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Numerical Range in C*-Algebras

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Abstract. Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e., $\mathcal{S} = \{\varphi \in \mathcal{A}^* : \varphi \geqslant 0, \varphi(1) = 1\}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range is defined by

$$V(a) := \{\varphi(a) : \varphi \in \mathcal{S}\}.$$

We prove that if $V(a)$ is a disc with center at the origin, then $\|a+a^*\| = \|a - a^*\|$.

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1. Introduction

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We can write

$$T = A + iB, \tag{1}$$

where A and B are Hermitian operators. Such a decomposition is unique; we have

$$A = \frac{1}{2}(T + T^*), B = \frac{1}{2i}(T - T^*). \tag{2}$$

The elements A, B are called the real and imaginary parts of T, denoted by $Re(T)$ and $Im(T)$, respectively, and the decomposition (1) is called the Cartesian decomposition of T.
The *numerical range* of T is the set

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$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

in the complex plane, where $\langle ., .\rangle$ denotes the inner product in $\mathcal{H}$. In other words, $W(T)$ is the image of the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ of $\mathcal{H}$ under the (bounded) quadratic form $x \mapsto \langle Tx, x\rangle$.

Some properties of the numerical range follow easily from the definition. Recall that, the numerical range is unchanged under the unitary equivalence of operators: $W(T) = W(U^*TU)$ for any unitary operator U. It also behaves nicely under the operation of taking the adjoint of an operator: $W(T^*) = \{\overline{z} : z \in W(T)\}$. One of the most fundamental properties of the numerical range is it's convexity, stated by the famous Toeplitz-Hausdorff Theorem. Other important property of $W(T)$ is that its closure contains the spectrum of the operator. Also, $W(T)$ is a connected set and it is compact in the finite dimensional case.

2. Numerical Range and Norm

Suppose E is a bounded convex subset of the plane. For $0 \leqslant \theta < 2\pi$ define

$$p_E(\theta) := sup\{Re(e^{-i\theta}z) : z \in E\}. \tag{3}$$

Note that for $z \in \mathbb{C}$, the number $Re(e^{-i\theta}z)$ is the real dot product of the plane vectors $e^{i\theta}$ and z, i.e., the signed length of the projection of z in the direction of $e^{i\theta}$. Thus the set

$$\prod_\theta = \{z \in \mathbb{C} : Re(e^{-i\theta}z) \leqslant p_E(\theta)\},$$

is a closed half-plane that contains E and intersects ∂E. The boundary L_θ of $\prod_\theta$ is called the support line of E perpendicular to $e^{i\theta}$. The magnitude of $p_E(\theta)$ is the orthogonal distance from the origin to L_θ. The function $p_E(\theta) : [0, 2\pi) :\longrightarrow \mathbb{R}$ defined by (3) is called the support function of E. The Hahn-Banach theorem insures that the closure of E is the intersection of all the half-planes $\prod_\theta$ as θ runs from 0 to 2π. Hence two bounded convex sets with the same support function have the same closures(see [3]).

In our applications the set E will always contain the origin in its closure, in which case $p_E \geqslant 0$. We will be particularly interested in the support function of a standard ellipse.

Proposition 2.1. *Suppose $a, b > 0$ and E is the elliptical disc determined by the inequality $\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1$. Then $p_E(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$. $(0 \leqslant \theta < 2\pi)$.*

Proof. We parameterize the boundary of E by the complex equation $z(t) = a\cos t + ib\sin t$, with $0 \leqslant t < 2\pi$. So

$$\begin{aligned} p_E(\theta) &= sup\{Re(e^{-i\theta}z) : z \in E\} \\ &= sup\{a\cos\theta\cos t + b\sin\theta\sin t, 0 \leqslant t < 2\pi\}. \end{aligned}$$

Put $f(t) = a\cos\theta\cos t + b\sin\theta\sin t, 0 \leqslant t < 2\pi$. Since f is twice differentiable so, by second derivative test, it has a local maximum at $\tan t = \frac{b}{a}\tan\theta$. After a little calculation with right triangles this yields the equations

$$\cos t = \frac{a\cos\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \quad , \quad \sin t = \frac{b\sin\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$$

and then by substituting, $p_E(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$. $(0 \leqslant \theta < 2\pi)$. $\square$

We note in closing that this result persists in the limiting case $b = 0$. In this case E is the real segment $[-a, a]$, for which the definition of support function yields $p_E(\theta) = a|\cos(\theta)|$. If $a = b$, then $p_E(\theta) = a$, indeed if E is a disc with center at the origin then the function $p_E(\theta)$ is constant for all θ.

Proposition 2.2. *If T is a bounded linear operator on a Hilbert space $\mathcal{H}$ such that $\overline{W}(T)$ is a disc with center at the origin. Then*

$$\|Re(T)\| = \|Im(T)\|.$$

Proof. We compute the support function p_T of $W(T)$ in this standard fashion:

$$\begin{aligned} p_T(\theta) : &= sup\{Re(e^{-i\theta}z) : z \in W(T)\} \\ &= sup\{Re(e^{-i\theta} < Tf, f >) : f \in \mathcal{H}, \|f\| = 1\} \\ &= sup\{< H_\theta f, f >: f \in \mathcal{H}, \|f\| = 1\} \end{aligned}$$

where $H_\theta := Re(e^{-i\theta}T) = \frac{1}{2}(e^{-i\theta}T + e^{i\theta}T^*)$.
Since H_θ is a self-adjoint operator on $\mathcal{H}$ and $W(T)$ is a disc with center at the origin, then the last calculation show that for each $0 \leqslant \theta < 2\pi$,

$$p_T(\theta) = sup\{| < H_\theta f, f > | : f \in \mathcal{H}, \|f\| = 1\} = \|H_\theta\|.$$

Now, Proposition 2. implies that, $p_T(\theta)$ and also $\|H_\theta\|$ is constant for all θ. In particular, $\|H_0\| = \|H_{\frac{\pi}{2}}\|$ or

$$\|T + T^*\| = \|T - T^*\|.$$

This completes the proof. □

Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e., $\mathcal{S} = \{\varphi \in \mathcal{A}^* : \varphi \geqslant 0, \varphi(1) = 1\}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range is defined by

$$V(a) := \{\varphi(a) : \varphi \in \mathcal{S}\}.$$

It is well known that $V(a)$ is non empty, compact and convex subset of the complex plane and $V(\alpha 1 + \beta a) = \alpha + \beta V(a)$ where $a \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$, and if $z \in V(a)$, $|z| \leqslant \|a\|$ (for further details see [2]).
As an example, let $\mathcal{A}$ be the C*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $A \in \mathcal{A}$. It is well known that $V(A)$ is the closure of $W(A)$, where

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

is the usual numerical range of the operator A.

Theorem 2.3. *Let $a \in \mathcal{A}$ be such that $V(a)$ be a disc with center at the origin. Then*

$$\|Re(a)\| = \|Im(a)\|$$

.

Proof. Let ρ be a state of $\mathcal{A}$. Then there exists a cyclic representation φ_ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\rho$ and a unit cyclic vector x_ρ for φ_ρ such that

$$\rho(a) = \langle \varphi_\rho(a)x_\rho, x_\rho \rangle, \ a \in \mathcal{A}.$$

By Gelfand-Naimark Theorem, the direct sum $\varphi : a \mapsto \sum_{\rho\in\mathcal{S}} \oplus\varphi_\rho(a)$ is a faithful representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H} = \sum_{\rho\in\mathcal{S}} \oplus\mathcal{H}_\rho$ (see [6]). Therefore, for each $\rho \in \mathcal{S}, \rho(a) \in W(\varphi_\rho(a)) \subset W(\varphi(a))$, and hence $V(a)$ is contained in $W(\varphi(a))$. On the other hand if x is a unit vector of $\mathcal{H}$, then the formula $\rho(b) = \langle \varphi(b)x, x \rangle, b \in \mathcal{A}$ defines a state on $\mathcal{A}$ and hence $\rho(a) = \langle \varphi(a)x, x \rangle \in V(a)$. So it follows that

$$W(T_a) = V(a), \tag{4}$$

where $T_a = \varphi(a)$. (see also Theorem 3 of [1]).

Since $\varphi(Re(a)) = Re(T_a)$, $\varphi(Im(a)) = Im(T_a)$ and φ is isometry, thus by equation (4) and Proposition (2.) the proof is completed. □

Example 2.4. Let $\mathbb{U}$ denote the open unit disc in the complex plane. Recall that the *Hardy space* H^2 consists the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the n-th Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $< f, g >:= \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$< f, g >= \frac{1}{2\pi i} \int_{\partial\mathbb{U}} f(z)\overline{g(z)} \frac{dz}{z},$$

where $\partial\mathbb{U}$ is positively oriented and f and g are defined a.e. on $\partial\mathbb{U}$ via radial limits.

Each holomorphic self map φ of $\mathbb{U}$ induces on H^2 a *composition operator* C_φ defined by the equation $C_\varphi f = f \circ \varphi (f \in H^2)$. A consequence of a famous theorem of J. E. Littlewood [7] asserts that C_φ is a bounded operator. (see also [9] and [4]). In fact,

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leqslant \|C_\varphi\| \leqslant \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

In the case $\varphi(0) \neq 0$ Joel H. Shapiro has been shown that the second inequality changes to equality if and only if φ is an inner function.
A *conformal automorphism* is a univalent holomorphic mapping of $\mathbb{U}$ onto itself. Each such map is linear fractional, and can be represented as a product $w.\alpha_p$, where

$$\alpha_p(z) := \frac{p-z}{1-\overline{p}z}, (z \in \mathbb{U}),$$

for some fixed $p \in \mathbb{U}$ and $w \in \partial\mathbb{U}$ (see [8]).
The map α_p interchanges the point p and the origin and it is a self-inverse automorphism of $\mathbb{U}$.

Each conformal automorphism is a bijection map from the sphere $\mathbb{C}\bigcup\{\infty\}$ to itself with two fixed points (counting multiplicity). An automorphism is called:

- *elliptic* if it has one fixed point in the disc and one outside the closed disc,
- *hyperbolic* if it has two distinct fixed point on the boundary $\partial\mathbb{U}$, and
- *parabolic* if there is one fixed point of multiplicity 2 on the boundary $\partial\mathbb{U}$.

For $r \in \mathbb{U}$, a r-*dilation* is a map of the form $\delta_r(z) = rz$ and we call r the dilation parameter of δ_r and in the case that $r > 0, \delta_r$ is called *positive dilation*. A *conformal* r-*dilation* is a map that is conformally conjugate to an r-dilation, i.e., a map $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where $r \in \mathbb{U}$ and α is a conformal automorphism of $\mathbb{U}$.
For $w \in \partial\mathbb{U}$, an w-*rotation* is a map of the form $\rho_w(z) = wz$. We call w the rotation parameter of ρ_w. A straightforward calculation shows that

every elliptic automorphism φ of $\mathbb{U}$ must have the form

$$\varphi = \alpha_p \circ \rho_w \circ \alpha_p,$$

for some $p \in \mathbb{U}$ and some $w \in \partial\mathbb{U}$.
In [3] the shapes of the numerical range for composition operators induced on H^2 by some conformal automorphisms of the unit disc specially parabolic and hyperbolic are investigated. The authors proved, among other things, the following results:

1. If φ is a conformal automorphism of $\mathbb{U}$ is either parabolic or hyperbolic then $W(C_\varphi)$ is a disc centered at the origin.
2. If φ is a hyperbolic automorphism of $\mathbb{U}$ with antipodal fixed points and it is conformally conjugate to a positive dilation $z \mapsto rz$ $(0 < r < 1)$ then $W(C_\varphi)$ is the open disc of radius $1/\sqrt{r}$ centered at the origin.
3. If φ is elliptic and conformally conjugate to a rotation $z \mapsto \omega z$ $(|\omega| = 1)$ and ω is not a root of unity then $\overline{W}(C_\varphi)$ is a disc centered at the origin.

So, we have the following consequences:

Proposition 2.5. *If φ is a conformal automorphism of $\mathbb{U}$, except finite order elliptic automorphism, then*

$$\|C_\varphi + C_\varphi^*\| = \|C_\varphi - C_\varphi^*\|.$$

Also C_φ is not self adjoint. If φ is a finite order elliptic automorphism with rotation parameter w of order k, then

$$\sigma(C_\varphi) = \{1, w, w^2, ..., w^{k-1}\}.$$

If $w \neq \pm 1$, then $\sigma(C_\varphi)$ is not a subset of $\mathbb{R}$ and so C_φ is not self adjoint.

Corollary 2.6. *C_φ is Hermitian if and only if $\varphi(z) = z$ or $-z$.*

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