Journal of Mathematical Extension
Vol. 16, No. 8, (2022) (6)1-18
URL: https://doi.org/10.30495/JME.2022.1695
ISSN: 1735-8299
Original Research Paper

Dynamics and Neimark-Sacker Bifurcation of a Modified Nicholson-Bailey Model

M. H. Akrami*
Yazd University

A. Atabaigi
Razi university

Abstract. In this paper, the dynamics of a modified Nicholson-Bailey model as a discrete dynamical system has been studied. Local dynamics in a neighborhood of boundary fixed points are investigated. It is also proved that the model has a unique positive fixed point and a Neimark-Sacker bifurcation emerges at this fixed point. Some numerical simulations are presented to illustrate the analytical results.

AMS Subject Classification: 37N25; 39A28
Keywords and Phrases: Discrete dynamical systems, Neimark-Sacker bifurcation, Nicholson-Bailey model

1 Introduction

Many of natural phenomena are described by mathematical models and dynamical systems is a powerful tool for analyzing these models. Nowadays, mathematical modelling in population dynamics attract much attention of researchers. A relation between populations of host and parasitoid become interesting research subject for many scientist, ecologist,

Received: May 2020; Accepted: May 2021
*Corresponding Author

biologist, etc. See [2, 3, 5, 11, 12, 15, 16, 17] and references therein for instances.
One of the most popular models for many theoretical and experimental studies in ecology is Nicholson-Bailey host-parasitoid model [3]. In this model, parasitoids search randomly for host and the host population grows exponentially in the absence of parasitoids. Nicholson and Bailey assume that the number of encounters are distributed randomly among the available hosts, therefore they used the Poisson distribution: $P(n) = \frac{e^{-\mu}\mu^n}{n!}$, where n is the number of encounters and μ is the mean of encounters per host in one generation. Actually, they used the zero term of a Poisson distribution with parameter the mean number of parasitoid attacks per host, that is, $P(0) = e^{-\mu}$ and $\mu =$encounters$/H_n = \gamma P_n$. Hence, their model is of the form

$$\begin{aligned} H_{n+1} &= rH_n e^{-\gamma P_n}, \\ P_{n+1} &= sH_n(1 - e^{-\gamma P_n}), \end{aligned}$$

where, H_n and P_n represent the densities of the host and parasitoid population at year n, respectively. Each host that survives to the end of the season produces r hosts next year. In 1975 Beddington et al [4] proposed a model to improve the Nicholson-Bailey model. These authors added the effect of carrying capacity K (i.e. the maximum population size that can be supported by the available and potential limited resources) and represented the following model

$$\begin{aligned} H_{n+1} &= H_n e^{r(1-H_n/K)-\gamma P_n}, \\ P_{n+1} &= sH_n(1 - e^{-\gamma P_n}). \end{aligned}$$

In this model, in the absence of the parasitoid, the equation of the host converts to the famous Ricker model. Some authors investigated dynamics, stabilities and bifurcation of Bedington's model[1, 14]. In [2], authors investigated stability and bifurcations of the following host-parasite model

$$\begin{aligned} &H_{n+1} = rH_n(1 - H_n)e^{-\beta P_n}, \\ &P_{n+1} = H_n(1 - e^{-\beta P_n}), \end{aligned}$$

where, in the absence of parasite population, the equation of the host becomes the famous logistic model. In this model a Neimark-Sacker and

a period-doubling bifurcation occur.
Khan and Qureshi [15], studied the following model

$$x_{n+1} = \frac{bx_n}{1+dx_n}e^{-ay_n},$$
$$y_{n+1} = cx_n(1-e^{-ay_n}).$$

They investigated the boundedness, existence and uniqueness of the positive equilibrium point, local asymptotic stability and global stability of the unique positive equilibrium point, and the rate of convergence of positive solutions of the system. But they did not study the possible bifurcations of the model.
In [7], the author investigated the qualitative behavior of system

$$H_{n+1} = \frac{rH_n}{1+kH_n}e^{-aP_n},$$
$$P_{n+1} = cH_n(1-e^{-aP_n}),$$

and the local and global asymptotic stability of the unique positive equilibrium point and rate of convergence of the model were also analyzed. Moreover, the author proved that the system undergoes a Neimark-Sacker bifurcation by taking c as the bifurcation parameter.
In this paper, motivated by the above works we propose the following more general discrete-time mathematical model for host and parasitoid populations:

$$H_{n+1} = \frac{\mu K H_n}{K+(\mu-1)H_n}e^{-\gamma P_n},$$
$$P_{n+1} = H_n(1-e^{-\gamma P_n}), \tag{1}$$

where $K > 0$ and $\mu > 1$. Here, in the absence of parasite population, the equation of the host becomes the Beverton-Holt model [6]. We study the positive fixed point and investigate the dynamics and bifurcation in the vicinity of this point. We choose γ as bifurcation parameter and we prove that the model exhibits a Neimark-Sacker bifurcation by varying the parameter γ.
The rest of paper is organized as follows. We first obtain fixed points of

the model (1) in Section 2. In Section 3, we investigate the local dynamics of the fixed points of (1). In the forth Section, we study Neimark-Sacker bifurcation for the model (1) by choosing γ as the bifurcation parameter. In Section 5, some numerical simulations are presented to support our analytical arguments . In the last Section, a brief conclusion is given.

2 Host-parasitoid system

In this section, we focus on the fixed points of the following system

$$\begin{aligned} x_{n+1} &= \frac{\mu K x_n}{K+(\mu-1)x_n} e^{-\gamma y_n} := f(x_n, y_n), \\ y_{n+1} &= x_n(1-e^{-\gamma y_n}) := g(x_n, y_n), \end{aligned} \tag{2}$$

where x_n and y_n are the host and parasite populations at n-th generation respectively. It is not hard to see that every positive solution of the model (2) satisfy

$$x_{n+1} \leq \frac{\mu K}{\mu - 1}, \qquad y_{n+1} \leq x_n.$$

Thus, for $\mu > 1$ the set $[0, \frac{\mu K}{\mu-1}] \times [0, \frac{\mu K}{\mu-1}]$ is an invariant set for the model (2).

For this model, we can have at most three fixed points at $(0,0)$, $(K,0)$ and (x^*, y^*) where

$$y^* = \frac{1}{\gamma} h(x^*), \quad h(x) := \ln\left(\frac{\mu K}{K+(\mu-1)x}\right). \tag{3}$$

From the second equation of (2), one can see that

$$x^* = \frac{y^*}{1-e^{-\gamma y^*}},$$

thus, we have $y^* < x^*$. Let

$$F(x) = \frac{h(x)}{\gamma(1-e^{-h(x)})} - x.$$

Then the x-component of the positive fixed points are the positive roots of $F(x)$. In addition, we have the following observations

$$F(0) = \frac{\mu}{\mu-1}\frac{1}{\gamma}\ln\mu > 0, \qquad F(\frac{\mu K}{\mu-1}) = -\frac{\mu}{\gamma}\ln\frac{\mu}{\mu+1} - \frac{\mu K}{\mu-1} < 0.$$

On the other hand

$$F'(x) = \frac{h'(x)(1-e^{-h(x)}) + h'(x)e^{-h(x)}h(x)}{\gamma(1-e^{-h(x)})^2} - 1 < 0,$$

since

$$h'(x) = -\frac{\mu-1}{K+(\mu-1)x} < 0.$$

Therefore, we have the following proposition:

Proposition 2.1. *For any γ, $K > 0$ and $\mu > 1$, the model (2) has a unique positive fixed point.*

3 Local dynamics analysis

There are two boundary fixed points $(0,0)$ and $(K,0)$. The Jacobian matrix associated with (2) is given by

$$J(x,y) := \begin{pmatrix} \frac{\mu K^2 e^{-\gamma y}}{(K+(\mu-1)x)^2} & -\frac{\mu K x\gamma e^{-\gamma y}}{K+(\mu-1)x} \\ 1-e^{-\gamma y} & x\gamma e^{-\gamma y} \end{pmatrix}. \tag{4}$$

At $(0,0)$ we have

$$J(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

Therefore, as $\mu > 1$ the origin is an unstable fixed point. At $(K,0)$ we have

$$J(K,0) = \begin{pmatrix} \frac{1}{\mu} & -K\gamma \\ 0 & K\gamma \end{pmatrix}. \tag{6}$$

Depending on the location of the eigenvalues in the complex plane w.r.t. the unit circle, we can deduce the following results on the stability of $(K,0)$:

Proposition 3.1. *Let $\mu > 1$. The following hold:*

(1) if $K\gamma < 1$, then $(K, 0)$ is an attracting node;

(2) if $K\gamma > 1$, then $(K, 0)$ is a saddle.

Proposition 3.2. *When $\mu > 1$ and $K\gamma = 1$, we have a non-hyperbolic fixed point at $(K, 0)$ which is asymptotically stable.*

Proof. Let $K\gamma = 1$, first, we bring $(K, 0)$ to the origin by using linear transformation $(x, y) \to (x + K, y)$. This yields the following two-dimensional map defined by

$$\tilde{f}(x,y) = \frac{x}{\mu} - y + \frac{1}{2}\gamma\, y^2 - \frac{\gamma\, xy}{\mu} - \frac{(\mu-1)\, x^2}{\mu^2 K} + O(3),$$
$$\tilde{g}(x,y) = \gamma\, xy + y - \frac{1}{2}\gamma\, y^2 + \frac{1}{6}\gamma^2 y^3 - \frac{1}{2}\gamma^2 x y^2 + O(4).$$

Next, we need to diagonalize the linear part of this equation, i.e $\begin{pmatrix} \frac{1}{\mu} & -1 \\ 0 & 1 \end{pmatrix}$.
In this way, we use the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ 0 & 1-\mu \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X + \mu Y \\ (1-\mu) Y \end{pmatrix}.$$

Then, we obtain

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\mu} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \gamma XY + \frac{\gamma}{2}(3\mu - 1) Y^2 + O(3) \\ \frac{1-\mu}{\mu K} XY + \frac{1-\mu}{\mu^2 K} X^2 + \frac{(\mu-1)^2}{2K} Y^2 + O(3) \end{pmatrix}. \quad (7)$$

Now, the center manifold is given by

$$X = h(Y) = \alpha Y^2 + \beta Y^3 + O(Y^4) = \frac{1}{2}\frac{\gamma^2 \mu\, (3\,\mu - 1)}{\mu - 1} Y^2 + O(3).$$

Then, the dynamics on the center manifold is given by the following scalar map:

$$Y \mapsto Y - \frac{1}{2}\frac{\gamma^2\, (3\,\mu - 1)}{K} Y^3 + O(4),$$

which shows that $(K, 0)$ is asymptotically stable. □

Positive fixed point

Now, we pay our attention to the positive fixed point (x^*, y^*) which exists for $\mu > 1$. The Jacobian matrix of the map (2) computed at the positive fixed point is given by:

$$J^* := \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix} = \begin{pmatrix} \frac{K}{K+(\mu-1)x^*} & -\gamma x^* \\ \frac{(\mu-1)(K-x^*)}{\mu K} & \frac{(K+(\mu-1)x^*)}{\mu K}\gamma x^* \end{pmatrix}. \quad (8)$$

Then,

$$tr(J^*) = \frac{K}{K + (\mu - 1)\, x^*} + \frac{(K + (\mu - 1)\, x^*)}{\mu\, K}\gamma\, x^*,$$

$$\det(J^*) = \frac{\gamma x^*}{\mu\, K}\left(\mu\, (K - x^*) + x^*\right). \quad (9)$$

Since $tr(J^*) > 0$, by the Jury test [8], the positive fixed point is locally asymptotically stable if $2 > 1 + \det(J^*) > tr(J^*)$. In the region

$$\Omega = \{(K, \gamma) :\ tr(J^*) < 1 + \det(J^*) < 2\},$$

of the parameter space, we have that (x^*, y^*) is locally attractive. It follows from Equation (3) that

$$K = \frac{x^{*2}\, (\mu - 1)}{\mu(x^* - y^*) - x^*}.$$

Under the assumption $\det(J^*) < 1$ and using Equation (9), we get

$$0 < \gamma < g_1(x^*) := \frac{K\mu}{x^*\, (K\mu + x^*(1 - \mu))}. \quad (10)$$

On the other hand, it is easy to see that $tr(J^*) < 1+\det(J^*)$ if and only if

$$0 < \gamma < g_2(x^*) := -\frac{K\mu}{(K - 2\, x^*)\, (x^*(\mu - 1) + K)}. \quad (11)$$

Now, keeping $\mu > 1$ fixed, we get

$$\Omega = \{(x^*, \gamma) : 0 < x^* < \frac{\mu K}{\mu - 1},\ 0 < \gamma < \min\{g_1(x^*), g_2(x^*)\}\}.$$

For a picture of Ω, see Figure 1.

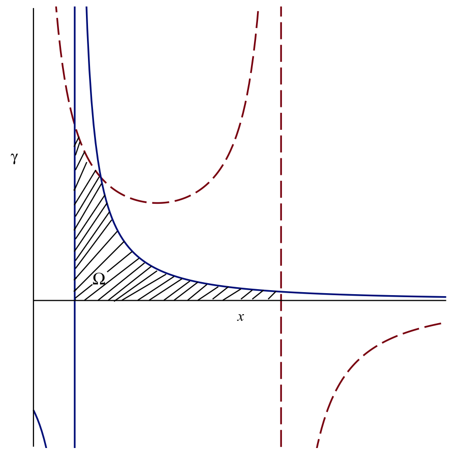

Figure 1: Shape of Ω. The solid blue line is the graph of $g_2(x)$ and dashed red curve is the graph of $g_1(x)$. The vertical solid blue line is $x = K/2$ and the vertical dashed red line is $x = K\mu/(\mu - 1)$.

4 The Neimark-Sacker bifurcation

The Hopf bifurcation is a well known phenomenon for a system of ordinary differential equations in two or more dimensions, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle appears. In the discrete setting, The Neimark-Sacker bifurcation is the analogue of the Hopf bifurcation [13]. Here, we seek conditions for the model (2) to have a non-hyperbolic fixed point with a pair of complex conjugate eigenvalues of modulus 1. This happens surely at the positive fixed point (x^*, y^*). The associated Jacobian matrix J^* in (8) has two complex conjugate eigenvalues with modulus 1 in the case $\det(J^*) = 1$ and $0 < tr(J^*) < 2$. Hence, the candidate for the bifurcation curve is as follows

$$\gamma = \frac{K\mu}{x^* \left(K\mu + x^*(1-\mu)\right)}. \tag{12}$$

We fix μ and K and take $\gamma > 0$ as a parameter and write it as $\gamma = \gamma^* + \lambda$, where

$$\gamma^* = \frac{K\mu}{x^*\left(K\mu + x^*(1-\mu)\right)}. \tag{13}$$

Then using (9), it is not hard to prove the following:

Lemma 4.1. *Let $\gamma = \gamma^*$ and $\omega = \cos^{-1}\left(\frac{tr(J^*)}{2}\right)$, then the eigenvalues of the Jacobian matrix at the positive fixed point (x^*, y^*) are $\lambda, \bar{\lambda} = \mathrm{e}^{\pm i\omega}$. Moreover, λ satisfies the following:*

(i) $\lambda^j \neq 1, \quad j = 1, 2, 3, 4$;

(ii) $\frac{d|\lambda|}{d\gamma}|_{\gamma=\gamma^*} = \frac{1}{2}\sqrt{\frac{\mu(K-x^*)+x^*}{\gamma^*\mu K}} > 0.$

In order to study the Neimark-Sacker bifurcation, it is convenient to introduce a new coordinate system that are better adapted to the neighborhood of the fixed point (x^*, y^*). To begin with we set

$$x = x^* + \bar{x}, \quad y = y^* + \bar{y}, \quad \gamma = \gamma^* + \sigma,$$

to obtain the coordinates $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ whose origin coincides with $(x, y) = (x^*, y^*)$. In these new variables the model (2) is transformed to the system

$$\begin{aligned}
\bar{x}_{n+1} &= a_{10}\bar{x}_n + a_{01}\bar{y}_n + a_{11}\bar{y}_n\bar{x}_n + a_{02}\bar{y}_n^2 + a_{20}\bar{x}_n^2 \\
&\quad + a_{30}\bar{x}_n^3 + a_{21}\bar{x}_n^2\bar{y}_n + a_{12}\bar{x}_n\bar{y}_n^2 + a_{03}\bar{y}_n^3 + O(4), \\
\bar{y}_{n+1} &= b_{10}\bar{x}_n + b_{01}\bar{y}_n + b_{11}\bar{y}_n\bar{x}_n + b_{02}\bar{y}_n^2 + b_{20}\bar{x}_n^2 \\
&\quad + b_{30}\bar{x}_n^3 + b_{21}\bar{x}_n^2\bar{y}_n + b_{12}\bar{x}_n\bar{y}_n^2 + b_{03}\bar{y}_n^3 + O(4),
\end{aligned} \tag{14}$$

where $O(4)$ denotes nonlinear terms of degree four and higher, and

$$\begin{aligned}
&a_{10} = c^*d^*,\ a_{01} = -d^*x^*(\gamma^* + \sigma),\ a_{11} = -d^*c^*(\gamma^* + \sigma),\\
&a_{02} = \frac{1}{2}\, d^*x^*(\gamma^* + \sigma)^2,\ a_{20} = -\frac{K(\mu-1)}{(K+(\mu-1)x^*)^2}d^*,\\
&a_{30} = \frac{K(\mu-1)^2}{(K+(\mu-1)x^*)^3}d^*,\ a_{21} = -a_{20}d^*(\gamma^* + \sigma),\\
&a_{12} = \frac{1}{2}d^*c^*(\gamma^* + \sigma)^2,\ a_{03} = -\frac{1}{6}d^*x^*(\gamma^* + \sigma)^3,\\
&b_{10} = 1 - f^*,\ b_{01} = x^*f^*(\gamma^* + \sigma),\ b_{11} = f^*(\gamma^* + \sigma),\\
&b_{02} = -\frac{1}{2}x^*f^*(\gamma^* + \sigma)^2,\ b_{20} = 0,\ b_{30} = 0,\ b_{21} = 0,\\
&b_{12} = -\frac{1}{2}f^*(\gamma^* + \sigma)^2,\ b_{03} = \frac{1}{6}x^*f^*(\gamma^* + \sigma)^3,
\end{aligned}$$

with

$$d^* = \frac{\mu\, K\, \mathrm{e}^{-(\gamma^*+\sigma)\, y^*}}{K + (\mu-1)\, x^*}, \quad c^* = 1 - \frac{(\mu-1)\, x^*}{x^*\,(\mu-1) + K}, \quad f^* = \mathrm{e}^{-(\gamma^*+\sigma)y^*}.$$

Of course, the eigenvalues of J^* in (8) are the same as the eigenvalues of the Jacobian of the map (14) at the fixed point $(0,0)$, so from the 2×2 matrix

$$J = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix},$$

we have

$$\begin{aligned}
tr(J) &= a_{10} + b_{01}, \quad \det(J) = a_{10}b_{01} - a_{01}b_{10},\\
\lambda, \bar{\lambda} &= \frac{tr(J) \pm \sqrt{tr(J)^2 - 4\det(J)}}{2} := \alpha \pm \mathrm{i}\beta \quad (\text{for} \quad \sigma \approx 0).
\end{aligned}$$

Let

$$T = \begin{pmatrix} 0 & a_{01} \\ \beta & \alpha - a_{10} \end{pmatrix},$$

then T is invertible. Now, we set $\sigma = 0$ and apply the following translation

$$\begin{pmatrix} \bar{x}_n \\ \bar{y}_n \end{pmatrix} = \begin{pmatrix} 0 & a_{01} \\ \beta & \alpha - a_{10} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

which brings the linear part into normal form. Then system (14) becomes of the following form:

$$\begin{aligned} u_{n+1} &= \alpha u_n - \beta v_n + c_{11} v_n u_n + c_{02} v_n^2 + c_{20} u_n^2 \\ &\quad + c_{30} u_n^3 + c_{21} u_n^2 v_n + c_{12} u_n v_n^2 + c_{03} v_n^3 + O(4) := \bar{f}(u_n, v_n), \\ v_{n+1} &= \beta u_n + \alpha v_n + e_{11} v_n u_n + e_{02} v_n^2 + e_{20} u_n^2 \\ &\quad + e_{30} u_n^3 + e_{21} u_n^2 v_n + e_{12} u_n v_n^2 + e_{03} v_n^3 + O(4) := \bar{g}(u_n . v_n), \end{aligned} \tag{15}$$

where

$$c_{11} = 2\,\alpha\,\beta\,a_{02} + \beta\,a_{01}a_{11} - 2\,\beta\,a_{02}a_{10},$$
$$c_{02} = \alpha^2 a_{02} + \alpha\,a_{01}a_{11} - 2\,\alpha\,a_{02}a_{10} + {a_{01}}^2 a_{20} - a_{01}a_{10}a_{11} + a_{02}{a_{10}}^2,$$
$$c_{20} = \beta^2 a_{02}, \quad c_{30} = \beta^3 a_{03}, \quad c_{21} = 3\,\alpha\,\beta^2 a_{03} + \beta^2 a_{01}a_{12} - 3\,\beta^2 a_{03}a_{10},$$
$$c_{12} = 3\,\alpha^2 \beta\,a_{03} + 2\,\alpha\,\beta\,a_{01}a_{12} - 6\,\alpha\,\beta\,a_{03}a_{10} + \beta\,{a_{01}}^2 a_{21} - 2\,\beta\,a_{01}a_{10}a_{12} + 3\,\beta\,a_{03}{a_{10}}^2,$$
$$c_{03} = \alpha^3 a_{03} + \alpha^2 a_{01}a_{12} - 3\,\alpha^2 a_{03}a_{10} + \alpha\,{a_{01}}^2 a_{21} - 2\,\alpha\,a_{01}a_{10}a_{12} + 3\,\alpha\,a_{03}{a_{10}}^2 + {a_{01}}^3 a_{30} - {a_{01}}^2 a_{10}a_{21} + a_{01}{a_{10}}^2 a_{12} - a_{03}{a_{10}}^3,$$
$$e_{11} = b_{11}a_{01}\beta + 2\,b_{02}\,(\alpha - a_{10})\,\beta, \quad e_{02} = b_{11}a_{01}\,(\alpha - a_{10}) + b_{02}\,(\alpha - a_{10})^2,$$
$$e_{20} = b_{02}\beta^2, \quad e_{30} = b_{03}\beta^3, \quad e_{21} = b_{12}a_{01}\beta^2 + 3\,b_{03}\,(\alpha - a_{10})\,\beta^2,$$
$$e_{12} = 0, \quad e_{03} = b_{12}a_{01}\,(\alpha - a_{10})^2 + b_{03}\,(\alpha - a_{10})^3.$$

In order to guarantee the Neimark-Sacker bifurcation for (15), we require that the following discriminatory quantity is not zero (Guckenheimer and Holmes [10]):

$$a = Re\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{11}\xi_{20}\right] - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\bar{\lambda}\xi_{21}), \tag{16}$$

where

$$\xi_{20} = \frac{1}{8}\left[(\bar{f}_{uu} - \bar{f}_{vv} + 2\bar{g}_{uv}) + \mathrm{i}(\bar{g}_{uu} - \bar{g}_{vv} - 2\bar{f}_{uv})\right]|_{(0,0)},$$
$$\xi_{11} = \frac{1}{4}\left[(\bar{f}_{uu} + \bar{f}_{vv}) + \mathrm{i}(\bar{g}_{uu} + \bar{g}_{vv})\right]|_{(0,0)},$$
$$\xi_{02} = \frac{1}{8}\left[(\bar{f}_{uu} - \bar{f}_{vv} - 2\bar{g}_{uv}) + \mathrm{i}(\bar{g}_{uu} - \bar{g}_{vv} + 2\bar{f}_{uv})\right]|_{(0,0)},$$
$$\xi_{21} = \frac{1}{16}\left[(\bar{f}_{uuu} + \bar{f}_{uvv} - \bar{f}_{vv} - 2\bar{g}_{uv}) + \mathrm{i}(\bar{g}_{uu} - \bar{g}_{vv} + 2\bar{f}_{uv})\right]|_{(0,0)}.$$

After calculating, we get

$$\xi_{20} = \frac{1}{4}\,a_{02}\beta^2 - \frac{1}{4}\,a_{20}{a_{01}}^2 - \frac{1}{4}\,a_{11}a_{01}\left(\alpha - a_{10}\right) - \frac{1}{4}\,a_{02}\left(\alpha - a_{10}\right)^2$$
$$+\frac{1}{4}\,b_{11}a_{01}\beta + \frac{1}{2}\,b_{02}\left(\alpha - a_{10}\right)\beta + \frac{\mathrm{i}}{8}\left(2\,b_{02}\beta^2 - 2\,b_{11}a_{01}\left(\alpha - a_{10}\right)\right.$$
$$\left.-2\,b_{02}\left(\alpha - a_{10}\right)^2 - 2\,a_{11}a_{01}\beta - 4\,a_{02}\left(\alpha - a_{10}\right)\beta\right),$$
$$\xi_{11} = \frac{1}{2}\,a_{02}\beta^2 + \frac{1}{2}\,a_{20}{a_{01}}^2 + \frac{1}{2}\,a_{11}a_{01}\left(\alpha - a_{10}\right) + \frac{1}{2}\,a_{02}\left(\alpha - a_{10}\right)^2$$
$$+\frac{\mathrm{i}}{4}\left(2\,b_{02}\beta^2 + 2\,b_{11}a_{01}\left(\alpha - a_{10}\right) + 2\,b_{02}\left(\alpha - a_{10}\right)^2\right),$$
$$\xi_{02} = \frac{1}{4}\,a_{02}\beta^2 - \frac{1}{4}\,a_{20}{a_{01}}^2 - \frac{1}{4}\,a_{11}a_{01}\left(\alpha - a_{10}\right)$$
$$-\frac{1}{4}\,a_2\left(\alpha - a_{10}\right)^2 - \frac{1}{4}\,b_{11}a_{01}\beta - \frac{1}{2}\,b_{02}\left(\alpha - a_{10}\right)\beta$$
$$+\frac{\mathrm{i}}{8}\left(2\,b_{02}\beta^2 - 2\,b_{11}a_{01}\left(\alpha - a_{10}\right) - 2\,b_{02}\left(\alpha - a_{10}\right)^2\right.$$
$$\left.+\,2\,a_{11}a_{01}\beta + 4\,a_{02}\left(\alpha - a_{10}\right)\beta\right),$$
$$\xi_{21} = \frac{3}{8}\,a_{03}\beta^3 + \frac{1}{8}\,a_{21}{a_{01}}^2\beta + \frac{1}{4}\,a_{12}a_{01}\left(\alpha - a_{10}\right)\beta$$
$$+\frac{3}{8}\,a_3\left(\alpha - a_{10}\right)^2\beta + \frac{1}{8}\,b_{12}a_{01}\beta^2$$
$$+\frac{3}{8}\,b_{03}\left(\alpha - a_{10}\right)\beta^2 + \frac{3}{8}\,b_{12}a_{01}\left(\alpha - a_{10}\right)^2 + \frac{3}{8}\,b_{03}\left(\alpha - a_{10}\right)^3$$
$$+\frac{\mathrm{i}}{16}\left(6\,b_{03}\beta^3 + 4\,b_{12}a_{01}\left(\alpha - a_{10}\right)\beta + 6\,b_{03}\left(\alpha - a_{10}\right)^2\beta - 2\,a_{12}a_{01}\beta^2\right.$$
$$-6\,a_{03}\left(\alpha - a_{10}\right)\beta^2 - 6\,a_{30}{a_{01}}^3 - 6\,a_{21}{a_{01}}^2\left(\alpha - a_{10}\right)$$
$$\left.-\,6\,a_{12}a_{01}\left(\alpha - a_{10}\right)^2 - 6\,a_{03}\left(\alpha - a_{10}\right)^3\right).$$

Analyzing the above and the Neimark-Sacker bifurcation conditions discussed in Guckenheimer and Holmes [10], we write our main result as follows:

Proposition 4.2. *If the condition $a \neq 0$ holds, then the map (2) undergoes Neimark-Sacker bifurcation at the positive fixed point (x^*, y^*) when the parameter γ varies in a small neighborhood of γ^*. Moreover,*

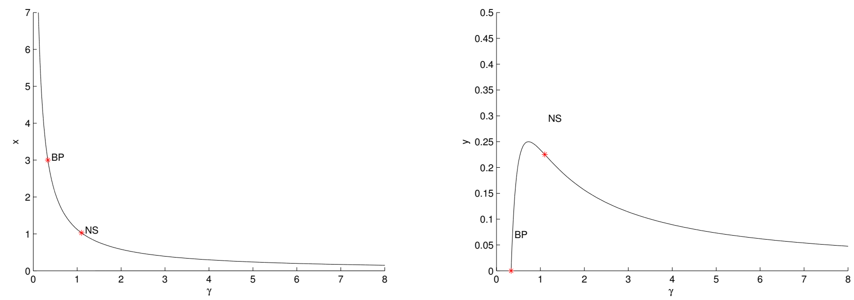

Figure 2: Fixed point continuations of system (2) by MatContM package.

if $a < 0$ (respectively $a > 0$), then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point (x^*, y^*) for $\gamma > \gamma^*$ (respectively $\gamma < \gamma^*$).

5 Numerical simulations

By following computations presented in the paper, we illustrate some numerical simulation in this section.

Example 5.1. Let $\mu = 1.5, \gamma = 1$ and $K = 3$, then the positive fixed point of (2) is $(x^*, y^*) \approx (1.121607910, 0.2340910474)$. We investigate the stability of the positive fixed point by numerical method using MatContM package on MATLAB [9]. We perform a fixed point continuation and choose γ as the bifurcation parameter. The fixed point curve detects a branch point(BP) corresponding to boundary fixed point $(K, 0) = (3, 0)$ and a Neimark-Sacker bifurcation corresponding to the positive fixed point (x^*, y^*). The positive fixed point is stable prior the NS bifurcation point and loses stability at $\gamma^* \simeq 1.0993918$. The results are shown in Figure 2. Moreover, Figure 3 shows that both host and parasite populations undergo Neimark-Sacker bifurcation as γ varies in the interval $[1, 2.6]$. For further confirmation of Neimark-Sacker at $\gamma^* \simeq 1.0993918$ phase portraits for system (2) in the neighbourhood of γ^* are shown in Figure 4.

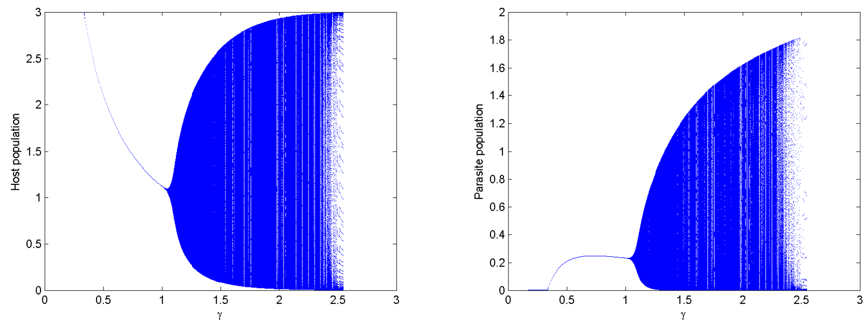

Figure 3: Neimark-Sacker bifurcation diagrams of system (2) with the initial values $(0.1, 0.1)$, $K = 3$ and $\mu = 1.5$.

Example 5.2. Fix $\mu = 1.5, \gamma = 1.09$ and $K = 3$, then the positive fixed point of (2) is $(x^*, y^*) \approx (1.035055488, 0.2259807334)$. After some calculations, we find that the Neimark-Sacker bifurcation appears from the fixed point at $\gamma^* = 1.091681932$, and the eigenvalues of the positive fixed point are $\lambda_{1,2} \approx 0.8673816100 + \mathrm{i}\, 0.4960932001$, so $|\lambda_{1,2}| = 1$. Moreover, we obtain $a \approx -0.05932393952 < 0$ and $(x^*, \gamma^*) \in \Omega$, therefore Proposition 4.2 is verified.

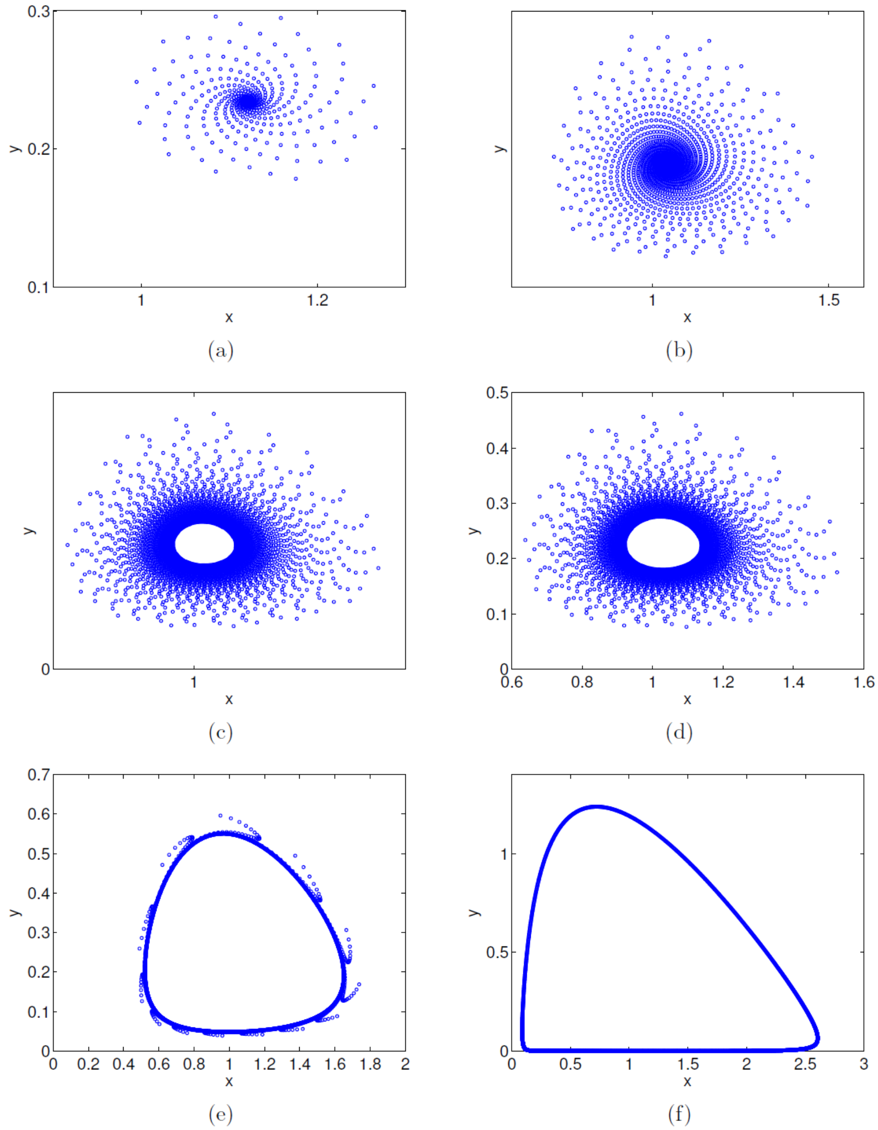

Figure 4: The phase portrait of system (2) with initial value $(x_0, y_0) = (0.5, 1)$ and $K = 3, \mu = 1.5$, (a) $\gamma = 1$, (b) $\gamma = 1.08$, (c) $\gamma = 1.099$, (d) $\gamma = 1.1$, (e) $\gamma = 1.15$, (f) $\gamma = 1.5$.

6 Conclusion

This paper deals with the stability and bifurcation analysis of a modified Nicholson-Bailey host-parasite model. It is shown that system (2) has two boundary equilibria. The existence and uniqueness of positive fixed point for $\mu > 1$ and $K, \gamma > 0$ is investigated. It is shown that the positive fixed point can undergoes Neimark-Sacker bifurcation. In this way, γ is chosen as the bifurcation parameter and the Neimark-Sacker bifurcation is analyzed both by theoretical argument and numerical simulations. For numerical simulations, MatContM package on MATLAB software is implemented.

References

[1] R. Asheghi, Bifurcations and dynamics of a discrete predator-prey system, *Journal of Biological Dynamics* 8(2014), 161-186.

[2] A. Atabaigi and M. H. Akrami, Dynamics and bifurcations of a host-parasite model, *International Journal of Biomathematics*, 10(2017), 1750089.

[3] V. A. Bailey, A. J. Nicholson and E. J. Williams, Interaction between hosts and parasites when some host individuals are more difficult to find than others, *J. Theor. Biol.* 3(1) (1962) 1-18.

[4] J. R. Beddington, C. A. Free, and J. H. Lawton, Dynamic complexity in predator-prey models framed in difference equations, *Nature* 225 (1975), 58-60.

[5] J. Bektešević, V. Hadžiabdić, S. Kalabušić, M. Mehuljić, E. Pilav, Dynamics of a class of host–parasitoid models with external stocking upon parasitoids, *Advances in Difference Equations* (2021), 1-37.

[6] R. J. H. Beverton, S. J. Holt, *On the Dynamics of Exploited Fish Populations, in: Fisheries Investigations, Series 2,* vol. 19, H. M. Stationery Office, London, 1957.

[7] Q. Din, Global stability and Neimark-Sacker bifurcation of a host-parasitoid model, *Int. J. Syst. Sci.* (2016), DOI:10.1080/00207721.2016.1244308.

[8] L. Edelstein-Keshet, *Mathematical Models in Biology*, SIAM, Philadelphia, PA, 2005.

[9] W. Govaerts, Yu. A. Kuznetsov, R. K. Ghaziani and H. G. E. Meijer, *Cl MatContM:A Toolbox for Continuation and Bifurcation of Cycles of Maps, Universiteit Gent, Belgium and Utrecht University*, The Netherlands 2008.

[10] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer, New York, 1983.

[11] O. A. Gümüs, Dynamical Consequences and Stability Analysis of a New Host-Parasitoid Model, *Gen. Math. Notes* (27)1, 2015, 9-15.

[12] M. P. Hassell and R. M. May, In insect host-parasite models, *J. Anim. Ecol.* 42 (1973) 693-726.

[13] A. N. W. Hone, M. V. Irle, and G. W. Thurura, On the Neimark-Sacker bifurcation in a discrete predator-prey system, *Journal of biological dynamics* (4)6, (2010) 594-606.

[14] S. Kapcak, V. Ufuktube, S. Elaydi, Stability and invariant manifolds of a generalized Beddington host-parasitoid model, *Journal of Biological Dynamics* 7(2013), 233-253.

[15] A. Q. Khan and M.N. Qureshi, Dynamics of a modified Nicholson-Bailey host-parasitoid model, *Advances in Difference Equations* 2015.

[16] G. Livadiotisa, L. Assas, B. Dennis, S. Elaydi, and E. Kwessi, A discrete-time host-parasitoid model with an Allee effect, *Journal of Biological Dynamics* 9(2014) 34-51.

[17] X. Ma, Q. Din, M., Rafaqat, N. Javaid, and Y. Feng, A density-dependent host-parasitoid model with stability, bifurcation and chaos control, *Mathematics* (8)4, (2020), 536.

Mohammad Hossein Akrami
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Yazd University, 89195-741
Yazd, Iran
E-mail: akrami@yazd.ac.ir

Ali Atabaigi
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Razi university
Kermanshah, Iran
E-mail: aliaelmi@gmail.com