theta = \{ z \in \mathbb{C} : \text{Re}(e^{-i\theta}z) \leq p_E(\theta) \}, \]

is a closed half-plane that contains \( E \) and intersects \( \partial E \). The boundary \( L_\theta \) of \( \Pi_\theta \) is called the support line of \( E \) perpendicular to \( e^{i\theta} \). The magnitude of \( p_E(\theta) \) is the orthogonal distance from the origin to \( L_\theta \).

The function \( p_E(\theta) : [0, 2\pi) \to \mathbb{R} \) defined by (3) is called the support function of \( E \). The Hahn-Banach theorem insures that the closure of \( E \) is the intersection of all the half-planes \( \Pi_\theta \) as \( \theta \) runs from 0 to \( 2\pi \).

Hence two bounded convex sets with the same support function have the same closures (see [3]).
In our applications the set $E$ will always contain the origin in its closure, in which case $p_E \geq 0$. We will be particularly interested in the support function of a standard ellipse.

**Proposition 2.1.** Suppose $a, b > 0$ and $E$ is the elliptical disc determined by the inequality $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. Then $p_E(\theta) = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$. ($0 \leq \theta < 2\pi$).

**Proof.** We parameterize the boundary of $E$ by the complex equation $z(t) = a \cos t + ib \sin t$, with $0 \leq t < 2\pi$. So

$$p_E(\theta) = \sup\{Re(e^{-i\theta} z) : z \in E\} = \sup\{a \cos \theta \cos t + b \sin \theta \sin t, 0 \leq t < 2\pi\}.$$

Put $f(t) = a \cos \theta \cos t + b \sin \theta \sin t, 0 \leq t < 2\pi$. Since $f$ is twice differentiable so, by second derivative test, it has a local maximum at $\tan t = \frac{b}{a} \tan \theta$. After a little calculation with right triangles this yields the equations

$$\cos t = \frac{a \cos \theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}, \quad \sin t = \frac{b \sin \theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}$$

and then by substituting, $p_E(\theta) = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$. ($0 \leq \theta < 2\pi$). □

We note in closing that this result persists in the limiting case $b = 0$. In this case $E$ is the real segment $[-a, a]$, for which the definition of support function yields $p_E(\theta) = a |\cos(\theta)|$. If $a = b$, then $p_E(\theta) = a$, indeed if $E$ is a disc with center at the origin then the function $p_E(\theta)$ is constant for all $\theta$.

**Proposition 2.2.** If $T$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ such that $\overline{W(T)}$ is a disc with center at the origin. Then

$$\|Re(T)\| = \|Im(T)\|.$$
Proof. We compute the support function $p_T$ of $W(T)$ in this standard fashion:

$$p_T(\theta) : = \sup \{ \Re(e^{-i\theta}z) : z \in W(T) \}$$

$$= \sup \{ \Re(e^{-i\theta} < Tf, f >) : f \in \mathcal{H}, \|f\| = 1 \}$$

$$= \sup \{ < H_\theta f, f > : f \in \mathcal{H}, \|f\| = 1 \}$$

where $H_\theta := \Re(e^{-i\theta}T) = \frac{1}{2}(e^{-i\theta}T + e^{i\theta}T^*)$.

Since $H_\theta$ is a self-adjoint operator on $\mathcal{H}$ and $W(T)$ is a disc with center at the origin, then the last calculation show that for each $0 \leq \theta < 2\pi$,

$$p_T(\theta) = \sup \{ | < H_\theta f, f > | : f \in \mathcal{H}, \|f\| = 1 \} = \|H_\theta\|.$$

Now, Proposition 2. implies that, $p_T(\theta)$ and also $\|H_\theta\|$ is constant for all $\theta$. In particular, $\|H_0\| = \|H_{\frac{\pi}{2}}\|$ or

$$\|T + T^*\| = \|T - T^*\|.$$

This completes the proof. □

Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e., $\mathcal{S} = \{ \varphi \in \mathcal{A}^* : \varphi \geq 0, \varphi(1) = 1 \}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range is defined by

$$V(a) := \{ \varphi(a) : \varphi \in \mathcal{S} \}.$$

It is well known that $V(a)$ is non empty, compact and convex subset of the complex plane and $V(\alpha 1 + \beta a) = \alpha + \beta V(a)$ where $a \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$, and if $z \in V(a), |z| \leq \|a\|$ (for further details see [2]).

As an example, let $\mathcal{A}$ be the C*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $A \in \mathcal{A}$. It is well known that $V(A)$ is the closure of $W(A)$, where

$$W(A) := \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \},$$

is the usual numerical range of the operator $A$. 

Theorem 2.3. Let $a \in \mathcal{A}$ be such that $V(a)$ be a disc with center at the origin. Then

$$\|\text{Re}(a)\| = \|\text{Im}(a)\|$$

. 

Proof. Let $\rho$ be a state of $\mathcal{A}$. Then there exists a cyclic representation $\varphi_{\rho}$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\rho}$ and a unit cyclic vector $x_{\rho}$ for $\varphi_{\rho}$ such that $\rho(a) = \langle \varphi_{\rho}(a)x_{\rho}, x_{\rho} \rangle$, $a \in \mathcal{A}$.

By Gelfand-Naimark Theorem, the direct sum $\varphi : a \mapsto \sum_{\rho \in \mathcal{S}} \otimes \varphi_{\rho}(a)$ is a faithful representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H} = \sum_{\rho \in \mathcal{S}} \otimes \mathcal{H}_{\rho}$ (see [6]). Therefore, for each $\rho \in \mathcal{S}$, $\rho(a) \in W(\varphi_{\rho}(a)) \subset W(\varphi(a))$, and hence $V(a)$ is contained in $W(\varphi(a))$. On the other hand if $x$ is a unit vector of $H$, then the formula $\rho(b) = \langle \varphi(b)x, x \rangle$, $b \in \mathcal{A}$ defines a state on $\mathcal{A}$ and hence $\rho(a) = \langle \varphi(a)x, x \rangle \in V(a)$. So it follows that

$$W(T_a) = V(a), \quad (4)$$

where $T_a = \varphi(a)$. (see also Theorem 3 of [1]).

Since $\varphi(\text{Re}(a)) = \text{Re}(T_a)$, $\varphi(\text{Im}(a)) = \text{Im}(T_a)$ and $\varphi$ is isometry, thus by equation (4) and Proposition (2.) the proof is completed. \qed

Example 2.4. Let $\mathbb{U}$ denote the open unit disc in the complex plane. Recall that the Hardy space $H^2$ consists of the functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$, with $\hat{f}(n)$ denoting the n-th Taylor coefficient of $f$. The inner product inducing the norm of $H^2$ is given by $< f, g > = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$. The inner product of two functions $f$ and $g$ in $H^2$ may also be computed by integration:

$$< f, g > = \frac{1}{2\pi i} \int_{\partial \mathbb{U}} f(z)g(z)\frac{dz}{z},$$

where $\partial \mathbb{U}$ is positively oriented and $f$ and $g$ are defined a.e. on $\partial \mathbb{U}$ via radial limits.

Each holomorphic self map $\varphi$ of $\mathbb{U}$ induces on $H^2$ a composition operator $C_{\varphi}$ defined by the equation $C_{\varphi}f = f \circ \varphi(f \in H^2)$. A consequence of
a famous theorem of J. E. Littlewood [7] asserts that $C_\varphi$ is a bounded operator. (see also [9] and [4]). In fact,

$$\sqrt{\frac{1}{1 - |\varphi(0)|^2}} \leq \|C_\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$ 

In the case $\varphi(0) \neq 0$ Joel H. Shapiro has been shown that the second inequality changes to equality if and only if $\varphi$ is an inner function.

A conformal automorphism is a univalent holomorphic mapping of $U$ onto itself. Each such map is linear fractional, and can be represented as a product $w \alpha_p$, where

$$\alpha_p(z) := \frac{p - z}{1 - \overline{p}z}, (z \in U),$$

for some fixed $p \in U$ and $w \in \partial U$ (see [8]).

The map $\alpha_p$ interchanges the point $p$ and the origin and it is a self-inverse automorphism of $U$.

Each conformal automorphism is a bijection map from the sphere $\mathbb{C} \cup \{\infty\}$ to itself with two fixed points (counting multiplicity). An automorphism is called:

- **elliptic** if it has one fixed point in the disc and one outside the closed disc,

- **hyperbolic** if it has two distinct fixed point on the boundary $\partial U$, and

- **parabolic** if there is one fixed point of multiplicity 2 on the boundary $\partial U$.

For $r \in U$, a $r$-dilation is a map of the form $\delta_r(z) = rz$ and we call $r$ the dilation parameter of $\delta_r$ and in the case that $r > 0$, $\delta_r$ is called positive dilation. A conformal $r$-dilation is a map that is conformally conjugate to an $r$-dilation, i.e., a map $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where $r \in U$ and $\alpha$ is a conformal automorphism of $U$.

For $w \in \partial U$, an $w$-rotation is a map of the form $\rho_w(z) = wz$. We call $w$ the rotation parameter of $\rho_w$. A straightforward calculation shows that
every elliptic automorphism $\varphi$ of $U$ must have the form

$$\varphi = \alpha_p \circ \rho_w \circ \alpha_p,$$

for some $p \in U$ and some $w \in \partial U$.

In [3] the shapes of the numerical range for composition operators induced on $H^2$ by some conformal automorphisms of the unit disc specially parabolic and hyperbolic are investigated. The authors proved, among other things, the following results:

1. If $\varphi$ is a conformal automorphism of $U$ is either parabolic or hyperbolic then $W(C_{\varphi})$ is a disc centered at the origin.

2. If $\varphi$ is a hyperbolic automorphism of $U$ with antipodal fixed points and it is conformally conjugate to a positive dilation $z \mapsto rz$ ($0 < r < 1$) then $W(C_{\varphi})$ is the open disc of radius $1/\sqrt{r}$ centered at the origin.

3. If $\varphi$ is elliptic and conformally conjugate to a rotation $z \mapsto \omega z$ ($|\omega| = 1$) and $\omega$ is not a root of unity then $\overline{W}(C_{\varphi})$ is a disc centered at the origin.

So, we have the following consequences:

**Proposition 2.5.** If $\varphi$ is a conformal automorphism of $U$, except finite order elliptic automorphism, then

$$\|C_{\varphi} + C_{\varphi}^*\| = \|C_{\varphi} - C_{\varphi}^*\|.$$  

Also $C_{\varphi}$ is not self adjoint. If $\varphi$ is a finite order elliptic automorphism with rotation parameter $w$ of order $k$, then

$$\sigma(C_{\varphi}) = \{1, w, w^2, ..., w^{k-1}\}.$$  

If $w \neq \pm 1$, then $\sigma(C_{\varphi})$ is not a subset of $\mathbb{R}$ and so $C_{\varphi}$ is not self adjoint.

**Corollary 2.6.** $C_{\varphi}$ is Hermitian if and only if $\varphi(z) = z$ or $-z$. 

References


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