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On Odd Duals of a Banach Algebra as a Banach Algebra

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Abstract. It is known that even duals of a Banach algebra A with one of Arens products are Banach algebras, these products are natural multiplications extending the one on A. But the essence of A^* is completely different. By defining new products, we investigate some algebraic and spectral properties of odd duals of A. We will show relations between these products and Arens products, weak or weak-star continuity, commutativity and unit elements of these algebras. We also determine the spectrum and multiplier algebra for A^*, and we calculate the quasi-inverses, spectrum and spectral radius for elements of these kinds of algebras.

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1 Introduction

Throughout this paper, A is a Banach algebra. The set of all non-zero characters on A is called the *spectrum of* A and denoted by $\sigma(A)$. The *spectrum* $\sigma_A(a)$ *of an element* $a \in A$ is defined as follows [3];

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(i) If A has unit element e,

$$\sigma_A(a) =: \{\lambda \in \mathbb{C} \ : \ \ \lambda e - a \notin invA\} \ ,$$

where $invA$ is the set of all invertible elements of A.

(ii) If A dose not have unit element, we define

$$\sigma_A(a) =: \sigma_{A^\#}(a, 0) \ ,$$

where $A^\# = A \oplus \mathbb{C}$ is the unitization of A.

The *spectral radius* of $a \in A$, is defined by [3]

$$r_A(a) =: \lim_{n\to\infty} ||a^n||^{\frac{1}{n}} \ .$$

The *quasi-product* of $a, b \in A$ is $a \circ b = a + b - ab$. An element $a \in A$ is *left [right] quasi-invertible* if there exists $b \in A$ such that $b \circ a = 0$ $[a \circ b = 0]$, and it is *quasi-invertible* if it is both left and right quasi-invertible. So, if a is quasi-invertible, there is a unique element $b \in A$ such that $a \circ b = b \circ a = 0$, b is the quasi-inverse of a, and is denoted by a^q. We write $q - invA$ for the set of all quasi-invertible elements of A [3, 4]. Clearly $a \circ b = 0$ if and only if $(-a, 1)(-b, 1) = e$ in $A^\#$, and so

$$\begin{aligned} &invA^\# = \{(-a, 1) \ : \ \ a \in q - invA\} \ , \\ &q - invA = \{a \in A \ : \ \ (-a, 1) \in invA^\#\} \ . \end{aligned} \tag{1}$$

We recall the first and second *Arens products* $\Box$ and $\Diamond$ on the second dual A^{**} of A are defined by

$$\begin{aligned} &(f \cdot a)(b) = f(ab) && (a \cdot f)(b) = f(ba) \\ &(F \cdot f)(a) = F(f \cdot a) && (f \cdot F)(a) = F(a \cdot f) \\ &(F \Box G)(f) = F(G \cdot f) && (F \Diamond G)(f) = G(f \cdot F) \ , \end{aligned} \tag{2}$$

for $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$. Each of these products makes A^{**} to a Banach algebra, and A is called *Arens regular* if two products $\Box$ and $\Diamond$ coincide [2, 6]. We have

$$\begin{aligned} &F \Box G = w^* - \lim_{\alpha} w^* - \lim_{\beta} (a_\alpha b_\beta)\hat{} \ , \\ &F \Diamond G = w^* - \lim_{\beta} w^* - \lim_{\alpha} (a_\alpha b_\beta)\hat{} \ , \end{aligned} \tag{3}$$

in which $\hat{a} \in A^{**}$ is defined by $\hat{a}(f) = f(a)$ for $f \in A^*$. We also take $\hat{A} =: \{\hat{a} \;:\; a \in A\}$, that is a subalgebra in A^{**}. Therefore, all even duals of A become Banach algebras with Arens products. But the essence of A^* is compelely different. For example if $A = C_0(X)$ for a locally compact Hausdorff space X, then $A^* = M(X)$ is the space of regular countably additive Borel measures on X. Now for A^* (and also for odd duals of A) the pointwise product is not well defined, so there are another products on A^* to make it into a Banach algebra. Let $a \in A$ with $||a|| \leqslant 1$ and $F \in A^{**}$ with $||F|| \leqslant 1$, and define two products $\bigcirc_a$ and $\bigcirc^F$ on A^* by

$$f \bigcirc_a g =: f(a)g \quad , \quad f \bigcirc^F g =: F(f)g \qquad (f, g \in A^*) \ . \tag{4}$$

It was shown in [12] that with each of above products, A^* is a Banach algebra. Also there are similar works in [1, 5, 8, 9, 10, 11]. These kinds of algebras can be a source of (counter$-$) examples for various purposes in functional analysis.

In this paper, $A^{(n)}$ $(n \in \mathbb{N})$ denotes the $n-$th dual of A. We use the symbols $(A^*, \bigcirc_a)$ and $(A^*, \bigcirc^F)$ for Banach algebras with products $\bigcirc_a$ and $\bigcirc^F$ as in (4), in section 2, we will investigate some properties of these algebras such as relations between these products and Arens products, weak or weak-star continuity, commutativity and unit elements of these algebras. In section 3, we determine the spectrum and multiplier algebra for $(A^*, \bigcirc_a)$ and $(A^*, \bigcirc^F)$. In section 4, we calculate the quasi-inverses, spectrum and spectral radius for elements of these kinds of algebras. Finally, we will present some examples by using Riesz representation theorem.

2 Some Algebraic and Analytic Properties

The first proposition shows the relations between the products defined in (2) and $\bigcirc_a$, $\bigcirc^F$. Its proof is straightforward.

Proposition 2.1. *Let A be a Banach algebra and let $n \in \{1, 2, \ldots\}$, then for $a, b \in A^{(2n-2)}$, $f, g \in A^{(2n-1)}$ and $F, G \in A^{(2n)}$ we have*

(i) $f \bigcirc_a g = f \bigcirc^{\hat{a}} g \qquad , \quad \hat{f} \bigcirc_{\hat{a}} \hat{g} = (f \bigcirc_a g)\hat{}\ ,$

(ii) $\hat{f} \bigcirc_F \hat{g} = (f \bigcirc^F g)\hat{} = \hat{f} \bigcirc^{\hat{F}} \hat{g}$,

(iii) $b \cdot f \bigcirc_a g = f \bigcirc_{ab} g$, $f \cdot b \bigcirc_a g = f \bigcirc_{ba} g$,

(iv) $f \bigcirc_a b \cdot g = b \cdot (f \bigcirc_a g)$, $f \bigcirc_a g \cdot b = (f \bigcirc_a g) \cdot b$,

(v) $F \cdot f \bigcirc_a g = f \cdot a \bigcirc^F g$, $f \cdot F \bigcirc_a g = a \cdot f \bigcirc^F g$,

(vi) $f \bigcirc_a F \cdot g = F \cdot (f \bigcirc_a g)$, $f \bigcirc_a g \cdot F = (f \bigcirc_a g) \cdot F$,

(vii) $f \bigcirc^F a \cdot g = a \cdot (f \bigcirc^F g)$, $f \bigcirc^F g \cdot a = (f \bigcirc^F g) \cdot a$,

(viii) $G \cdot f \bigcirc^F g = f \bigcirc^{F \Box G} g$, $f \cdot G \bigcirc^F g = f \bigcirc^{F \Diamond G} g$,

(ix) $f \bigcirc^F G \cdot g = G \cdot (f \bigcirc^F g)$, $f \bigcirc^F g \cdot G = (f \bigcirc^F g) \cdot G$.

It is known that the product in a Banach algebra is norm-continuous, the next proposition shows relations between weak or weak* topologies and products $\bigcirc^F$, $\bigcirc_a$.

Proposition 2.2. *Let A be a Banach algebra and let $n \in \{1, 2, \ldots\}$, then for $f, g \in A^{(2n-1)}$*

(i) the map $(a \mapsto f \bigcirc_a g)$ from $A^{(2n-2)}$ to $A^{(2n-1)}$ is linear and $w - w^*-$continuous,

(ii) the map $(F \mapsto f \bigcirc^F g)$ from $A^{(2n)}$ to $A^{(2n-1)}$ is linear and $w^* - w^*-$continuous,

(iii) the maps $(f \mapsto f \bigcirc_a g)$ and $(f \mapsto g \bigcirc_a f)$ from $A^{(2n-1)}$ to $A^{(2n-1)}$ are linear and $w^* - w^*-$continuous for $a \in A^{(2n-2)}$,

(iv) the maps $(f \mapsto f \bigcirc^F g)$ and $(f \mapsto g \bigcirc^F f)$ from $A^{(2n-1)}$ to $A^{(2n-1)}$ are linear and $w - w^*-$continuous and $w^* - w^*-$continuous, respectively, for $F \in A^{(2n)}$,

(v) for $F = w^* - \lim_\alpha \hat{a}_\alpha$ and $G = w^* - \lim_\beta \hat{b}_\beta$ in $A^{(2n)}$ we have the following formulas

$$\begin{aligned} f \bigcirc^{F} g &= w^* - \lim_{\alpha} f \bigcirc_{a_\alpha} g \ , \\ f \bigcirc^{F\Box G} g &= w^* - \lim_{\alpha} w^* - \lim_{\beta} f \bigcirc_{a_\alpha b_\beta} g \ , \\ f \bigcirc^{F\Diamond G} g &= w^* - \lim_{\beta} w^* - \lim_{\alpha} f \bigcirc_{a_\alpha b_\beta} g \ . \end{aligned}$$

Proof. It is easy to check that the maps in all parts are linear. Now for $F = w^* - \lim_{\alpha} F_\alpha$ and $a = w - \lim_{\beta} a_\beta$, where $(F_\alpha) \in A^{(2n)}$, $(a_\beta) \in A^{(2n-2)}$ and for $x \in A^{(2n-2)}$, one can write

$$\begin{aligned} \langle f \bigcirc_a g, x\rangle &= \langle f(a)g, x\rangle \\ &= \langle \lim_{\beta} f(a_\beta)g, x\rangle \\ &= \lim_{\beta} \langle f \bigcirc_{a_\beta} g, x\rangle \\ &= \langle w^* - \lim_{\beta} (f \bigcirc_{a_\beta} g), x\rangle \ , \end{aligned}$$

and this complete the proof of (i). Also we have

$$\begin{aligned} \langle f \bigcirc^{F} g, x\rangle &= \langle F(f)g, x\rangle \\ &= \langle \lim_{\alpha} F_\alpha(f)g, x\rangle \\ &= \lim_{\alpha} \langle f \bigcirc^{F_\alpha} g, x\rangle \\ &= \langle w^* - \lim_{\alpha} (f \bigcirc^{F_\alpha} g), x\rangle \ , \end{aligned}$$

and this proves (ii).
For parts (iii) and (iv) it is easy to check the following equations for the bounded net $(f_\alpha) \in A^{(2n-1)}$

$$\begin{aligned} (w^* - \lim_{\alpha} f_\alpha) \bigcirc_a g &= w^* - \lim_{\alpha} (f_\alpha \bigcirc_a g) \ , \\ g \bigcirc_a (w^* - \lim_{\alpha} f_\alpha) &= w^* - \lim_{\alpha} (g \bigcirc_a f_\alpha) \ , \\ (w - \lim f_\alpha) \bigcirc^{F} g &= w^* - \lim_{\alpha} (f_\alpha \bigcirc^{F} g) \ , \\ g \bigcirc^{F} (w^* - \lim_{\alpha} f_\alpha) &= w^* - \lim_{\alpha} (g \bigcirc^{F} f_\alpha) \ . \end{aligned}$$

Finally, part (v) is a consequence of part (ii), the formulas (3) and also part (i) of proposition 2.1. □

Now we investigate the third dual A^{***} of A. We use the symbol $(A^{**})^*$ for dual of A^{**} (with one of its Arens products), and the symbol $(A^*)^{**}$ as the second dual of A^*. One can find more details in [7].

Proposition 2.3. *Let A^{**} be the second dual of a Banach algebra A, with one of Arens products $\Box$ or $\Diamond$. Then we have*

(i) $\big((A^{**})^*, \bigcirc_{\hat{a}}\big) = (A^*, \bigcirc_a)^{**}$,

(ii) $\big((A^{**})^*, \bigcirc^{\hat{F}}\big) = (A^*, \bigcirc^F)^{**}$.

Note that the right sides of each above equations can be considered with each of Arens products $\Box$ or $\Diamond$.

Proof. Suppose that A^{**} has the first Arens product $\Box$ (the proof with the second Arense product $\Diamond$ is similar), and let $\varphi = w^* - \lim_\alpha \hat{g}_\alpha$, $\eta = w^* - \lim_\beta \hat{h}_\beta$, where (g_α) and (h_β) are bounded nets in A^*. Now for φ, η as elements of $\big((A^{**})^*, \bigcirc_{\hat{a}}\big)$ we have

$$\varphi \bigcirc_{\hat{a}} \eta = \varphi(\hat{a})\eta = \lim_\alpha g_\alpha(a)\eta \ .$$

On the other hand for φ, η as elements of $(A^*, \bigcirc_a)^{**}$ with first Arens product $\Box$ (the proof with second Arens product is similar), we can write

$$\begin{aligned} \varphi\Box\eta &= w^* - \lim_\alpha w^* - \lim_\beta (g_\alpha \bigcirc_a h_\beta)\hat{} \\ &= w^* - \lim_\alpha w^* - \lim_\beta g_\alpha(a)\hat{h}_\beta \\ &= \lim_\alpha g_\alpha(a)\eta \ , \end{aligned}$$

and this completes the proof. The proof of part (ii) is similar. $\Box$

The next two propositions show that how can the algebras $(A^*, \bigcirc_a)$ or $(A^*, \bigcirc^F)$ be commutative or unital.

Proposition 2.4. *Let A be a Banach algebra, then*

(i) $(A^*, \bigcirc^F)$ *is commutative if and only if F is one to one,*

(ii) $(A^*, \bigcirc_a)$ *is commutative if and only if $\hat{a}$ is one to one.*

Proof. (i) Let $(A^*, \bigcirc^F)$ be commutative and $F(f) = F(g)$ for some $f, g \in A^*$. Since $(A^*, \bigcirc^F)$ is commutative, we have $f \bigcirc^F g = g \bigcirc^F f$, so $F(f)g = F(g)f$. Hence $g = f$ and this proves that F is one to one. Conversely, let F be one to one. For each $f, g \in A^*$ we have

$$F(f \bigcirc^F g) = F\big(F(f)g\big) = F(f)F(g) = F(g)F(f) = F(g \bigcirc^F f) \ .$$

Hence $f \bigcirc^F g = g \bigcirc^F f$, and this proves the commutativity of $(A^*, \bigcirc^F)$.
(ii) This is a consequence of part (i) for $(A^*, \bigcirc_a) = (A^*, \bigcirc^{\hat{a}})$. □

Proposition 2.5. *Let A be a Banach algebra, then*

(i) $f \in (A^*, \bigcirc^F)$ $\big[f \in (A^*, \bigcirc_a)\big]$ *is left identity if and only if* $F(f) = 1$ $\big[f(a) = 1\big]$,

(ii) *if* $f \in (A^*, \bigcirc^F)$ $\big[f \in (A^*, \bigcirc_a)\big]$ *be right identity, then* $F(f) = 1$ $\big[f(a) = 1\big]$. *Also* $(A^*, \bigcirc^F)$ $\big[(A^*, \bigcirc_a)\big]$ *has right identity if and only if it is one dimensional.*

Proof. (i) Let $f \in (A^*, \bigcirc^F)$ be left identity, then for all $g \in A^*$, $f \bigcirc^F g = g$. So $F(f)g = g$, or $F(f) = 1$. Conversely, let $F(f) = 1$ and $g \in A^*$, then $f \bigcirc^F g = F(f)g = g$. Since $f \bigcirc^{\hat{a}} g = f \bigcirc_a g$, the proof for $(A^*, \bigcirc_a)$ is a consequence of the first part of proof.
(ii) Let $f \in (A^*, \bigcirc^F)$ be right identity, then for all $g \in A^*$, $g \bigcirc^F f = g$. So $F(g)f = g$, also for $g = f$ we have $F(f)f = f$. Hence $F(f) = 1$. On the other hand the equality $F(g)f = g$ shows that A^* is one dimensional. The converse is obvious. The proof for $(A^*, \bigcirc_a)$ is similar. □

3 Linear, Multiplicative and Multiplier Functions

Proposition 3.1. *Let A be a Banach algebra, then*

(i) *The linear functional* $F \in (A^*, \bigcirc^F)^*$ *is the only multiplicative functional on* $(A^*, \bigcirc^F)$.

(ii) *The linear functional* $\hat{a} \in (A^*, \bigcirc_a)^*$ *is the only multiplicative functional on* $(A^*, \bigcirc_a)$.

Proof. (i) For $f, g \in A^*$ we can write

$$F(f \bigcirc^F g) = F\big(F(f)g\big) = F(f)F(g) \ ,$$

so F is multiplicative. Now suppose that $G \in (A^*, \bigcirc^F)^*$ is multiplicative, then $G(f \bigcirc^F g) = G(f)G(g)$, so $F(f)G(g) = G(f)G(g)$ for all $f, g \in A^*$. Therefor $F(f) = G(f)$ for all $f \in A^*$, this completes the proof (i). The proof of part (ii) is similar. $\square$

Corollary 3.2. *For a Banach algebra A, we have*

(i) $\sigma(A^*, \bigcirc^F) = \{F\}$,

(ii) $\sigma(A^*, \bigcirc_a) = \{\hat{a}\}$.

It is known that the adjoint $T^* : B^* \to A^*$ of a bounded linear map $T : A \to B$ is bounded and linear. Now in the next proposition we investigate the multiplicativity of T^*.

Proposition 3.3. *Let $T : A \to B$ be a bounded linear map between Banach algebras A and B. Then for $f, g \in B^*$, $b \in B$ and $G \in B^{**}$ we have* $T^*(f \bigcirc_b g) = f(b)T^*(g) \quad , \quad T^*(f \bigcirc^G g) = G(f)T^*(g)$.
If also T is surjective, then $T^* : (B^*, \bigcirc_b) \to (A^*, \bigcirc_a)$ is multiplicative, where $b = T(a)$. If $T^{**} : A^{**} \to B^{**}$ is sujective, then $T^* : (B^*, \bigcirc^G) \to (A^*, \bigcirc^F)$ is multiplicative, where $G = T^{**}(F)$.

Proof. We can write

$$T^*(f \bigcirc_b g) = (f \bigcirc_b g)oT = f(b)goT = f(b)T^*(g) \ .$$

Thus for $b = T(a)$ we have

$$T^*(f \bigcirc_b g) = T^*(f \bigcirc_{T(a)} g) = f\big(T(a)\big)T^*(g) = \big(T^* f\big)(a)T^*(g) = T^* f \bigcirc_a T^* g \ ,$$

and this proves the multiplicativity of T^*. The rest of proof is similar.
$\square$

Proposition 3.4. *Let A be a Banach algebra. Then*

(i) $RM(A^*, \bigcirc_a) = RM(A^*, \bigcirc^F) = B(A^*)$,

(ii) for each $T \in LM(A^*, \bigcirc^F)$ $\big[T \in LM(A^*, \bigcirc_a)\big]$ *and* $f \in A^*$*, there exists* $\lambda_f \in \mathbb{C}$ *such that* $T(f) = \lambda_f f$*,*

where LM and RM stand for the set of all left and right multipliers, respectively, and $B(A^*)$ *denotes the set of bounded linear maps* $T : A^* \to A^*$*.*

Proof. (i) A direct varification shows that for $T \in B(A^*)$ and $f, g \in (A^*, \bigcirc^F)$

$$T(f \bigcirc^F g) = f \bigcirc^F T(g) \quad , \quad T(f \bigcirc_a g) = f \bigcirc_a T(g) \ .$$

(ii) Let $T \in LM(A^*, \bigcirc^F)$ and $f, g \in (A^*, \bigcirc^F)$, then we have

$$T(f \bigcirc^F g) = T(f) \bigcirc^F g \Rightarrow F(f)T(g) = F\big(T(f)\big)g \ ,$$

and for $f = g$, $T(f) = \dfrac{F\big(T(f)\big)}{F(f)} f$, in which $\lambda_f =: \dfrac{F\big(T(f)\big)}{F(f)}$. A similar calculating for $(A^*, \bigcirc_a)$ completes the proof. □

4 Spectral Theory in Nonunital Case

According to proposition 2.5 we will focus on nonunital Banach algebras $(A^*, \bigcirc^F)$ and $(A^*, \bigcirc_a)$.

Proposition 4.1. *Let A be a Banach algebra. Then*

(i) $q - inv(A^*, \bigcirc^F) = \{f \in A^* \ : \ F(f) \neq 1\}$,

(ii) $q - inv(A^*, \bigcirc_a) = \{f \in A^* \ : \ f(a) \neq 1\}$,

and for each $f \in q - inv(A^*, \bigcirc^F)$ $\big[f \in q - inv(A^*, \bigcirc_a)\big]$ *we have*

$$f^q = \frac{f}{F(f) - 1}\Big[f^q = \frac{f}{f(a) - 1}\Big] \ .$$

Proof. Let $f \in A^*$. Then $g \in A^*$ is right quasi-inverse for f if and only if $f + g = f \bigcirc^F g$, so we have $f + g = F(f)g$. Hence $g = \dfrac{f}{F(f) - 1}$ if $F(f) \neq 1$. For the left quasi-inverse g for f we have $f + g = g \bigcirc^F f$, so

$f+g=F(g)f$. Therefore $F(f)+F(g)=F(g)F(f)$ and then $F(g)=\dfrac{F(f)}{F(f)-1}$. By replacing $F(g)$ in $F+g=F(g)f$ we conclude that $g=\big(\dfrac{F(f)}{F(f)-1}-1\big)f=\dfrac{f}{F(f)-1}$. This proves part (i) and the equality $f^q=\dfrac{f}{F(f)-1}$. The rest of proof is obvious for $(A^*,\bigcirc_a)=(A^*,\bigcirc^{\hat{a}})$.
□

Corollary 4.2. *Let A be a Banach algebra. Then*
(i) $inv(A^*,\bigcirc^F)^\#=\{(f,\lambda)\ :\ \lambda\neq 0\ \ ,\ \ \lambda\neq -F(f)\}$,
(ii) $inv(A^*,\bigcirc_a)^\#=\{(f,\lambda)\ :\ \lambda\neq 0\ \ ,\ \ \lambda\neq -f(a)\}$,
and for each $(f,\lambda)\in inv(A^*,\bigcirc^F)^\#$ $\big[(f,\lambda)\in inv(A^*,\bigcirc_a)^\#\big]$ *we have*

$$(f,\lambda)^{-1}=\Big(\frac{-f}{\lambda\big(F(f)+\lambda\big)},\frac{1}{\lambda}\Big)\ \Big[(f,\lambda)^{-1}\Big(\frac{-f}{\lambda\big(f(a)+\lambda\big)},\frac{1}{\lambda}\Big)\Big]\ .$$

Proof. It is straightforward by using (1). □

Corollary 4.3. *For $f\in A^*$ we have*

$$\begin{aligned}\sigma_{(A^*,\bigcirc^F)}(f)&=\{0,F(f)\}\ ,\\ \sigma_{(A^*,\bigcirc_a)}(f)&=\{0,f(a)\}\ ,\end{aligned}$$

and for spectral radius of f we have

$$\begin{aligned}r_{(A^*,\bigcirc^F)}(f)&=|F(f)|\ ,\\ r_{(A^*,\bigcirc_a)}(f)&=|f(a)|\ .\end{aligned}$$

Proof. By corollary 4.2 one can write

$$\begin{aligned}\sigma_{(A^*,\bigcirc^F)}(f)=:\sigma_{(A^*,\bigcirc^F)^\#}(f,0)&=\{\lambda\in\mathbb{C}\ :\ \lambda(0,1)-(f,0)\notin inv(A^*,\bigcirc^F)^\#\}\\ &=\{\lambda\in\mathbb{C}\ :\ (-f,\lambda)\notin inv(A^*,\bigcirc^F)^\#\}\\ &=\{\lambda\in\mathbb{C}\ :\ \lambda=0\text{ or }\lambda=-F(-f)\}\\ &=\{0,F(f)\}\ ,\end{aligned}$$

and with a similar proof for $\sigma_{(A^*,\bigcirc_a)}(f)$. Alse we have

$$r_{(A^*,\bigcirc^F)}(f)=\lim_{n\to\infty}||f^n||^{\frac{1}{n}}=\lim_{n\to\infty}|F(f)|^{\frac{n-1}{n}}||f||^{\frac{1}{n}}=|F(f)|\ ,$$

and with a similar proof for $r_{(A^*,\bigcirc_a)}(f)$. □

Example 4.4. According to Riesz representation theorem we have

(i) the linear isometric isomorphism $H^* \approx H$, where H is a Hilbert space, so for each $T \in H^*$ there exists a unique $y_T \in H$ such that

$$T(x) = (y_T, x) \qquad (x \in H) ,$$

where $(\cdot, \cdot)$ denotes the inner product of H. Now if we consider one of the products $\bigcirc_a$ or $\bigcirc^F$ in H^*, with $a \in H$ and $F \in H^{**}$ we will have

$$y_{T \bigcirc_a S} = T(a) y_S \quad , \quad y_{T \bigcirc^F S} = F(T) y_S \qquad (T, S \in H^*) .$$

(ii) the linear isometric isomorphism $C_0(X)^* \approx M(X)$, where X is a locally compact Hausdorff space, so for each $T \in C_0(X)^*$ there exists a unique measure $\mu_T \in M(X)$ such that

$$T(f) = \int_X f \ d\mu \qquad \big(f \in C_0(X)\big) .$$

Now with one of the products $\bigcirc_h$ or $\bigcirc^\varphi$ in $C_0(X)^*$, where $h \in C_0(X)$ and $\varphi \in C_0(X)^{**}$ it is easy to obtain

$$\mu_{T \bigcirc_h S} = T(h) \mu_S \quad , \quad \mu_{T \bigcirc^\varphi S} = \varphi(T) \mu_S \qquad \big(T, S \in C_0(X)^*\big) .$$

(iii) the linear isometric isomorphism $(L^p)^* \approx L^q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 \leqslant p < +\infty$. For each $T \in (L^p)^*$ there exists a unique $g_T \in L^q$ such that

$$T(f) = \int f g_T \qquad (f \in L^p) .$$

Now for the products $\bigcirc_h$ and $\bigcirc^\varphi$ in $(L^p)^*$ we have

$$g_{T \bigcirc_h S} = T(h) g_S \quad , \quad g_{T \bigcirc^\varphi S} = \varphi(T) g_S \qquad \big(T, S \in (L^p)^*\big) ,$$

where $h \in L^p$ and $\varphi \in (L^p)^{**}$.

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