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# $S$-2-Absorbing Second Submodules

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**Abstract.** Let $R$ be a commutative ring with identity, $S$ be a multiplicatively closed subset of $R$, and let $M$ be an $R$-module. In this paper, we introduce and investigate some properties of the notion of $S$-2-absorbing second submodules of $M$ as a generalization of $S$-second submodules and strongly 2-absorbing second submodules of $M$. Also, we obtain some results concerning $S$-2-absorbing submodules of $M$.

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## 1 Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [14]. A non-zero submodule $N$ of $M$ is said to be *second* if for each $a \in R$, the homomorphism $N \stackrel{a}{\to} N$ is either surjective or zero [27]. A proper ideal $I$ of $R$ is called a *2-absorbing ideal* of $R$ if whenever

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$a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$ [10]. A proper submodule $N$ of $M$ is called a *2-absorbing submodule* of $M$ if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$ [13, 22]. A non-zero submodule $N$ of $M$ is said to be a *strongly 2-absorbing second submodule of* $M$ if whenever $a, b \in R$, $K$ is a submodule of $M$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$ [8].

A non-empty subset $S$ of $R$ is called a multiplicatively closed subset of $R$ if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $s\acute{s} \in S$ for all $s, \acute{s} \in S$ [26]. Let $S$ be a multiplicatively closed subset of $R$. A submodule $P$ of an $R$-module $M$ with $(P :_R M) \cap S = \emptyset$ is said to be an *$S$-prime submodule* of $M$ if there exists a fixed $s \in S$, and whenever $am \in P$, then $sa \in (P :_R M)$ or $sm \in P$ for each $a \in R$, $m \in M$ [24]. Particularly, an ideal $I$ of $R$ is said to be an *$S$-prime ideal* if $I$ is an $S$-prime submodule of the $R$-module $R$. A submodule $N$ of an $R$-module $M$ with $Ann_R(N) \cap S = \emptyset$ is said to be an *$S$-second submodule* of $M$ if there exists a fixed $s \in S$, and whenever $rN \subseteq K$, where $r \in R$ and $K$ is a submodule of $M$, then $rsN = 0$ or $sN \subseteq K$ [17].

Let $M$ be an $R$-module and $S$ be a multiplicatively closed subset of $R$. In [25], the authors introduced the notion of $S$-2-absorbing submodules of $M$ which is a generalization of $S$-prime submodules and 2-absorbing submodules and investigated some properties of this class of submodules. A submodule $P$ of $M$ is said to be an *$S$-2-absorbing* if $(P :_R M) \cap S = \emptyset$ and there exists a fixed $s \in S$ such that $abm \in P$ for some $a, b \in R$ and $m \in M$ implies that $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. In particular, an ideal $I$ of $R$ is said to be an *$S$-2-absorbing ideal* if $I$ is an $S$-2-absorbing submodule of the $R$-module $R$ [25]. Also, for the some recent works on $S$-version of some algebraic structures, we refer the reader to [1, 16, 23, 28].

The main purpose of this paper is to introduce the notion of $S$-2-absorbing second submodules of an $R$-module $M$ as a generalization of $S$-second submodules and strongly 2-absorbing second submodules of $M$. Also, this can be regarded as a dual notion of the $S$-2-absorbing submodules of $M$. We provide some information about this class of submodules. Moreover, we investigate some properties of $S$-2-absorbing submodules of $M$.

## 2 $S$-2-Absorbing Submodules

The following theorem gives a useful characterization of $S$-2-absorbing submodules.

**Theorem 2.1.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be a submodule of an $R$-module $M$ with $(N :_R M) \cap S = \emptyset$. Then $N$ is $S$-2-absorbing if and only if there is a fixed $s \in S$ such that for every $a, b \in R$, we have either $(N :_M s^2ab) = (N :_M s^2a)$ or $(N :_M s^2ab) = (N :_M s^2b)$ or $(N :_M s^3ab) = M$.

**Proof.** Let $N$ be an $S$-2-absorbing submodule of $M$ and $m \in (N :_M s^2ab)$. Then $(sa)(sb)m \in N$. Assume that $(N :_M s^3ab) \neq M$, that is, $s^3ab \notin (N :_R M)$. So by assumption, either $s^2am \in N$ or $s^2bm \in N$. This implies that $(N :_M s^2ab) \subseteq (N :_M s^2a) \cup (N :_M s^2b)$. Clearly, $(N :_M s^2a) \cup (N :_M s^2b) \subseteq (N :_M s^2ab)$. So, $(N :_M s^2a) \cup (N :_M s^2b) = (N :_M s^2ab)$. As $N$ is a submodule of $M$, it cannot be written as union of two distinct submodules. Thus $(N :_M s^2ab) = (N :_M s^2a)$ or $(N :_M s^2ab) = (N :_M s^2b)$. Conversely, let $a, b \in R$ and $m \in M$ such that $abm \in N$. Then $m \in (N :_R s^2ab)$. By given hypothesis, we have $(N :_M s^2ab) = (N :_M s^2a)$ or $(N :_M s^2ab) = (N :_M s^2b)$ or $(N :_M s^3ab) = M$. Thus $s^2am \in N$ or $s^2bm \in N$ or $s^3ab \in (N :_R M)$. Hence, $s^3am \in N$ or $s^3bm \in N$ or $s^3ab \in (N :_R M)$. Now by setting $s_1 = s^3$, we get the result. $\square$

**Lemma 2.2.** [9, Lemma 3.2] Let $N$ be a submodule of an $R$-module $M$ and $r \in R$. Then for every flat $R$-module $F$, we have $F \otimes (N :_M r) = (F \otimes N :_{F \otimes M} r)$.

**Theorem 2.3.** Let $S$ be a multiplicatively closed subset of $R$, $N$ be an $S$-2-absorbing submodule of an $R$-module $M$, and $F$ be a flat $R$-module. If $(F \otimes N :_R F \otimes M) \cap S = \emptyset$, then $F \otimes N$ is an $S$-2-absorbing submodule of $F \otimes M$.

**Proof.** Since $N$ is an $S$-2-absorbing submodule of $M$, by Theorem 2.1, we have either $(N :_M s^2ab) = (N :_M s^2a)$ or $(N :_M s^2ab) = (N :_M s^2b)$ or $(N :_M s^3ab) = M$ for $a, b \in R$. Assume that $(N :_M s^2ab) = (N :_M s^2a)$. Then by Lemma 2.2, we have

$$(F \otimes N :_{F \otimes M} s^2ab) = F \otimes (N :_M s^2ab) = F \otimes (N :_M s^2a) = (F \otimes N :_{F \otimes M} s^2a).$$

If $(N :_M s^3ab) = M$, then by Lemma 2.2, we have

$$(F \otimes N :_{F\otimes M} s^3ab) = F \otimes (N :_M s^3ab) = F \otimes M.$$

Hence by Theorem 2.1, $F \otimes N$ is $S$-2-absorbing submodule of $F \otimes M$. □

**Theorem 2.4.** Let $S$ be a multiplicatively closed subset of $R$ and $F$ be a faithfully flat $R$module. Then $N$ is an $S$-2-absorbing submodule of $M$ if and only if $F \otimes N$ is an $S$-2-absorbing submodule of $F \otimes M$.

**Proof.** Let $N$ be an $S$-2-absorbing submodule of $M$. Suppose $(F\otimes N :_R F \otimes M) \cap S \neq \emptyset$. Then there is an $t \in (F \otimes N :_R F \otimes M) \cap S$. Thus $F \otimes tM \subseteq F \otimes N$. Hence, $0 \to F \otimes tM \to F \otimes N$ is an exact sequence. Since $F$ is a faithfully flat, $0 \to tM \to N$ is an exact which implies that $tM \subseteq N$. Thus $(N :_R M) \cap S \neq \emptyset$, this is a contradiction. So $(F \otimes N :_R F \otimes M) \cap S = \emptyset$. Now by Theorem 2.3, we have $F \otimes N$ is an $S$-2-absorbing submodule of $F \otimes M$. Conversely, suppose $F \otimes N$ is an $S$-2-absorbing submodule of $F \otimes M$. Then $(F \otimes N :_R F \otimes M) \cap S = \emptyset$ implies that $(N :_R M) \cap S = \emptyset$. Let $a, b \in R$. Then by Theorem 2.1, we can assume that $(F \otimes N :_{F\otimes M} s^2ab) = (F \otimes N :_{F\otimes M} s^2a)$. By Lemma 2.2, we have

$$F\otimes(N :_M s^2ab) = (F\otimes N :_{F\otimes M} s^2ab) = (F\otimes N :_{F\otimes M} s^2a) = F\otimes(N :_M s^2a).$$

So, $0 \to F \otimes (N :_M s^2ab) \to F \otimes (N :_M s^2a) \to 0$ is an exact sequence. As $F$ is a faithfully flat, $0 \to (N :_M s^2ab) \to (N :_M s^2a) \to 0$ is an exact sequence. Thus $(N :_M s^2ab) = (N :_M s^2a)$ and so by Theorem 2.1, $N$ is $S$-2-absorbing. If $(F \otimes N :_{F\otimes M} s^3ab) = F \otimes M$, then $F \otimes (N :_M s^3ab) = (F \otimes N :_{F\otimes M} s^3ab) = F \otimes M$. So,

$$0 \to F \otimes (N :_M abs^3) \to F \otimes M \to 0$$

is an exact sequence. As $F$ is a faithfully flat, $0 \to (N :_M s^3ab) \to M \to 0$ is an exact sequence. Thus $(N :_M s^3an) = M$. Hence $N$ is an $S$-2-absorbing submodule of $M$. □

**Proposition 2.5.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be an $S$-2-absorbing submodule of an $R$-module $M$. Then the following statements hold for some $s \in S$.

(a) $(N :_M th) \subseteq (N :_M ts)$ or $(N :_M th) \subseteq (N :_M sh)$ for all $t, h \in S$.

(b) $((N :_R M) :_R th) \subseteq ((N :_R M) :_R ts)$ or $((N :_R M) :_R th) \subseteq ((N :_R M) :_R sh)$ for all $t, h \in S$.

**Proof.** (a) Let $N$ be an $S$-2-absorbing submodule of $M$. Then there is a fixed $s \in S$. Take an element $m \in (N :_M th)$, where $t, h \in S$. Then $stm \in N$ or $shm \in N$ or $sth \in (N :_R M)$. As $(N :_R M) \cap S = \emptyset$, we have $sth \notin (N :_R M)$. If for each $m \in (N :_M th)$, we have $stm \in N$ (resp. $shm \in N$), then we are done. So suppose that there are $m_1 \in (N :_M th)$ such that $stm_1 \notin N$ and $m_2 \in (N :_M th)$ such that $shm_2 \notin N$. Then we conclude that $shm_1 \in N$ and $stm_2 \in N$. Now $ht(m_1 + m_2) \in N$ implies that $hs(m_1 + m_2) \in N$ or $st(m_1 + m_2) \in N$. Thus $stm_1 \in N$ or $shm_2 \in N$, which is a desired contradiction.

(b) This follows from part (a). □

**Lemma 2.6.** Let $S$ be a multiplicatively closed subset of $R$ and $I$ be an $S$-2-absorbing ideal of $R$. Then $\sqrt{I}$ is an $S$-2-absorbing ideal of $R$ and there is a fixed $s \in S$ such that $sa^2 \in I$ for every $a \in \sqrt{I}$.

**Proof.** Clearly, as $I$ is an $S$-2-absorbing ideal of $R$, there is a fixed $s \in S$ such that $sa^2 \in I$ for every $a \in \sqrt{I}$. Now let $a, b, c \in R$ such that $abc \in \sqrt{I}$. Then $sa^2b^2c^2 = s(abc)^2 \in I$. Since $I$ is a $S$-2-absorbing ideal of $R$, we may assume that $s^2a^2b^2 \in I$. This implies that $sab \in \sqrt{I}$, as needed. □

Recall that an $R$-module $M$ is said to be a *multiplication module* if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$ [11].

Let $N$ be a proper submodule of an $R$-module $M$. Then the *$M$-radical* of $N$, denoted by $rad(N)$, is defined to be the intersection of all prime submodules of $M$ containing $N$ [20].

**Theorem 2.7.** Let $S$ be a multiplicatively closed subset of $R$ and $M$ a finitely generated multiplication $R$-module. If $N$ is an $S$-2-absorbing submodule of $M$, then $rad(N)$ is an $S$-2-absorbing submodule of $M$.

**Proof.** Since $N$ is an $S$-2-absorbing submodule of $M$, we have $(N :_R M)$ is a $S$-2-absorbing ideal of $R$ by [25, Proposition 3]. Thus by Lemma 2.6, $\sqrt{(N :_R M)}$ is an $S$-2-absorbing ideal of $R$. By [20, Theorem 4],

$(rad(N) :_R M)) = \sqrt{(N :_R M)}$. Therefore, $(rad(N) :_R M)$ is an $S$-2-absorbing ideal of $R$. Now the result follows from [25, Proposition 3]. □

## 3 $S$-2-Absorbing Second Submodules

**Definition 3.1.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be a submodule of an $R$-module $M$ such that $Ann_R(N) \cap S = \emptyset$. We say that $N$ is a *$S$-2-absorbing second submodule of $M$* if there exists a fixed $s \in S$ and whenever $abN \subseteq K$, where $a, b \in R$ and $K$ is a submodule of $M$, implies either that $saN \subseteq K$ or $sbN \subseteq K$ or $sabN = 0$. In particular, an ideal $I$ of $R$ is said to be an *$S$-2-absorbing second ideal* if $I$ is an $S$-2-absorbing second submodule of the $R$-module $R$. By a *$S$-2-absorbing second module*, we mean a module which is a $S$-2-absorbing second submodule of itself.

**Lemma 3.2.** Let $S$ be a multiplicatively closed subset of $R$, $I$ an ideal of $R$, and let $N$ be an $S$-2-absorbing second submodule of $M$. Then there exists a fixed $s \in S$ and whenever $a \in R$, $K$ is a submodule of $M$, and $IaN \subseteq K$, then $asN \subseteq K$ or $IsN \subseteq K$ or $Ias \subseteq Ann_R(N)$.

**Proof.** Let $asN \not\subseteq K$ and $Ias \not\subseteq Ann_R(N)$. Then there exists $b \in I$ such that $absN \neq 0$. Now as $N$ is a $S$-2-absorbing second submodule of $M$, $baN \subseteq K$ implies that $bsN \subseteq K$. We show that $IsN \subseteq K$. To see this, let $c$ be an arbitrary element of $I$. Then $(b + c)aN \subseteq K$. Hence, either $(b + c)sN \subseteq K$ or $(b + c)as \in Ann_R(N)$. If $(b + c)sN \subseteq K$, then since $bsN \subseteq K$ we have $csN \subseteq K$. If $(b + c)as \in Ann_R(N)$, then $cas \notin Ann_R(N)$, but $caN \subseteq K$. Thus $csN \subseteq K$. So, we conclude that $sIN \subseteq K$, as requested. □

**Lemma 3.3.** Let $S$ be a multiplicatively closed subset of $R$, $I$ and $J$ be two ideals of $R$, let and $N$ be a $S$-2-absorbing second submodule of $M$. Then there exists a fixed $s \in S$ and whenever $K$ is a submodule of $M$ and $IJN \subseteq K$, then $sIN \subseteq K$ or $sJN \subseteq K$ or $IJs \subseteq Ann_R(N)$.

**Proof.** Let $IsN \not\subseteq K$ and $JsN \not\subseteq K$. We show that $IJs \subseteq Ann_R(N)$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $asN \not\subseteq K$ but $aJN \subseteq K$. Now Lemma 3.2 shows that $aJs \subseteq Ann_R(N)$

and so $(I\backslash(K:_R N))Js \subseteq Ann_R(N)$. Similarly there exists $b \in (J\backslash(K:_R N))$ such that $Ibs \subseteq Ann_R(N)$ and also $I(J \setminus (K:_R N))s \subseteq Ann_R(N)$. Thus we have $abs \in Ann_R(N)$, $ads \in Ann_R(N)$ and $cbs \in Ann_R(N)$. As $a+c \in I$ and $b+d \in J$, we have $(a+c)(b+d)N \subseteq K$. Therefore, $(a+c)sN \subseteq K$ or $(b+d)sN \subseteq K$ or $(a+c)(b+d)s \in Ann_R(N)$. If $(a+c)sN \subseteq K$, then $csN \not\subseteq K$. Hence $c \in I \setminus (K:_R N)$ which implies that $cds \in Ann_R(N)$. Similarly if $(b+d)sN \subseteq K$, we can deduce that $cds \in Ann_R(N)$. Finally if $(a+c)(b+d)s \in Ann_R(N)$, then $(ab+ad+cb+cd)s \in Ann_R(N)$ so that $cds \in Ann_R(N)$. Therefore, $IJs \subseteq Ann_R(N)$. □

Let $M$ be an $R$-module. A proper submodule $N$ of $M$ is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$. Thus the intersection of all completely irreducible submodules of $M$ is zero. [18].

**Remark 3.4.** Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$ [5].

Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. $S$-2-absorbing second submodules of $M$ can be characterized in various ways as we demonstrate in the following theorem.

**Theorem 3.5.** Let $S$ be a multiplicatively closed subset of $R$. For a submodule $N$ of an $R$-module $M$ with $Ann_R(N) \cap S = \emptyset$ the following statements are equivalent:

(a) $N$ is an $S$-2-absorbing second submodule of $M$;

(b) There exists a fixed $s \in S$ such that $s^2abN = s^2aN$ or $s^2abN = s^2bN$ or $s^3abN = 0$ for each $a, b \in R$;

(c) There exists a fixed $s \in S$ and whenever $abN \subseteq L_1 \cap L_2$, where $a, b \in R$ and $L_1, L_2$ are completely irreducible submodules of $M$, implies either that $absN = 0$ or $saN \subseteq L_1 \cap L_2$ or $sbN \subseteq L_1 \cap L_2$.

(d) There exists a fixed $s \in S$, and $IJN \subseteq K$ implies either that $sIJ \subseteq Ann_R(N)$ or $sIN \subseteq K$ or $sJN \subseteq K$ for each ideals $I, J$ of $R$ and submodule $K$ of $M$.

**Proof.** $(b) \Rightarrow (a)$ Let $a, b \in R$ and $K$ be a submodule of $M$ with $abN \subseteq K$. By part (b), there exists a fixed $s \in S$ such that $s^2abN = s^2aN$ or $s^2abN = s^2bN$ or $s^3abN = 0$. Thus either $s^3abN = 0$ or $s^3aN \subseteq s^2aN = abs^2N \subseteq s^2K \subseteq K$ or $s^3bN \subseteq s^2bN = abs^2N \subseteq s^2K \subseteq K$. Therefore, by setting $\acute{s} := s^3$, we have either $\acute{s}aN \subseteq K$ or $\acute{s}bN \subseteq K$ or $\acute{s}abN = 0$, as needed.

$(a) \Rightarrow (c)$ This is clear.

$(c) \Rightarrow (b)$ By part (c), there exists a fixed $s \in S$. Assume that there are $a, b \in R$ such that $s^2abN \neq s^2aN$ and $s^2abN \neq s^2bN$. Then there exist completely irreducible submodules $L_1$ and $L_2$ of $M$ such that $s^2abN \subseteq L_1$, $s^2abN \subseteq L_2$, $s^2aN \not\subseteq L_1$, and $s^2bN \not\subseteq L_2$ by Remark 3.6. Now $(as)(bs)N = s^2abN \subseteq L_1 \cap L_2$ implies that either $s^2aN \subseteq L_1 \cap L_2$ or $s^2bN \subseteq L_1 \cap L_2$ or $s^3abN = 0$ by part (c). If $s^2aN \subseteq L_1 \cap L_2$ or $s^2bN \subseteq L_1 \cap L_2$, then $s^2aN \subseteq L_1$ or $s^2bN \subseteq L_2$, which are contradiction. Thus $s^3abN = 0$, as required.

$(a) \Rightarrow (d)$ By Lemma 3.3.

$(d) \Rightarrow (a)$ Take $a, b \in R$ and $K$ a submodule of $M$ with $abN \subseteq K$. Now, put $I = Ra$ and $J = Rb$. Then we have $IJN \subseteq K$. By assumption, there is a fixed $s \in S$ such that either $sIJ = s(Ra)(Rb) \subseteq Ann_R(N)$ or $sIN \subseteq K$ or $sJN \subseteq K$ and so either $sab \in Ann_R(N)$ or $saN \subseteq K$ or $sbN \subseteq K$, as needed. □

**Remark 3.6.** Let $M$ be an $R$-module and $S$ a multiplicatively closed subset of $R$. Clearly, every $S$-second submodule of $M$ and every strongly 2-absorbing second submodule $N$ of $M$ with $Ann_R(N) \cap S = \emptyset$ is an $S$-2-absorbing second submodule of $M$. But the converse is not true in general, as we can see in the following examples.

**Example 3.7.** Consider $\mathbb{Z}_4$ as an $\mathbb{Z}$-module. Clearly, $\mathbb{Z}_4$ is not a strongly 2-absorbing second $\mathbb{Z}$-module. Set $S := \mathbb{Z} \setminus 2\mathbb{Z}$. Then for each $s \in S$, $2\mathbb{Z}_4 = 2s\mathbb{Z}_4 \neq s\mathbb{Z}_4 = \mathbb{Z}_4$ and $2s\mathbb{Z}_4 \neq 0$ implies that $\mathbb{Z}_4$ is not an $S$-second $\mathbb{Z}$-module. But, if we consider $s = 1$, and $n, m \in \mathbb{Z}$, then we have three cases:

**Case 1** If $n \neq 2k$ and $m \neq 2k$ for each $k \in \mathbb{N}$, then

$$nm(1)^2\mathbb{Z}_4 = \mathbb{Z}_4 = (1)^2n\mathbb{Z}_4 = (1)^2m\mathbb{Z}_4.$$

**Case 2** If $n = 2k_1$ and $m = 2k_2$ for some $k_1, k_2 \in \mathbb{N}$, then $nm(1)^3\mathbb{Z}_4 = 0$.

**Case 3** If $n = 2k_1$ for some $k_1 \in \mathbb{N}$ and $m \neq 2k$ for each $k \in \mathbb{N}$, then

$$nm(1)^2\mathbb{Z}_4 = \bar{2}\mathbb{Z}_4 = (1)^2 n\mathbb{Z}_4.$$

Thus by Theorem 3.5 $(b) \Rightarrow (a)$, $\mathbb{Z}_4$ is an $S$-2-absorbing second $\mathbb{Z}$-module.

**Example 3.8.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{pq}$, where $p \neq q$ are prime numbers. Then $M$ is not a strongly 2-absorbing second $\mathbb{Z}$-module since

$$pqM = \mathbb{Z}_{p^\infty} \oplus 0 \neq \mathbb{Z}_{p^\infty} \oplus p\mathbb{Z}_{pq} = pM,$$

$$pqM = \mathbb{Z}_{p^\infty} \oplus 0 \neq \mathbb{Z}_{p^\infty} \oplus q\mathbb{Z}_{pq} = qM,$$

and $pqM = \mathbb{Z}_{p^\infty} \oplus 0 \neq 0$. Now, take the multiplicatively closed subset $S = \mathbb{Z} \setminus \{0\}$ and put $s = pq$. Then $s^2pqM = \mathbb{Z}_{p^\infty} \oplus 0 = s^2pM$ implies that $M$ is an $S$-2-absorbing second $\mathbb{Z}$-module by Theorem 3.5 $(b) \Rightarrow (a)$.

The following lemma is known, but we write it here for the sake of completeness.

**Lemma 3.9.** Let $M$ be an $R$-module, $S$ a multiplicatively closed subset of $R$, and $N$ be a finitely generated submodule of $M$. If $S^{-1}N \subseteq S^{-1}K$ for a submodule $K$ of $M$, then there exists an $s \in S$ such that $sN \subseteq K$.

**Proof.** This is straightforward. $\square$

Let $S$ be a multiplicatively closed subset of $R$. Recall that the saturation $S^*$ of $S$ is defined as $S^* = \{x \in R : x/1 \text{ is a unit of } S^{-1}R\}$. It is obvious that $S^*$ is a multiplicatively closed subset of $R$ containing $S$ [19].

**Proposition 3.10.** Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an R-module. Then we have the following.

(a) If $N$ is a strongly 2-absorbing second submodule of $M$ such that $S \cap Ann_R(N) = \emptyset$, then $N$ is a $S$-2-absorbing second submodule of $M$. In fact if $S \subseteq u(R)$ and $N$ is an $S$-2-absorbing second submodule of $M$, then $N$ is a strongly 2-absorbing second submodule of $M$.

(b) If $S_1 \subseteq S_2$ are multiplicatively closed subsets of $R$ and $N$ is an $S_1$-2-absorbing second submodule of $M$, then $N$ is an $S_2$-2-absorbing second submodule of $M$ in case $Ann_R(N) \cap S_2 = \emptyset$.

(c) $N$ is an $S$-2-absorbing second submodule of $M$ if and only if $N$ is an $S^*$-2-absorbing second submodule of $M$

(d) If $N$ is a finitely generated $S$-2-absorbing second submodule of $M$, then $S^{-1}N$ is a strongly 2-absorbing second submodule of $S^{-1}M$.

**Proof.** (a) and (b) These are clear.

(c) Assume that $N$ is an $S$-2-absorbing second submodule of $M$. We claim that $Ann_R(N) \cap S^* = \emptyset$. To see this assume that there exists an $x \in Ann_R(N) \cap S^*$ As $x \in S^*$, $x/1$ is a unit of $S^{-1}R$ and so $(x/1)(a/s) = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Now we have that $us = uxa \in Ann_R(N) \cap S$, a contradiction. Thus, $Ann_R(N) \cap S^* = \emptyset$. Now as $S \subseteq S^*$, by part (b), $N$ is an $S^*$-2-absorbing second submodule of $M$. Conversely, assume that $N$ is an $S^*$-2-absorbing second submodule of $M$. Let $rtN \subseteq K$ for some $r, t \in R$. As $N$ is an $S^*$-2-absorbing second submodule of $M$, there is a fixed $x \in S^*$ such that $xrt \in Ann_R(N)$ or $xrN \subseteq K$ or $xtN \subseteq K$. As $x/1$ is a unit of $S^{-1}R$, there exist $u, s \in S$ and $a \in R$ such that $us = uxa$. Then note that $(us)rt = uaxrt \in Ann_R(N)$ or $us(txN) \subseteq K$ or $us(rxN) \subseteq K$. Therefore, $N$ is a $S$-2-absorbing second submodule of $M$.

(d) As $N$ is an $S$-2-absorbing second submodule of $M$, there is a fixed $s \in S$. If $S^{-1}N = 0$, then as $N$ is finitely generated, there is an $t \in S$ such that $t \in Ann_R(N)$ by Lemma 3.9. Thus $Ann_R(N) \cap S \neq \emptyset$, a contradiction. So, $S^{-1}N \neq 0$. Now let $a/t, b/h \in S^{-1}R$. As $N$ is an $S$-2-absorbing second submodule of $M$, we have either $abs^2N = as^2N$ or $abs^2N = bs^2N$ or $abs^3N = 0$. This implies that either $(a/t)(b/h)S^{-1}N = (a/t)S^{-1}N$ or $(a/t)(b/h)S^{-1}N = (b/h)S^{-1}N$ or $(a/t)(b/h)S^{-1}N = 0$, as needed. □

The following example shows that the converse of Proposition 3.10 (d) is not true in general.

**Example 3.11.** Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take the submodule $N = \mathbb{Z} \oplus \mathbb{Z} \oplus 0$ and the multiplicatively closed subset $S = \mathbb{Z} \setminus \{0\}$. Now, take $s \in S$. Then there exist prime numbers $p \neq q$ such that $gcd(p, s) = gcd(q, s) = 1$. Then one can see that $s^2pqN \neq s^2pN$, $s^2pqN \neq s^2qN$, and $s^3pqN \neq 0$. Thus $N$ is not an $S$-2-absorbing second submodule of $M$. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field, $S^{-1}(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})$ is a vector space and so the non-zero submodule $S^{-1}N$ is a strongly 2-absorbing second submodule of $S^{-1}(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})$.

**Theorem 3.12.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be a submodule of an $R$-module $M$ such that $Ann_R(N) \cap S = \emptyset$. Then $N$ is an $S$-2-absorbing second submodule of $M$ if and only if $s^3N$ is a strongly 2-absorbing second submodule of $M$ for some $s \in S$.

**Proof.** Let $s^3N$ be a strongly 2-absorbing second submodule of $M$ for some $s \in S$ and $a, b \in R$. Then $abs^3N = as^3N$ or $abs^3N = bs^3N$ or $abs^3N = 0$ by [8, Theorem 3.3]. Hence $s^6abN = s^6aN$ or $s^6abN = s^6bN$ or $s^9abN = 0$. Set $t := s^3$. Then $t^2abN = t^2aN$ or $t^2abN = t^2bN$ or $t^3abN = 0$. Thus by Theorem 3.5 $(b) \Rightarrow (a)$, $N$ is an $S$-2-absorbing second submodule of $M$. Conversely, suppose that $N$ is an $S$-2-absorbing second submodule of $M$ and $a, b \in R$. Then for some $s \in S$ we have $s^2abN = s^2aN$ or $s^2abN = s^2bN$ or $s^3abN = 0$ by Theorem 3.5 $(b) \Rightarrow (a)$. This implies that $abs^3N = as^3N$ or $abs^3N = bs^3N$ or $abs^3N = 0$. We not that $Ann_R(N) \cap S = \emptyset$, implies that $s^3N \neq 0$. Therefore, by [8, Theorem 3.3], $s^3N$ is a strongly 2-absorbing second submodule of $M$. □

**Proposition 3.13.** Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. Let $N \subset K$ be two submodules of $M$ and $K$ be a $S$-2-absorbing second submodule of $M$. Then $K/N$ is a $S$-2-absorbing second submodule of $M/N$.

**Proof.** This is straightforward. □

**Proposition 3.14.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be an $S$-2-absorbing second submodule of an $R$-module $M$. Then we have the following.

(a) $Ann_R(N)$ is a $S$-2-absorbing ideal of $R$.

(b) If $K$ is a submodule of $M$ such that $(K :_R N) \cap S = \emptyset$, then $(K :_R N)$ is a $S$-2-absorbing ideal of $R$.

(c) There exists a fixed $s \in S$ such that $s^n N = s^{n+1} N$, for all $n \geq 3$.

**Proof.** (a) Let $a, b, c \in R$ and $abc \in Ann_R(N)$. Then there exists a fixed $s \in S$ and $abN \subseteq abN$ implies that $asN \subseteq abN$ or $bsN \subseteq abN$ or $sabN = 0$. If $sabN = 0$, then we are done. If $asN \subseteq abN$, then $casN \subseteq cabN = 0$. In other case, we do the same.

(b) Let $a, b, c \in R$ and $abc = (ac)b \in (K :_R N)$. Then there exists a fixed $s \in S$ such that $acsN \subseteq K$ or $cbsN \subseteq bsN \subseteq K$ or $sabc \in Ann_R(N) \subseteq (K :_R N)$, as needed.

(c) As $N$ is an $S$-2-absorbing second submodule of $M$, there exists a fixed $s \in S$. It is enough to show that $s^3N = s^4N$. It is clear that $s^4N \subseteq s^3N$. Since $N$ is an $S$-2-absorbing second submodule, $(s^2)(s^2)N = s^4N \subseteq s^4N$ implies that either $s^3N \subseteq s^4N$ or $s^5N = 0$. If $s^5N = 0$, then $s^5 \in Ann_R(N) \cap S = \emptyset$ which is a contradiction. Thus $s^3N \subseteq s^4N$, as desired. □

**Proposition 3.15.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be a $S$-2-absorbing second submodule of $M$. Then the following statements hold for some $s \in S$.

(a) $tsN \subseteq thN$ or $hsN \subseteq thN$ for all $t, h \in S$.

(b) $(Ann_R(N) :_R th) \subseteq (Ann_R(N) :_R ts)$ or $(Ann_R(N) :_R th) \subseteq (Ann_R(N) :_R sh)$ for all $t, h \in S$.

**Proof.** (a) Let $N$ be a $S$-2-absorbing second submodule of $M$. Then there is a fixed $s \in S$. Let $L$ be a completely irreducible submodule of $M$ such that $thN \subseteq L$, where $t, h \in S$. Then $tsN \subseteq L$ or $shN \subseteq L$ or $sth \in Ann_R(N)$. As $Ann_R(N) \cap S = \emptyset$, we have $sth \notin Ann_R(N)$. If for each completely irreducible submodule of $M$, we have $tsN \subseteq L$ (resp. $shN \subseteq L$), then we are done by Remark 3.6. So suppose that there are completely irreducible submodules $L_1$ and $L_2$ of $M$ such that $tsN \not\subseteq L_1$ and $shN \not\subseteq L_2$. Then since $N$ is a $S$-2-absorbing second submodule of $M$, we conclude that $hsN \subseteq L_1$ and $stN \subseteq L_2$. Now $htN \subseteq L_1 \cap L_2$ implies that $hsN \subseteq L_1 \cap L_2$ or $stN \subseteq L_1 \cap L_2$. Thus $tsN \subseteq L_1$ or $shN \subseteq L_2$, a contradiction.

(b) This follows from Proposition 3.15 (a) and Proposition 2.5 (a). □

An $R$-module $M$ is said to be a *comultiplication module* if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$, equivalently, for each submodule $N$ of $M$, we have $N = (0 :_M Ann_R(N))$ [3].

**Lemma 3.16.** Let $S$ be a multiplicatively closed subset of $R$ and $N$ be a submodule of a comultiplication $R$-module $M$. If $Ann_R(N)$ is a $S$-2-absorbing ideal of $R$, then $N$ is a $S$-2-absorbing second submodule of $M$.

**Proof.** Let $a, b \in R$, $K$ be a submodule of $M$, and $abN \subseteq K$. Then we have $Ann_R(K)abN = 0$. As $Ann_R(N)$ is a $S$-2-absorbing ideal of $R$, there is a fixed $s \in S$ and so $sAnn_R(K)aN = 0$ or $sAnn_R(K)bN = 0$ or $sabN = 0$. If $sabN = 0$, we are done. If $sAnn_R(K)aN = 0$ or $sAnn_R(K)bN = 0$, then $Ann_R(K) \subseteq Ann_R(saN)$ or $Ann_R(K) \subseteq Ann_R(sbN)$. Hence, $saN \subseteq K$ or $sbN \subseteq K$ since $M$ is a comultiplication $R$-module. □

The following example shows that the Lemma 3.16 is not satisfied in general.

**Example 3.17.** By [3, 3.9], the $\mathbb{Z}$-module $\mathbb{Z}$ is not a comultiplication $\mathbb{Z}$-module. Take the multiplicatively closed subset $S = \mathbb{Z} \setminus \{0\}$. The submodule $p\mathbb{Z}$ of $\mathbb{Z}$, where $p$ is a prime number, is not $S$-2-absorbing second submodule. But $Ann_{\mathbb{Z}}(p\mathbb{Z}) = 0$ is an $S$-2-absorbing ideal of $\mathbb{Z}$.

**Proposition 3.18.** Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. Then we have the following.

(a) If $M$ is a multiplication $S$-2-absorbing second $R$-module, then every submodule $N$ of $M$ with $(N :_R M) \cap S = \emptyset$ is a $S$-2-absorbing submodule of $M$.

(b) If $M$ is a comultiplication $R$-module such that the zero submodule of $M$ is a $S$-2-absorbing submodule, then every submodule $N$ of $M$ with $Ann_R(N) \cap S = \emptyset$ is a $S$-2-absorbing second submodule of $M$.

**Proof.** (a) Let $M$ is a multiplication $S$-2-absorbing second $R$-module and $N$ be a submodule of $M$ with $(N :_R M) \cap S = \emptyset$. Then by Proposition 3.14 (b), $(N :_R M)$ is an $S$-2-absorbing ideal of $R$. Now the result follows from [25, Proposition 3].

(b) Let $M$ is a comultiplication $R$-module with the zero submodule of $M$ is a $S$-2-absorbing submodule and $N$ be a submodule of $M$ with $Ann_R(N) \cap S = \emptyset$. We show that $Ann_R(N)$ is an $S$-2-absorbing ideal of $R$. To see this let $a, b, c \in R$ and $abc = (ac)b \in Ann_R(N)$. Then there exists a fixed $s \in S$ such that $acsN = 0$ or $cbsN \subseteq bsN = 0$ or $sabc \in Ann_R(M) \subseteq Ann_R(N)$. Thus $Ann_R(N)$ is an $S$-2-absorbing ideal of $R$. Now the result follows from Lemma 3.16. □

An $R$-module $M$ satisfies the *double annihilator conditions* (DAC for short) if for each ideal $I$ of $R$ we have $I = Ann_R((0:_M I))$ [15]. An $R$-module $M$ is said to be a *strong comultiplication module* if $M$ is a comultiplication $R$-module and satisfies the DAC conditions [7].

**Theorem 3.19.** Let $M$ be a strong comultiplication $R$-module and $N$ be a submodule of $M$ such that $Ann_R(N) \cap S = \emptyset$, where $S$ is a multiplicatively closed subset of $R$. Then the following are equivalent:

(a) $N$ is an $S$-2-absorbing second submodule of $M$;

(b) $Ann_R(N)$ is a $S$-2-absorbing ideal of $R$;

(c) $N = (0:_M I)$ for some $S$-2-absorbing ideal $I$ of $R$ with $Ann_R(N) \subseteq I$.

**Proof.** $(a) \Rightarrow (b)$ This follows from Proposition 3.14 (a).

$(b) \Rightarrow (c)$ As $M$ is a comultiplication $R$-module, $N = (0:_M Ann_R(N))$. Now the result is clear.

$(c) \Rightarrow (a)$ As $M$ satisfies the DAC conditions, $Ann_R((0:_M I)) = I$. Now the result follows from Lemma 3.16. □

**Lemma 3.20.** Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. If $N$ is an $S$-second submodule of $M$. Then there exists a fixed $s \in S$ such that $abN \subseteq K$, where $a, b \in R$ and $K$ is a submodule of $M$ implies that either $sa \in Ann_R(N)$ or $sb \in Ann_R(N)$ or $sN \subseteq K$

**Proof.** Let $N$ be an $S$-second submodule of $M$ and $abN \subseteq K$, where $a, b \in R$ and $K$ is a submodule of $M$. Then $aN \subseteq (K:_M b)$. Since $N$ is an $S$-second submodule of $M$, there exists a fixed $s \in S$ such that $sa \in Ann_R(N)$ or $sbN \subseteq K$. Now, we will show that $sbN \subseteq K$ implies that $sb \in Ann_R(N)$ or $sN \subseteq K$. Assume that $bN \subseteq (K:_M s)$. Since $N$

is an $S$-second submodule, we get either $sb \in Ann_R(N)$ or $s^2N \subseteq K$. If $sb \in Ann_R(N)$, then we are done. So assume that $s^2N \subseteq K$. By [17, Lemma 2.13 (a)], we know that $sN \subseteq s^2N$. Thus we have $sN \subseteq K$. □

**Theorem 3.21.** Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. Then the sum of two $S$-second submodules is an $S$-2-absorbing second submodule of $M$.

**Proof.** Let $N_1, N_2$ be two $S$-second submodules of $M$ and $N = N_1+N_2$. Let $abN \subseteq K$ for some $a, b \in R$ and submodule $K$ of $M$. Since $N_1$ is an $S$-second submodule submodule of $M$, there exists a fixed $s_1 \in S$ such that $s_1a \in Ann_R(N_1)$ or $s_1b \in Ann_R(N_1)$ or $s_1N_1 \subseteq K$ by Lemma 3.20. Also, as $N_2$ is an $S$-second submodule of $M$, there exists a fixed $s_2 \in S$ such that $s_2a \in Ann_R(N_2)$ or $s_2b \in Ann_R(N_2)$ or $s_2N_2 \subseteq K$ by Lemma 3.20. Without loss of generality, we may assume that $s_1a \in Ann_R(N_1)$ and $s_2N_2 \subseteq K$. Now, put $s = s_1s_2 \in S$. This implies that $saN \subseteq K$ and hence $N$ is an S-2-absorbing second submodule of $M$. □

The following example shows that sum of two $S$-2-absorbing second submodules is not necessarily $S$-2-absorbing second submodule.

**Example 3.22.** Consider $M = \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{q^n}$ as $\mathbb{Z}$-module, where $n \in \mathbb{N}$ and $p, q$ are distinct prime numbers. Set $S = \{x \in \mathbb{Z} : gcd(x, pq) = 1\}$. Then $S$ is a multiplicatively closed subset of $\mathbb{Z}$. One can see that $\mathbb{Z}_{p^n} \oplus 0$ and $0 \oplus \mathbb{Z}_{q^n}$ both are $S$-2-absorbing second submodules. However $p^nM \subseteq 0 \oplus \mathbb{Z}_{q^n}$, $p^{n-1}xM \not\subseteq 0 \oplus \mathbb{Z}_{q^n}$, $pxM \not\subseteq 0 \oplus \mathbb{Z}_{q^n}$, and $xp^nM \neq 0$ for each $x \in S$ implies that $M$ is not an $S$-2-absorbing second $\mathbb{Z}$-module.

Let $M$ be an $R$-module. The idealization $R(+)M = \{(a, m) : a \in R, m \in M\}$ of $M$ is a commutative ring whose addition is componentwise and whose multiplication is defined as $(a, m)(b, \acute{m}) = (ab, a\acute{m} + bm)$ for each $a, b \in R$, $m, \acute{m} \in M$ [21]. If $S$ is a multiplicatively closed subset of $R$ and $N$ is a submodule of $M$, then $S(+)N = \{(s, n) : s \in S, n \in N\}$ is a multiplicatively closed subset of $R(+)M$ [2].

**Proposition 3.23.** Let $M$ be an $R$-module and let $I$ be an ideal of $R$ such that $I \subseteq Ann_R(M)$. Then the following are equivalent:

(a) $I$ is a strongly 2-absorbing second ideal of $R$;

(b) $I(+)0$ is a strongly 2-absorbing second ideal of $R(+)M$.

**Proof.** This is straightforward. □

**Theorem 3.24.** Let $S$ be a multiplicatively closed subset of $R$, $M$ be an $R$-module, and $I$ be an ideal of $R$ such that $I \subseteq Ann_R(M)$ and $I \cap S = \emptyset$. Then the following are equivalent:

(a) $I$ is an $S$-2-absorbing second ideal of $R$;

(b) $I(+)0$ is an $S(+)0$-2-absorbing second ideal of $R(+)M$;

(c) $I(+)0$ is an $S(+)M$-2-absorbing second ideal of $R(+)M$.

**Proof.** $(a) \Rightarrow (b)$ Let $(a,m),(b,\acute{m}) \in R(+)M$. As $I$ is an $S$-2-absorbing second ideal of $R$, there exists a fixed $s \in S$ such that $abs^2I = as^2I$ or $abs^2I = bs^2I$ or $abs^3I = 0$. If $abs^3I = 0$, then $(a,m)(b,\acute{m})(s^3,0)(I(+)0) = 0$. If $abs^2I = as^2I$, then we claim that $(a,m)(b,\acute{m})(s^2,0)(I(+)0) = (a,m)(s^2,0)(I(+)0)$. To see this let $(s^2x,0)(a,m) = (s^2,0)(x,0)(a,m) \in (s^2,0)(a,m)(I(+)0)$. As $abs^2I = as^2I$, we have $s^2ax = abs^2y$ for some $y \in I$. Thus as $y \in I \subseteq Ann_R(M)$,

$$(s^2,0)(x,0)(a,m) = (s^2xa,0) = (abs^2y,0) = (s^2,0)(y,0)(a,m)(b,\acute{m}).$$

Hence, $(s^2,0)(x,0)(a,0) \in (a,m)(b,\acute{m})(s^2,0)(I(+)0)$ and so we have $(a,m)(s^2,0)(I(+)0) \subseteq (a,m)(b,\acute{m})(s^2,0)(I(+)0)$. Since the inverse inclusion is clear we reach the claim.

$(b) \Rightarrow (c)$ Since $S(+)0 \subseteq S(+)M$, the result follows from Proposition 3.10 (b).

$(c) \Rightarrow (a)$ Let $a,b \in R$. As $I(+)0$ is an $S(+)M$-2-absorbing second ideal of $R(+)M$, there exists a fixed $(s,m) \in S(+)M$ such that

$$(a,0)(b,0)(s,m)^2(I(+)0) = (a,0)(s,m)^2(I(+)0)$$

or

$$(a,0)(b,0)(s,m)^2(I(+)0) = (b,0)(s,m)^2(I(+)0)$$

or

$$(a,0)(b,0)(s,m)^3(I(+)0) = 0.$$

If $(ab, 0)(s, m)^3(I(+)0) = 0$, then for each $abs^3x \in abs^3I$ we have

$$0 = (ab, 0)(s, m)^3(x, 0) = (ab, 0)(s^3, 3sm)(x, 0) = (abs^3, 3absm)(x, 0)$$
$$= (abs^3, 0)(x, 0) = (abs^3x, 0).$$

Thus $abs^3I = 0$. If $(ab, 0)(s, m)^2(I(+)0) = (a, 0)(s, m)^2(I(+)0)$, then we claim that $abs^2I = as^2I$. To see this, let $s^2xa \in s^2aI$. Then for some $y \in I$, as $x \in I \subseteq Ann_R(M)$ we have

$$(s^2ax, 0) = (s^2ax, 2sxm) = (s, m)^2(a, 0)(x, 0) = (s, m)^2(aby, 0)$$
$$= (s^2aby, 2sabmy) = (s^2aby, 0).$$

Hence, $s^2ax \in abs^2I$ and so $s^2aI \subseteq s^2abI$ Thus $s^2aI = s^2abI$. Similarly, if $(ab, 0)(s, m)^2(I(+)0) = (b, 0)(s, m)^2(I(+)0)$, then $s^2bI \subseteq s^2abI$, and so $s^2bI = s^2abI$. □

Let $R_i$ be a commutative ring with identity, $M_i$ be an $R_i$-module for each $i = 1, 2, ..., n$, and $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times ... \times M_n$ and $R = R_1 \times R_2 \times ... \times R_n$. Then $M$ is clearly an $R$-module with componentwise addition and scalar multiplication. Also, if $S_i$ is a multiplicatively closed subset of $R_i$ for each $i = 1, 2, ..., n$, then $S = S_1 \times S_2 \times ... \times S_n$ is a multiplicatively closed subset of $R$. Furthermore, each submodule $N$ of $M$ is of the form $N = N_1 \times N_2 \times ... \times N_n$, where $N_i$ is a submodule of $M_i$ for each $i = 1, 2, ..., n$.

**Theorem 3.25.** Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ be a multiplicatively closed subset of $R$, where $R_i$ is a commutative ring with $1 \neq 0$ and $S_i$ is a multiplicatively closed subset of $R_i$ for each $i = 1, 2$. Let $M = M_1 \times M_2$ be an $R$-module, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. Suppose that $N = N_1 \times N_2$ is a submodule of $M$. Then the following conditions are equivalent:

(a) $N$ is an $S$-2-absorbing second submodule of $M$;

(b) Either $Ann_{R_1}(N_1) \cap S_1 \neq \emptyset$ and $N_2$ is a $S_2$-2-absorbing second submodule of $M_2$ or $Ann_{R_2}(N_2) \cap S_2 \neq \emptyset$ and $N_1$ is a $S_1$-2-absorbing second submodule of $M_1$ or $N_1$ is an $S_1$-second submodule of $M_1$ and $N_2$ is an $S_2$-second submodule of $M_2$ .

**Proof.** $(a) \Rightarrow (b)$ Let $N = N_1 \times N_2$ be a $S$-2-absorbing second submodule of $M$. Then $Ann_R(N) = Ann_{R_1}(N_1) \times Ann_{R_2}(N_2)$ is an $S$-2-absorbing ideal of $R$ by Proposition 3.14 (a). Thus, either $Ann_R(N_1) \cap S_1 = \emptyset$ or $Ann_R(N_2) \cap S_2 = \emptyset$. Assume that $Ann_R(N_1) \cap S_1 \neq \emptyset$. We show that $N_2$ is an $S_2$-2-absorbing second submodule of $M_2$. To see this, let $t_2 r_2 N_2 \subseteq K_2$ for some $t_2, r_2 \in R_2$ and a submodule $K_2$ of $M_2$. Then $(1, t_2)(1, r_2)(N_1 \times N_2) \subseteq M_1 \times K_2$. As $N$ is an $S$-2-absorbing second submodule of $M$, there exists a fixed $(s_1, s_2) \in S$ such that $(s_1, s_2)(1, r_2)(N_1 \times N_2) \subseteq M_1 \times K_2$ or $(s_1, s_2)(1, t_2)(N_1 \times N_2) \subseteq M_1 \times K_2$ or $(s_1, s_2)(1, t_2)(1, r_2)(N_1 \times N_2) = 0$. It follows that either $s_2 r_2 N_2 \subseteq K_2$ or $s_2 t_2 N_2 \subseteq K_2$ or $s_2 t_2 r_2 N_2 = 0$ and so $N_2$ is an $S_2$-2-absorbing second submodule of $M_2$. Similarly, if $Ann_{R_2}(N_2) \cap S_2 \neq \emptyset$, then one can see that $N_1$ is an $S_1$-2-absorbing second submodule of $M_1$. Now assume that $Ann_R(N_1) \cap S_1 = \emptyset$ and $Ann_R(N_2) \cap S_2 = \emptyset$. We will show that $N_1$ is an $S_1$-second submodule of $M_1$ and $N_2$ is an $S_2$-second submodule of $M_2$. First, note that there exists a fixed $s = (s_1, s_2) \in S$ satisfying $N$ to be an $S$-2-absorbing second submodule of $M$. Suppose that $N_1$ is not an $S_1$-second submodule of $M_1$. Then there exists $a \in R_1$ and a submodule $K_1$ of $M_1$ such that $aN_1 \subseteq K_1$ but $s_1 a \notin Ann_R(N_1)$ and $s_1 N_1 \not\subseteq K_1$. On the other hand $Ann_R(N_2) \cap S_2 = \emptyset$ and $s_2 \notin Ann_R(N_2)$ so that $s_2 N_2 \neq 0$. Thus by Remark 3.6, there exists a completely irreducible submodule $L_2$ of $M_2$ such that $s_2 N_2 \not\subseteq L_2$. Also note that

$$(a, 1)(1, 0)N = (a, 1)(1, 0)(N_1 \times N_2) = aN_1 \times 0 \subseteq K_1 \times 0 \subseteq K_1 \times L_2.$$

As $N$ is an $S$-2-absorbing second submodule of $M$, either $(s_1, s_2)(1, 0)N \subseteq K_1 \times L_2$ or $(s_1, s_2)(a, 1)N \subseteq K_1 \times L_2$ or $(s_1, s_2)(a, 1)(1, 0)N = 0$. Hence, we conclude that either $s_1 N_1 \subseteq K_1$ or $s_2 N_2 \subseteq L_2$ or $s_1 a N_1 = 0$, which them are contradictions. Thus, $N_1$ is an $S_1$-second submodule of $M_1$. Similar argument shows that $N_2$ is an $S_2$-second submodule of $M_2$.

$(b) \Rightarrow (a)$ Assume that $N_1$ is an $S_1$-2-absorbing second submodule of $M_1$ and $Ann_{R_2}(N_2) \cap S_2 \neq \emptyset$. we will show that $N$ is an $S$-2-absorbing second submodule of $M$. Then there exists an $s_2 \in Ann_{R_2}(N_2) \cap S_2$. Let $(r_1, r_2)(t_1, t_2)(N_1 \times N_2) \subseteq K_1 \times K_2$ for some $t_i, r_i \in R_i$ and submodule $K_i$ of $M_i$, where $i = 1, 2$. Then $r_1 t_1 N_1 \subseteq K_1$. As $N_1$ is an $S_1$-2-absorbing second submodule of $M_1$, there exists a fixed $s_1 \in S_1$ such that $s_1 r_1 N_1 \subseteq K_1$ or $s_1 t_1 N_1 \subseteq K_1$ or $s_1 r_1 t_1 N_1 = 0$. Now we set $s = (s_1, s_2)$.

Then $s(r_1, r_2)(N_1 \times N_2) \subseteq K_1 \times K_2$ or $s(t_1, t_2)(N_1 \times N_2) \subseteq K_1 \times K_2$ or $s(r_1, r_2)(t_1, t_2)(N_1 \times N_2) = 0$. Therefore, $N$ is an $S$-2-absorbing second submodule of $M$. Similarly one can show that if $N_2$ is an $S_2$-2-absorbing second submodule of $M_2$ and $Ann_{R_1}(N_1) \cap S_1 \neq \emptyset$, then $N$ is an $S$-2-absorbing second submodule of $M$. Now assume that $N_1$ is an $S_1$-second submodule of $M_1$ and $N_2$ is an $S_2$-second submodule of $M_2$ . Let $a, b \in R_1$, $x, y \in R_2$, $K_1$ is a submodule of $M_1$ and $K_2$ is a submodule of $M_2$ such that

$$(a, x)(b, y)N = (a, x)(b, y)(N_1 \times N_2 \subseteq K_1 \times K_2.$$

Then we have $abN_1 \subseteq K_1$ and $xyN_2 \subseteq K_2$. Since $N_1$ is an $S_1$-second submodule of $M_1$, there exists a fixed $s_1 \in S_1$ such that either $s_1 a \in Ann_{R_1}(N_1)$ or $s_1 b \in Ann_{R_1}(N_1)$ or $s_1 N_1 \subseteq K_1$ by Lemma 3.20. Similarly, there exists a fixed $s_2 \in S_2$ such that either $s_2 x \in Ann_{R_2}(N_2)$ or $s_2 y \in Ann_{R_2}(N_2)$ or $s_2 N_2 \subseteq K_2$ by Lemma 3.20. Also without loss of generality, we may assume that $s_1 a \in Ann_{R_1}(N_1)$ and $s_2 N_2 \subseteq K_2$ or $s_1 a \in Ann_{R_1}(N_1)$ and $s_2 x \in Ann_{R_2}(N_2)$ or $s_1 N_1 \subseteq K_1$ and $s_2 N_2 \subseteq K_2$. If $s_1 a \in Ann_{R_1}(N_1)$ and $s_2 N_2 \subseteq K_2$, then we have

$$(s_1, s_2)(a, x)(N_1 \times N_2) = s_1 a N_1 \times s_2 x N_2 \subseteq 0 \times K_2 \subseteq K_1 \times K_2.$$

If $s_1 a \in Ann_{R_1}(N_1)$ and $s_2 x \in Ann_{R_2}(N_2)$, then $(s_1, s_2)(a, x)(b, y)(N_1 \times N_2) = 0$. If $s_1 N_1 \subseteq K_1$ and $s_2 N_2 \subseteq K_2$, then

$$(s_1, s_2)(a, x)(b, y)(N_1 \times N_2) \subseteq (s_1, s_2)N \subseteq K_1 \times K_2.$$

Hence, $N$ is an $S$-2-absorbing second submodule of $M$. $\square$

The following example shows that if $N_1$ is an $S_1$-2-absorbing second submodule of $M_1$ and $N_2$ is an $S_2$-2-absorbing second submodule of $M_2$, then $N_1 \times N_2$ may not be an $S_1 \times S_2$-2-absorbing second submodule of $M_1 \times M_2$ in general.

**Example 3.26.** Consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}_9$ and $M_2 = \mathbb{Z}_4$. Let $S_1 = \mathbb{Z} \backslash 3\mathbb{Z}$ and $S_2 = \mathbb{Z} \backslash 2\mathbb{Z}$. Then $M_1$ and $M_2$ are $S_1$ and $S_2$-2-absorbing second modules (see Example 3.7). But $M = M_1 \times M_2$ is not an $S = S_1 \times S_2$-2-absorbing second module since $(1, 2)(3, 1)M \subseteq \bar{3}\mathbb{Z}_9 \times \bar{2}\mathbb{Z}_4$ but for each $s = (s_1, s_2) \in S$, $s(3, 1)M \not\subseteq \bar{3}\mathbb{Z}_9 \times \bar{2}\mathbb{Z}_4$, $s(1, 2)M \not\subseteq \bar{3}\mathbb{Z}_9 \times \bar{2}\mathbb{Z}_4$, and $s(1, 2)(3, 1)M \neq 0$.

**Theorem 3.27.** Let $M = M_1 \times M_2 \times ... \times M_n$ be an $R = R_1 \times R_2 \times ... \times R_n$-module and $S = S_1 \times S_2 \times ... \times S_n$ be a multiplicatively closed subset of $R$, where $M_i$ is an $R_i$-module and $S_i$ is a multiplicatively closed subset of $R_i$ for each $i = 1, 2, ..., n$. Let $N = N_1 \times N_2 \times ... \times N_n$ be a submodule of $M$. Then the following are equivalent:

(a) $N$ is an $S$-2-absorbing second submodule of $M$;

(b) $N_k$ is an $S_k$-2-absorbing second submodule of $M_k$ for some $k \in \{1, 2, ..., n\}$ and $Ann_{R_t}(N_t) \cap S_t \neq \emptyset$ for each $t \in \{1, 2, ..., n\} \setminus \{k\}$ or $N_{k_1}$ is an $S_{k_1}$-second submodule of $M_{k_1}$ and $N_{k_2}$ is an $S_{k_2}$-second submodule of $M_{k_2}$ for some $k_1, k_2 \in \{1, 2, ..., n\}$ $(k_1 \neq k_2)$ and $Ann_{R_t}(N_t) \cap S_t \neq \emptyset$ for each $t \in \{1, 2, ..., n\} \setminus \{k_1, k_2\}$.

**Proof.** We apply induction on $n$. For $n = 1$, the result is true. If $n = 2$, then the result follows from Theorem 3.25. Now assume that parts (a) and (b) are equal when $k < n$. We shall prove $(b) \Leftrightarrow (a)$ when $k = n$. Put $R = \acute{R} \times R_n$, $M = \acute{M} \times M_n$, and $S = \acute{S} \times S_n$, where $\acute{R} = R_1 \times R_2 \times ... \times R_{n-1}$, $\acute{M} = M_1 \times M_2 \times ... \times M_{n-1}$, and $\acute{S} = S_1 \times S_2 \times ... \times S_{n-1}$. Also, $N = \acute{N} \times N_n$, where $\acute{N} = N_1 \times N_2 \times ... \times N_{n-1}$. Then by Theorem 3.25, $N$ is an $S$-2-absorbing second submodule of $M$ if and only if $Ann_{\acute{R}}(\acute{N}) \cap \acute{S} \neq \emptyset$ and $N_n$ is an $S_n$-2-absorbing second submodule of $M_n$ or $\acute{N}$ is an $\acute{S}$-2-absorbing second submodule of $\acute{M}$ and $Ann_{R_n}(N_n) \cap S_n \neq \emptyset$ or $\acute{N}$ is an $\acute{S}$-second submodule of $\acute{M}$ and $N_n$ is an $S_n$-second submodule of $M_n$. Now the rest follows from induction hypothesis and [17, Theorem 2.12]. □

For a submodule $N$ of an $R$-module $M$ the *second radical* (or second socle) of $N$ is defined as the sum of all second submodules of $M$ contained in $N$ and it is denoted by $sec(N)$ (or $soc(N)$). In case $N$ does not contain any second submodule, the second radical of $N$ is defined to be (0) (see [12] and [4]).

**Theorem 3.28.** Let $M$ be a finitely generated comultiplication $R$-module. If $N$ is a $S$-2-absorbing second submodule of $M$, then $sec(N)$ is a $S$-2-absorbing second submodule of $M$.

**Proof.** Let $N$ be a $S$-2-absorbing second submodule of $M$. By Proposition 3.14 (a), $Ann_R(N)$ is an $S$-2-absorbing ideal of $R$. Thus by

Lemma 2.6, $\sqrt{Ann_R(N)}$ is an $S$-2-absorbing ideal of $R$. By [6, 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Therefore, $Ann_R(sec(N))$ is an $S$-2-absorbing ideal of $R$. Now the result follows from Lemma 3.16. $\square$

**Proposition 3.29.** Let $S$ be a multiplicatively closed subset of $R$ and $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.

(a) If $N$ is an $S$-2-absorbing second submodule of $M$, then $f(N)$ is an $S$-2-absorbing second submodule of $\acute{M}$.

(b) If $\acute{N}$ is an $S$-2-absorbing second submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is an $S$-2-absorbing second submodule of $M$.

**Proof.** (a) As $Ann_R(N) \cap S = \emptyset$ and $f$ is a monomorphism, we have $Ann_R(f(N)) \cap S = \emptyset$. Let $a, b \in R$. Since $N$ is an $S$-2-absorbing second submodule of $M$, there exists a fixed $s \in S$ such that $s^2abN = s^2aN$ or $s^2abN = s^2bN$ or $s^3abN = 0$. Thus $s^2abf(N) = s^2af(N)$ or $s^2abf(N) = s^2bf(N)$ or $s^3abf(N) = 0$, as needed.

(b) $Ann_R(\acute{N}) \cap S = \emptyset$ implies that $Ann_R(f^{-1}(\acute{N})) \cap S = \emptyset$. Now let $a, b \in R$. As $\acute{N}$ is an $S$-2-absorbing second submodule of $\acute{M}$, there exists a fixed $s \in S$ such that $s^2ab\acute{N} = s^2a\acute{N}$ or $s^2ab\acute{N} = s^2b\acute{N}$ or $s^3ab\acute{N} = 0$. Therefore, $s^2abf^{-1}(\acute{N}) = s^2af^{-1}(\acute{N})$ or $s^2abf^{-1}(\acute{N}) = s^2bf^{-1}(\acute{N})$ or $s^3abf^{-1}(\acute{N}) = 0$, as requested. $\square$

**Theorem 3.30.** Let $S$ be a multiplicatively closed subset of $R$ and let $M$ be an $R$-module. If $E$ is an injective $R$-module and $N$ is an $S$-2-absorbing submodule of $M$ such that $Ann_R(Hom_R(M/N, E)) \cap S \neq \emptyset$, then $Hom_R(M/N, E)$ is a $S$-2-absorbing second $R$-module.

**Proof.** Let $a, b \in R$. Since $N$ is an $S$-2-absorbing submodule of $M$, there is a fixed $s \in S$ such that either $(N :_M abs^2) = (N :_M as^2)$ or $(N :_M abs^2) = (N :_M bs^2)$ or $(N :_M abs^3) = M$ by Theorem 2.1. Since $E$ is an injective $R$-module, by replacing $M$ with $M/N$ in [5, Theorem 3.13 (a)], we have $Hom_R(M/(N :_M a), E) = aHom_R(M/N, E)$. Therefore,

$$abs^2Hom_R(M/N, E) = Hom_R(M/(N :_M abs^2), E) =$$

$$Hom_R(M/(N :_M as^2), E) = as^2Hom_R(M/N, E)$$

or

$$abs^2 Hom_R(M/N, E) = Hom_R(M/(N :_M abs^2), E) =$$
$$Hom_R(M/(N :_M bs^2), E) = bs^2 Hom_R(M/N, E)$$

or

$$abs^3 Hom_R(M/N, E) = Hom_R(M/(N :_M abs^3), E) =$$
$$Hom_R(M/M, E) = 0,$$

as needed □

**Theorem 3.31.** Let $M$ be a $S$-2-absorbing second $R$-module and $F$ be a right exact linear covariant functor over the category of $R$-modules. Then $F(M)$ is a $S$-2-absorbing second $R$-module if $Ann_R(F(M)) \cap S \neq \emptyset$.

**Proof.** This follows from [5, Lemma 3.14] and Theorem 3.5 $(a) \Leftrightarrow (b)$. □

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