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Original Research Paper

Strong Convergence of Split-Step Forward Methods for Stochastic Differential Equations Driven by $S\alpha S$ processes

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Abstract. We consider stochastic differential equation driven by α-stable processes. Three methods of drifting split-step Euler, diffused split-step Euler and three-stage Milstein for approximation of solution are used. The strong convergence of these three methods is proven and the upper bounds of their stabilities are obtained and depicted.

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1 Introduction

We consider one-dimensional time-independent Stable Lévy SDE's of the following form

$$\begin{cases} dX(t) = \mu(X(t))\ dt + \sigma(X(t))\ dS_\alpha(t), & t \in [t_0, T] \\ X(t_0) = X_0. \end{cases} \tag{1}$$

where $X(t)$ is a real-valued stochastic process and μ, σ are real well-defined functions. $\{S_\alpha(t),\ t \in [t_0, T]\}$ is $S\alpha S$ process with $\alpha \in (1, 2]$ (for more details see [1, Page 30]). In this article, numerical methods on a given time interval $[t_0, T]$ are fixed by schemes based on equidistance time discretization points $t_n = t_0 + nh$, $n = 0, 1, \ldots, N$ with step size $h = \frac{T-t_0}{N}$, $N = 1, 2, \ldots$. We focus our attention that convergence in the strong sense. An approximation X_n convergence strongly to the exact solution $X(t_n)$ with order $p > 0$ if there exist constants $h_0, c \in (0, +\infty)$, such that for all $h \in (0, h_0)$

$$\mathbb{E}|X(t_n) - X_n| \leq ch^p.$$

For SDE (1), the well-known Euler–Maruyama (EM) method is given by [1, Page 157]

$$X_{n+1} = X_n + \mu(X_n)h + \sigma(X_n)\Delta S_\alpha(t_n)$$

where $\Delta S_\alpha(t_n) = S_\alpha(t_{n+1}) - S_\alpha(t_n)$, two split-step forward methods takes attention of consideration.

The first one is drifting split-step Euler (DRSSE) method [5]

$$\begin{cases} \widehat{X}_n = X_n + h\mu(X_n) \\ X_{n+1} = \widehat{X}_n + \Delta S_\alpha(t_n)\sigma(\hat{X}_n) \end{cases}$$

The second type is called diffused split-step Euler(DISSE) method [5]:

$$\begin{cases} \widehat{X}_n = X_n + \sigma(X_n)\Delta S_\alpha(t_n) \\ X_{n+1} = \widehat{X}_n + h\mu(X_n) \end{cases}$$

Following classical three-stage Milstein (TSM) method we define the following TSM method.

$$\begin{cases} \widehat{X}_{n1} = X_n - \frac{1}{2}h^{\frac{2}{\alpha}}\ \sigma(X_n)\ \sigma'(X_n) \\ \widehat{X}_{n2} = \widehat{X}_{n1} + h\ \mu(\widehat{X}_{n1}) \\ X_{n+1} = \widehat{X}_{n2} + \Delta S_\alpha\ \sigma(\widehat{X}_{n2}) + \frac{1}{2}h^{\frac{1}{\alpha}}\sigma(\widehat{X}_{n2})\ \sigma'(\widehat{X}_{n2})\ \Delta S_\alpha \end{cases}$$

The following assumptions are required when considering the convergence properties of spiliting shemes for SDE

Assumption 1.1 ([10, 5]). *The functions $\mu(\cdot)$, $\sigma(\cdot)$ and $\sigma(\cdot)\sigma'(\cdot)$ satisfy Lipschitz condition, i.e, there exist positive real k_1 such that*

$$|\mu(x_1)-\mu(x_2)|\vee|\sigma(x_1)-\sigma(x_2)|\vee|\sigma(x_1)\sigma'(x_1)-\sigma(x_2)\sigma'(x_2)| \leq k_1|x_1-x_2|$$

Assumption 1.2 ([10, 5]). *The functions $\mu(\cdot)$, $\sigma(\cdot)$ and $\sigma(\cdot)\sigma'(\cdot)$ satisfy $\beta-$growth condition for some $\beta \in (1,\alpha)$, i.e, for some constant $k_2 \in \mathbb{R}^+$*

$$|\mu(x)|^\beta \vee |\sigma(x)|^\beta \vee |\sigma(x)\sigma'(x)|^\beta \leq k_2(1+|x|^\beta)$$

In the Assumption 1.1 and Assumption 1.2, $a \vee b$ means $\max\{a,b\}$.

Lemma 1.3. *If μ and σ satisfy in Assumption 1.1 and if $E|X_0| < \infty$ then $E(\bar{X}_k) < \infty$ for $k = 0,1,\ldots,N$ where*

$$\begin{aligned} \overline{X}_k &= \overline{X}_{t_{k-1},X_{k-1}}(t_k) \\ &= \overline{X}_{k-1} + \mu(\overline{X}_{t_{k-1}})h + \sigma(\overline{X}_{t_{k-1}})\Delta S_\alpha(t_{k-1}), \quad k = 1,\ldots,N \\ \overline{X}_0 &= X_0 = X(t_0) \\ \overline{X}_{t,x}(t+h) &= x + \mu(X(t))h + \sigma(X(t))\Delta S_\alpha(t) \end{aligned}$$

Proof. First note that

$$\begin{aligned} \mathbb{E}(X_{t,x}(t+h) - \overline{X}_{t,x}(t+h)) = \mathbb{E}\Bigg[&\int_t^{t+h}[\mu(X(s)) - \mu(X(t))]\ ds \\ &+ \int_t^{t+h}[\sigma(X(s)) - \sigma(X(t))]\ dS_\alpha(s)\Bigg] \end{aligned}$$

Therefore

$$\begin{aligned}\left|\mathbb{E}(X_{t,x}(t+h)-\overline{X}_{t,x}(t+h))\right| &\leq \mathbb{E}\Big[\int_t^{t+h}[|\mu(X(s))-\mu(X(t))| \\ &\quad + |\sigma(X(s))-\sigma(X(t))|]\ ds\Big] \\ &\leq k\int_t^{t+h}[1+|t-s|]\ ds \\ &\leq k(1+h)h\end{aligned}$$

Now we have

$$\begin{aligned}\mathbb{E}\left|\overline{X}_{k+1}\right| =&\mathbb{E}\left|\overline{X}_{t_k,\overline{X}_k}(t_{k+1})\right| \\ =&\mathbb{E}\left|\overline{X}_{t_k,\overline{X}_k}(t_{k+1})-\overline{X}_k+\overline{X}_k-X_{t_k,\overline{X}_k}(t_{k+1})+X_{t_k,\overline{X}_k}(t_{k+1})\right| \\ =&\mathbb{E}\Big|\overline{X}_k+\big(X_{t_k,\overline{X}_k}(t_{k+1})-\overline{X}_k\big) \\ &+\big(\overline{X}_{t_k,\overline{X}_k}(t_{k+1})-X_{t_k,\overline{X}_k}(t_{k+1})\big)\Big| \\ &\leq \mathbb{E}\left|\overline{X}_k\right|+\mathbb{E}\left|X_{t_k,\overline{X}_k}(t_{k+1})-\overline{X}_k\right| \\ &+\mathbb{E}\left|\overline{X}_{t_k,\overline{X}_k}(t_{k+1})-X_{t_k,\overline{X}_k}(t_{k+1})\right|\end{aligned}$$

So if $\mathbb{E}\left|\overline{X}_k\right| < \infty$ then so is $\mathbb{E}\left|\overline{X}_{k+1}\right|$. In other words if $\mathbb{E}\left|X_0\right| < \infty$ then $\mathbb{E}\left(\left|\overline{X}_k\right|\right) < \infty$ for $k = 1, 2, \ldots, N$.

The finiteness of the second part of inequality is veryfied as follows:

$$\begin{aligned}\mathbb{E}\left|X_{t_k,\overline{X}_k}(t_{k+1})-\overline{X}_k\right| =&\mathbb{E}\Big|\overline{X}_k+\int_{t_k}^{t_{k+1}}\mu(X(s))\ ds \\ &+\int_{t_k}^{t_{k+1}}\sigma(X(s))\ dS_\alpha(s)-\overline{X}_k\Big| \\ =&\mathbb{E}\Big|\int_{t_k}^{t_{k+1}}\mu(X(s))\ ds \\ &+\int_{t_k}^{t_{k+1}}\sigma(X(s))\ dS_\alpha(s)\Big| \leq kh\end{aligned}$$

□

Lemma 1.4. *If conditions of Assumption 1.3 satisfy, then*

$$\mathbb{E}\left|\overline{X}_k\right| \leq k(1+\mathbb{E}\left|\overline{X}_0\right|)$$

Proof. By using the conditional version of Assumption 1.3, we have

$$\mathbb{E}\left|X_{t_k,\overline{X}_k}(t_{k+1}) - \overline{X}_{t_k,\overline{X}_k}(t_{k+1})\right| \leq k(1+\mathbb{E}\left|\overline{X}_k\right|)h$$

consider again the inequality , we obtain

$$\mathbb{E}\left|X_{t_k,\overline{X}_k}(t_{k+1}) - \overline{X}_k\right| \leq k(1+\mathbb{E}\left|\overline{X}_k\right|)h$$

or equivalently

$$\mathbb{E}\left|\overline{X}_{k+1}\right| \leq E\left|\overline{X}_k\right| + k(1+\mathbb{E}\left|\overline{X}_k\right|)$$

Now using the well known resulting inequality given in Assumption 1.5 we obtain the result. □

Lemma 1.5 ([2, Page 15])**.** *Suppose that for arbitrrary $N \in \mathbb{N}$ and $k = 0, 1, \ldots, N$ we have*

$$u_{k+1} \leq (1+Ah)u_k + Bh^p \tag{2}$$

where $h = \frac{T}{N}$, $A, B \geq 0$, $p \geq 1$, $u_k \geq 0$, $k = 0, 1, \ldots, N$ then

$$u_k \leq e^{AT}u_0 + \frac{B}{A}(e^{AT}-1)h^{p-1}. \tag{3}$$

2 Strong Convergence of DRSSE and DISSE

We now obtain the strong convergence of split step Euler method, under Assumption 1.1

Theorem 2.1. *Let X_k be the numerical approximation $X(t_k)$ after k steps with step size $h = \frac{T}{N}$, $N = 1, 2, \ldots$ $E\left|X_k\right| < \infty$. Apply one of DRESE or DFSSE methods to the given SDE, under Assumption 1.1, then for all $k = 0, 1, \ldots, N$ we have*

$$\mathbb{E}\left(\left|X_k - X(t_k)\right| \Big| X(t_0) = X_0\right) = O(h^{\frac{1}{2}})$$

Proof. We denote Euler-Maruyama approximation step by

$$E^{\mathrm{EM}} = X_k + \mu(X_k)h + \sigma(X_k)\Delta S_\alpha(t_k), \quad k = 0, 1, \ldots, N-1$$

then there exists constants $k_1, k > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left(|X(t_{k+1}) - X_{k+1}| \,\middle|\, X_k = X(t_k)\right) \\
&\leq \mathbb{E}\left(\left|X(t_{k+1}) - E^{\mathrm{EM}}_{k+1}\right| \,\middle|\, X_k = X(t_k)\right) \\
&+ \mathbb{E}\left(\left|E^{\mathrm{EM}}_{k+1} - X(t_{k+1})\right| \,\middle|\, X_k = X(t_k)\right) \\
&\leq k\left(1 + \mathbb{E}\left|X_k\right|\right) + \mathbb{E}\left(\left|E^{\mathrm{EM}}_{k+1} - X(t_{k+1})\right| \,\middle|\, X_k = X(t_k)\right)
\end{aligned}$$

Now

$$\begin{aligned}
&\mathbb{E}\left(\left|E^{\mathrm{EM}}_{k+1} - X(t_{k+1})\right| \,\middle|\, X_k = X(t_k)\right) \\
&= \begin{cases} \text{for DRSSE method} \\ \mathbb{E}\left(|\sigma(X_{t_k}) - \sigma(X_k + h\mu(X_k))| \; |\Delta S_\alpha(t_k)| \,\middle|\, X_k = X_{t_k}\right) \\ \\ \text{for DFSSE method} \\ \mathbb{E}\left(|\sigma(X_{t_k}) - \sigma(X_k + h\mu(X_k))| \; |\Delta S_\alpha(t_k)| \,\middle|\, X_k = X_{t_k}\right) \end{cases} \\
&\leq k_1\left(1 + \mathbb{E}\left|X_k\right|\right) h^{\frac{1}{2}}
\end{aligned}$$

Therefore the inequality is less than or equal to $k\left(1 + E\left|X_0\right|\right) h^{\frac{1}{2}}$. □

Lemma 2.2. *Suppose the one-step approximation $\overline{X}_{t,x}(t+h)$ has order of accuracy p_1 for the mathematical expectation of the deviation and order of accuracy p_2 for the $\beta-$growth deviation $(1 \leq \beta < \alpha < 2)$ more precicely, for arbitrary $t_0 \leq t \leq t_0 + T - h,\ x \in \mathbb{R}$ the following inequalty hold:*

$$|\mathbb{E}(X_{t,x}(t+h) - \overline{X}_{t,x}(t+h))| \leq k(1 + |x|^\beta)^{\frac{1}{\beta}} h^{\frac{1}{p_1}} \tag{4}$$

$$\left[\mathbb{E}|X_{t,x}(t+h) - \overline{X}_{t,x}(t+h)|^\beta\right]^{\frac{1}{\beta}} \leq k(1 + |x|^\beta)^{\frac{1}{\beta}} h^{\frac{1}{p_2}} \tag{5}$$

Proof. By using Minkowski's inequality, we modified [2, Theorem 1.1] for α-stable motion with $\beta \in [1, \alpha]$ $(1 < \alpha \leq 2)$. □

Lemma 2.3. *Let for all natural number N and for all $k = 0, 1, \dots, N$ we have $\mathbb{E}\big(|X_k|^\beta\big) < \infty$, where β is defined in Assumption 2.2. Then the following inequality hold:*

$$\mathbb{E}\left(|X_k|^\beta\right) \le k\left(1 + \mathbb{E}\,|X_0|^\beta\right)$$

Proof. Suppose that $\mathbb{E}|x_k|^\beta < \infty$. Then using conditional version of (5) we obtain

$$\mathbb{E}|X_{t_k,x_k}(t_{k+1}) - \overline{X}_{t,x_k}(t_{k+1})| \le k^\beta\left(1 + \mathbb{E}|x_k|^\beta\right)^\beta h^{p_2} \tag{6}$$

It is well-known that is a random variable X has bounded $\beta-$th moment, then the solution $X_{t,x}(t+\theta)$ also has bounded $\beta-$th moment. Therefore $\mathbb{E}|X_{t,x_k}(t_{k+1})|^\beta < \infty$ which implies $\mathbb{E}|X_{k+1}|^\beta < \infty$. Since $\mathbb{E}|X_k|^\beta < \infty$ we have proved the existance of all $\mathbb{E}|X_k|^\beta < \infty,\ k = 0, \dots, N$. Consider the inequality

$$\big(\mathbb{E}|X_{k+1}|^\beta\big)^{\frac{1}{\beta}} \le \big(\mathbb{E}|X_k|^\beta\big)^{\frac{1}{\beta}} + \big(\mathbb{E}|X_{t_k,\overline{x}_k}(t_{k+1}) - X_k|\big)^{\frac{1}{\beta}} \tag{7}$$

$$+ \left(\mathbb{E}|X_{t_k,\overline{x}_k}(t_{k+1}) - \overline{X}_{t_k,\overline{x}_k}(t_{k+1})|\right)^{\frac{1}{\beta}} \tag{8}$$

we have

$$(\mathbb{E}|X_{t_k,\overline{x}_k}(t_{k+1}) - X_k|^\beta \le kh(1 + \mathbb{E}|X_k|^\beta) \tag{9}$$

It is not difficult to prove the inequality

$$\mathbb{E}\Big|\mathbb{E}(X_{t_k,\overline{X}_k} - \overline{X}_k|X(t_k))\Big|^\beta \le kh^\beta(1 + \mathbb{E}|X_k|^\beta) \tag{10}$$

Applying the inequality (6), (8) and (9) to inequality (10) and recalling that $p_1 \ge 1,\ p_2 \ge \frac{1}{2}$, we arrive at the inequality (for $h \le 1$)

$$\mathbb{E}|X_{k+1}|^\beta \le \mathbb{E}|X_k|^\beta + kh(1 + \mathbb{E}|X_k|^\beta) = kh + (1 + kh)\mathbb{E}|X_k|^\beta \tag{11}$$

Again using Assumption 1.5 we get to result. $\square$

Theorem 2.4. *Let X_k be the numerical approximation to $X(t_k)$ at time T after k steps with step size $h = \frac{T}{N}, N = 1, 2, \dots, \mathbb{E}\,|X_k|^\beta < \infty$, where $1 \le \beta < \alpha < 2$. Apply one of split-step Euler methods to the SDE* (1) *under Assumption 1.2, then for all $k = 0, 1, \dots, N$ we have*

$$\left[\mathbb{E}\,|X_k - X(t_k)|^\beta\,\Big|X_0 = X(t_0)\right]^{\frac{1}{\beta}} = O(h^{\frac{1}{\beta}})$$

Proof. Let X^{E}_{k+1} stand for the local Euler approximation

$$X^{\mathrm{E}}_{k+1} = X_k + h\mu(X_k) + \sigma(X_k)\Delta S_{\alpha k}, \quad k = 0, 1, \ldots, N-1$$

then $A_1 = \mathbb{E}^{\frac{1}{\beta}}\left(\left|X(t_{k+1}) - X^{\mathrm{E}}_{k+1}\right|^{\beta}\middle|X_0 = X(t_0)\right)$. By using Minkowski's inequality, we have

$$\begin{aligned} A_1 \le& \mathbb{E}^{\frac{1}{\beta}}\left(\left|X_{t_{k+1}} - X_{k+1}\right|^{\beta}\middle|X_k = X(t_k)\right) \\ &+ \mathbb{E}^{\frac{1}{\beta}}\left(\left|X^{\mathrm{E}}_{k+1} - X(t_{k+1})\right|^{\beta}\middle|X_k = X(t_k)\right) \end{aligned}$$

But Lemma attain that

$$\begin{aligned} \mathbb{E}^{\frac{1}{\beta}}\left(\left|X(t_{k+1}) - X^{\mathrm{E}}_{k+1}\right|^{\beta}\middle|X_k = X(t_k)\right) \le& k\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right] \\ \le& k\left[1 + \mathbb{E}\left|X_k\right|\right]^{\frac{1}{\beta}} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}^{\frac{1}{\beta}}\left[\left|X(t_{k+1}) - X^{\mathrm{E}}_{k+1}\right|^{\beta}\middle|X_k = X(t_k)\right] \\ &= \begin{cases} \text{for DRSSE method} \\ \mathbb{E}^{\frac{1}{\beta}}\left[\left|\left(\sigma(X_k) - \sigma(X^{\mathrm{E}}_k)\right)^{\beta}\Delta S_{\alpha k}\right|^{\beta}\middle|X_k = X(t_k)\right] \\ \text{for DFSSE method} \\ \mathbb{E}^{\frac{1}{\beta}}\left[\left|\left(\mu(X_k) - \mu(X^{\mathrm{E}}_k)\right)^{\beta}\Delta S_{\alpha k}\right|^{\beta}\middle|X_k = X(t_k)\right] \end{cases} \end{aligned}$$

then

$$\mathbb{E}^{\frac{1}{\beta}}\left[\left|X^{\mathrm{E}}_{k_1} - X_{k+1}\right|^{\beta}\middle|X_k = X(t_k)\right] \le k_1\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h^{\frac{3}{\beta}}$$

therefore

$$A \le k\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h + k_1\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h^{\frac{3}{\beta}}$$

but $\frac{3}{2} \le \frac{3}{\beta} < 3$ and therefore

$$\begin{aligned} \frac{k\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h + k_1\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h^{\frac{3}{\beta}}}{h^{\frac{1}{\beta}}} =& k\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h^{1-\frac{1}{\beta}} \\ &+ k_1\left[1 + \mathbb{E}\left|X_k\right|^{\beta}\right]^{\frac{1}{\beta}} h^{\frac{2}{\beta}} \end{aligned}$$

which tends to zero as $h \to 0$. The proof is complet □

Theorem 2.5. *Let X_k be the numerical approximation to $X(t_k)$ at time t_k after k steps with step size $h = \frac{T}{N}, N = 1, 2, \ldots$. Apply the three-stage Milstein method to the SDE (1) under Assumption 1.1, and Assumption 1.2, then for all $k = 0, 1, \ldots, N$ we have*

$$\left(\mathbb{E}\left[|X_k - X(t_k)|^\beta \,\middle|\, X_0 = X(t_0)\right]\right)^{\frac{1}{\beta}} = O(h),$$

where $1 \leq \beta < \alpha < 2$.

Proof. Denote the local Milstein approximation step

$$X_{n+1}^{\mathrm{M}} = \widehat{X}_{n_2} + \Delta S_\alpha \ \sigma(\widehat{X}_{n_2}) + \frac{1}{2}\sigma(\widehat{X}_{n_2})\sigma'(\widehat{X}_{n_2})(\Delta S_\alpha)^2$$

then there exist some constant $k > 0$ such that

$$\begin{aligned} H_1 =& \left|\mathbb{E}\left[X(t_{n+1}) - X(t_n)\,\middle|\, X_n = X(t_n)\right]\right| \\ =& \left|\mathbb{E}\left[X(t_{n+1}) - X_{n+1}^{\mathrm{M}}\,\middle|\, X_n = X(t_n)\right]\right. \\ &\left. + \,\mathbb{E}\left[X_{n+1}^{\mathrm{M}} - X_{n+1}\,\middle|\, X_n = X(t_n)\right]\right| \\ &\leq k\left(1 + \mathbb{E}\,|X_n|\right) h + H_2 \end{aligned}$$

with

$$H_2 = \left|\mathbb{E}\left[X_{n+1}^{\mathrm{M}} - X_{n+1}\,\middle|\, X_n = X(t_n)\right]\right|$$

$$\leq \begin{cases} \left|\mathbb{E}\left[\Delta S_\alpha \left(\sigma(X_n) - \sigma(\overline{X}_{n2})\right) \,\middle|\, X_n = X(t_n)\right]\right| \\ + \left|\mathbb{E}\left[\frac{1}{2}\left(\Delta S_\alpha\right)^2 \sigma(X_n)\sigma'(X_n) - \sigma(\overline{X}_{n2})\sigma'(\overline{X}_{n2})\,\middle|\, X_n = X(t_n)\right]\right| \\ + \left|\mathbb{E}\left[h\left(\mu(X_n) - \mu(\overline{X}_{n1})\right) \,\middle|\, X_n = X(t_n)\right]\right|, \\ \text{if TSM method is used.} \end{cases}$$

$$\leq k_1 h \left(\left|\mathbb{E}\left[X_n - \overline{X}_{n1}\,\middle|\, X_n = X(t_n)\right]\right| + \left|\mathbb{E}\left[X_n - \overline{X}_{n2}\,\middle|\, X_n = X(t_n)\right]\right|\right)$$

$$\leq k\left(1 + \mathbb{E}\,|X_n|\right) h$$

similary we check estimate for local β-th mean of tree-stage Milstain and obtain for $n = 0, 1, \ldots, N-1$ by standard arguments

$$\begin{aligned} H_3 &= \left(\mathbb{E}\left[|X(t_{n+1}) - X_{n+1}|^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ &\leq \left(\mathbb{E}\left[\left|X(t_{n+1}) - X^{\mathrm{M}}_{n+1}\right|^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ &\quad + \left(\mathbb{E}\left[\left|X^{\mathrm{M}}_{n+1} - X_{n+1}\right|^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ &\quad \leq k\left(1 + \mathbb{E}\,|X_n|\right)^{\frac{1}{\beta}} h^{\frac{3}{\beta}} + H_4 \end{aligned}$$

with

$$\begin{aligned} H_4 &= \left(\mathbb{E}\left[\left|X^{\mathrm{M}}_{n+1} - X_{n+1}\right|^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ &\leq \begin{cases} \left(\mathbb{E}\left[\left|\Delta S_\alpha\left(\sigma(X_n) - \sigma(\overline{X}_{n2})\right)\right|^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ + \left(\mathbb{E}\left[\left|\frac{1}{2}\Delta S_\alpha\left(\sigma(X_n)\sigma'(X_n) - \sigma(X_{n2})\sigma'(X_{n2})\right)\right| \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ + \left(\mathbb{E}\left[\left|h\left(\mu(X_n) - \mu(X_{n1})\right)\right|^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}}, \\ \\ \text{if TSM method is used.} \end{cases} \\ &\leq k_1 h^{\frac{3}{\beta}}\left(\mathbb{E}\left[\left(X_n - \overline{X}_{n1}\right)^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \\ &\quad + \left(\mathbb{E}\left[\left(X_n - \overline{X}_{n2}\right)^{\beta} \,\middle|\, X_n = X(t_n)\right]\right)^{\frac{1}{\beta}} \leq k\left(1 + \mathbb{E}\,|X_n|^{\beta}\right) h^{\frac{3}{\beta}} \end{aligned}$$

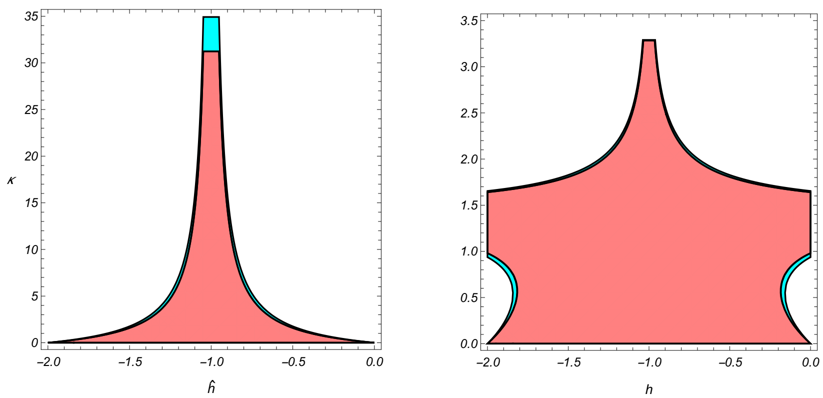
□

The last inequality is obtained under Assumption 1.2 and Assumption 2.3 the exponent $p_2 = \frac{3}{\beta}$ together with $p_1 = \beta$ and apply it to finally prove the strong order $\gamma = 1$ of the three-stage Milstain methods as was claimed in theorem.

3 Stability properties

The stability of the methods are considered in this subsection we apply one-step scheme to the scalar linear test equation

$$dX(t) = \lambda X(t)\; dt + \mu X(t)\; dS_\alpha(t), \qquad X(t_0) = X_0 \tag{12}$$



(a) Upper bound of DRSSE and DISSE methods for $\alpha = 1.99$ (Gray), $\alpha = 1.5$ (Cyan), and $\alpha = 1.01$ (Pink).

(b) Upper bound of TSM method for $\alpha = 1.99$ (Gray), $\alpha = 1.5$ (Cyan), and $\alpha = 1.01$ (Pink).

Figure 1: Stability bounds for $\alpha = 1.99$, $\alpha = 1.5$, and $\alpha = 1.01$.

which is represented by

$$X_{n+1} = R(\hat{h}, \kappa, h, S)X_n$$

where $S = S_\alpha(t_n)$ is stable random variable $S \sim S\alpha S$ with dispersion h, $\hat{h} = \lambda h$, and $\kappa = h^{\frac{1}{\alpha}}\mu$. However, motivated by [10, 8, 5], we can extend the definition of stability and introduce absolute-value (AV) stability for α-stable motion.

Definition 3.1. The numerical method is to be AV-stable for $\hat{h}, \kappa$ and h if

$$\overline{R}(\hat{h}, \kappa, h) = \mathbb{E}\Big|R(\hat{h}, \kappa, h, S)\Big| < 1$$

$\overline{R}(\hat{h}, \kappa, h)$ is called AV-stability function of the numerical method.

Applying one of DRSSE or DISSE to (12) we obtain

$$\begin{aligned} X_{n+1} =& (1 + \hat{h})(1 + \kappa S)X_n \\ =& R_1(\hat{h}, \kappa, h, S)X_n \end{aligned}$$

Then bound of AV-stability function of these methods is given by

$$\begin{aligned}\overline{R}_1 = \mathbb{E}|R_1| =&|1+\hat{h}|\ \mathbb{E}(|1+\kappa S|)\\ \leq&|1+\hat{h}|\ (1+|\kappa|\ \mathbb{E}|S|)\\ =&|1+\hat{h}|\ \big(1+\frac{2}{\pi}\kappa\ \Gamma(\frac{2}{\alpha})\big)\end{aligned}$$

Now applying TSM method to (12) we obtain

$$\begin{aligned}X_{n+1} =&\big(1+\hat{h}\big)\big(1-\frac{1}{2}\kappa^2\big)\big(1+\kappa S+\frac{1}{2}\kappa^2 S\big)X_n\\ =&R_2(\hat{h},\kappa,h,S)X_n\end{aligned}$$

$$\begin{aligned}\overline{R}_2 = \mathbb{E}|R_2| =&|1+\hat{h}||1-\frac{1}{2}\kappa^2||1+\kappa S+\frac{1}{2}\kappa^2 S|\\ \leq&|1+\hat{h}||1-\frac{1}{2}\kappa^2|\big(1+(\kappa+\frac{1}{2}\kappa^2)|S|\big)\frac{2}{\pi}h^{\alpha}\Gamma(\frac{2}{\alpha})\\ =&|1+\hat{h}||1-\frac{1}{2}\kappa^2|\Big(1+(\kappa+\frac{1}{2}\kappa^2)\frac{2}{\pi}\Gamma(\frac{2}{\alpha})\Big)\end{aligned}$$

References

[1] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of Alpha-Stable Stochastic Processes*, CRC Press, 1993.

[2] G. N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Springer Science & Business Media, 1994.

[3] G. Samoradnitsky, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Routledge, 2017.

[4] J. Nolan, *Stable distributions: Models for Heavy-Tailed Data*, Birkhauser New York, 2018.

[5] P. Wang and Y. Li, Split-step forward methods for stochastic differential equations, *Journal of computational and applied mathematics*, (2010), 233(10): 2641–2651.

[6] P. Zanzotto, On solutions of one-dimensional stochastic differential equations driven by stable Lévy motion, *Stochastic processes and their applications*, (1997), 68(2):209–228.

[7] P. A. Zanzotto, On stochastic differential equations driven by a cauchy process and other stable Lévy motions, *Annals of probability*, (2002), 802–825.

[8] S. Sinch, Split-step forward milstein method for stochastic differential equations, *International Journal of Numerical Analysis and Modeling*, (2012), 9(4):970–981.

[9] V. Kurenok, On driftless one-dimensional sde's with respect to stable levy processes. *Lithuanian Mathematical Journal*, (2007), 47(4):423– 435.

[10] X. Mao, *Stochastic differential equations and applications*, Elsevier, (2007).

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