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Original Research Paper

Some New Extension of Levinson's Integral Inequalities

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Abstract. In 1964, Levinson [4] proved integral inequalities concerning generalization of Hardy's inequalities. In this paper two results are given. First one is extension of the Levinson Integral inequalities via convexity and the second is for the Levinson Integral inequalities of Hardy, this inequalities are established for $p < 1$ and some related inequalities are also considered with a sharp constant.

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1 Introduction

Let f is a measurable function defined on the interval $(0, \infty)$. The Hardy operator $\mathcal{H}$, its adjoint $\mathcal{H}^*$ and its dual $\widetilde{\mathcal{H}}$ are defined by

$$\mathcal{H}(x) = \frac{1}{x}\int_0^x f(t)dt, \quad \mathcal{H}^*(x) = \int_x^\infty \frac{f(t)}{t}dt, \quad \widetilde{\mathcal{H}}(x) = \frac{1}{x}\int_x^\infty f(t)dt.$$

In 1964 , N. Levinson [4] proved the following theorems [Theorem 2, Theorem 3, Theorem4, Theorem5].

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Theorem 1.1. *On an open interval, finite or infinite, let $\phi(u) \geq 0$ be defined and have a second derivative $\phi'' \geq 0$. For some $p > 1$ let*

$$\phi\,\phi'' \geq \left(1 - \frac{1}{p}\right)(\phi').$$

At the ends of the interval let take its limiting values, finite or infinite. Then if for $0 < x < \infty$ the range of values of $f(x)$ lie in the closed interval of] definition of ϕ. Let $\lambda(x) > 0$, $x > 0$, continuous and monotone non-decreasing and let $\Lambda(x) = \int_0^x \lambda(t)dt$ exist. Then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \phi(f(x))dx.$$

Theorem 1.2. *Let ϕ and f be as in Theorem 1. Let $\lambda(x) > 0$, $x > 0$, and be absolutely continuous and $\Lambda(x) = \int_0^x \lambda(t)dt$ exist. Let there exist $K > 0$ such that for almost all $x > 0$*

$$\frac{\lambda'\Lambda}{\lambda^2} + p - 1 \geq \frac{p}{K}.$$

Then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)dx \leq K^p \int_0^\infty \phi(f(x))dx.$$

Theorem 1.3. *Let $p > 1$, $f(x) \geq 0$, let $\lambda(x) > 0$, $x > 0$, and let λ be absolutely continuous. Let*

$$\frac{p-1}{p} + \frac{x\lambda'}{\lambda} \geq \frac{1}{K}$$

for almost all x for some $K > 0$. If

$$H(x) = \frac{1}{x\lambda(x)}\int_0^x \lambda(t)f(t)dt,$$

then

$$\int_0^\infty H^p(x)dx \leq K^p \int_0^\infty f^p(x)dx. \tag{1}$$

Theorem 1.4. *Let $p > 1$, $f(x) \geq 0$, let $\lambda(x) > 0$, and let λ be absolutely continuous. Let*

$$\frac{x\lambda'}{\lambda} - \frac{p-1}{p} \geq \frac{1}{K}$$

for some $K > 0$. If

$$J(x) = \frac{\lambda(x)}{x} \int_x^\infty \frac{f(t)}{\lambda(t)} dt,$$

then

$$\int_0^\infty J^p(x)dx \leq K^p \int_0^\infty f^p(x)dx. \tag{2}$$

A large number of authors have studies the Levinson type integral inequalities and a many papers have been appeared in the literature which deals with the simple proofs see [2]-[8]. The objective of this paper is to obtain new types of the Levinson integral inequality of Hardy which will be useful in applications by using some elementary methods of analysis with the parameter $p < 1$.
We give the following corollary of the reverse Hölder's inequality [1] which will be used frequently in the proof.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $p < 0$, we suppose that f, g are measurable on Ω. If $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$ (p' is the conjugate parameter), then*

$$\int_\Omega |f.g|dt \geq \|f\|_{L_p} \|g\|_{L_{p'}}. \tag{3}$$

And we adopt the usual convention $\frac{\infty}{\infty} = \frac{0}{0} = 0$ and we use the notation $\widetilde{H}$ and $\widetilde{J}$ for the dual of H and J defined above.

2 Main Results

Throughout the paper, we assume that the integrals exist and are finite (i.e, convergent) and the functions are non-negative integrable on $(0, +\infty)$.

- **Levinson's inequality via concavity**

Theorem 2.1. *Suppose $f(x) \geq 0$, $\phi(x) > 0$ is a concave function and $\Lambda \in \mathcal{C}^2(0.\infty)$ such that $\Lambda' = \lambda$ and $\lambda(\infty) = \infty$. Let $0 < p < 1$, $K > 0$, if*

$$0 < \frac{\lambda' \Lambda}{\lambda^2} + p - 1 \leq \frac{p}{K}, \tag{4}$$

then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)} \int_0^x \lambda(t) f(t) dt\right) dx \geq K^p \int_0^\infty \phi(f(x)) dx. \tag{5}$$

Proof. Let

$$G(x) = \frac{1}{\Lambda(x)} \int_0^x \lambda(t) f(t) dt,$$

and

$$g(x) = \lambda^{-\frac{1}{p}}(x) \int_0^x \lambda(t) f(t) dt,$$

we have

$$g'(x) = -\frac{1}{p} \frac{\lambda'(x)}{\lambda(x)} g(x) + \lambda^{1-\frac{1}{p}}(x) f(x),$$

integrating by part, we get

$$\begin{aligned}
\int_0^\infty G^p(x) dx &= \int_0^\infty \lambda(x) \Lambda^{-p}(x) g^p(x) dx \\
&= \left[\frac{\Lambda^{1-p}(x) g^p(x)}{1-p}\right]_0^\infty - \frac{p}{1-p} \int_0^\infty \Lambda^{1-p}(x) g'(x) g^{p-1}(x) dx \\
&= \frac{p}{1-p} \int_0^\infty -\Lambda^{1-p}(x) \lambda^{1-\frac{1}{p}}(x) f(x) g^{p-1}(x) dx \\
&+ \frac{1}{1-p} \int_0^\infty \frac{\Lambda^{1-p}(x) \lambda'(x)}{\lambda(x)} g^p(x) dx \\
&= \frac{1}{1-p} \int_0^\infty \frac{\lambda'(x)}{\lambda^2(x)} \Lambda(x) G^p(x) dx - \frac{p}{1-p} \int_0^\infty G^{p-1}(x) f(x) dx,
\end{aligned}$$

hence

$$\int_0^\infty \left(\frac{1}{1-p}\frac{\lambda'(x)\Lambda(x)}{\lambda^2(x)} - 1\right) G^p(x)dx = \frac{p}{1-p}\int_0^\infty G^{p-1}(x)f(x)dx.$$

Using the assumption (4), we have

$$\int_0^\infty G^p(x)dx \geq K \int_0^\infty G^{p-1}(x)f(x)dx.$$

Applied the reverse Hölder integral inequality on the right hand side of the last inequality for $\frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\int_0^\infty G^p(x)dx \geq K \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty G^p(x)\right)^{\frac{1}{p'}},$$

thus

$$\int_0^\infty G^p(x)dx \geq K^p \int_0^\infty f^p(x)dx,$$

which is same as

$$\int_0^\infty \left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)^p dx \geq K^p \int_0^\infty f^p(x)dx. \tag{6}$$

let $\varphi = \phi^p$,as a result $\varphi'' = p\phi^{p-2}\left(\phi\phi'' - (1-p)(\phi')^2\right)$. Since ϕ is concave function, then $\varphi'' < 0$ where $\varphi(x) > 0$, consequently φ is concave. Thus by Jensens inequality

$$\phi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right) \geq \frac{1}{\Lambda(x)}\int_0^x \lambda(t)\phi(f(t))dt,$$

then

$$\varphi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)^{\frac{1}{p}} \geq \frac{1}{\Lambda(x)}\int_0^x \lambda(t)\varphi^{\frac{1}{p}}(f(t))dt,$$

therefore

$$\int_0^\infty \varphi\left(\frac{1}{R(x)}\int_0^x \lambda(t)f(t)dt\right) dx \geq \int_0^\infty \left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)\varphi^{\frac{1}{p}}(f(t))dt\right)^p dx. \tag{7}$$

On the another hand applying (6) to $\varphi^{\frac{1}{p}}(f)$ instead of f, we get

$$\int_0^\infty \left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)\varphi^{\frac{1}{p}}(f(t))dt\right)^p dx \geq K^p \int_0^\infty \varphi(f(x))dx, \tag{8}$$

using (7) and (8) to complete the proof. □

Theorem 2.2. *Suppose $f(x) \geq 0$, $\phi(x) > 0$ is a concave function and $\Lambda \in \mathcal{C}^2(0.\infty)$ such that $\Lambda' = \lambda$. Let $0 < p < 1$, $K > 0$, if*

$$0 < 1 - p - \frac{\lambda'\Lambda}{\lambda^2} \leq \frac{p}{K}, \tag{9}$$

then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)}\int_x^\infty \lambda(t)f(t)dt\right)dx \geq K^p \int_0^\infty \phi(f(x))dx. \tag{10}$$

Proof. The proof is similar to the proof of the theorem 2.1 □

Theorem 2.3. *Suppose $f(x) > 0$, $\phi(x) > 0$ is a convex function and $\Lambda \in \mathcal{C}^2(0.\infty)$ such that $\Lambda' = \lambda$ and $\lambda(\infty) = \infty$. Let $p < 0$, $K > 0$, if*

$$0 < 1 - p - \frac{\lambda'\Lambda}{\lambda^2} \leq \frac{-p}{K}, \tag{11}$$

and

$$\phi\phi'' \geq (1 - \frac{1}{p})(\phi')^2, \tag{12}$$

then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)dx \leq K^p \int_0^\infty \phi(f(x))dx. \tag{13}$$

Proof. Let

$$G(x) = \frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt,$$

and

$$g(x) = \lambda^{-\frac{1}{p}}(x)\int_0^x \lambda(t)f(t)dt,$$

using integration by part for the integral $\int_0^\infty \lambda(x)\Lambda^{-p}(x)g^p(x)dx$, we deduce that

$$\int_0^\infty \left(1 - \frac{1}{1-p}\frac{\lambda'(x)\Lambda(x)}{\lambda^2(x)}\right) G^p(x)dx = \frac{-p}{1-p}\int_0^\infty G^{p-1}(x)f(x)dx.$$

Using the assumption (11), we deduce that

$$\int_0^\infty G^p(x)dx \geq K\int_0^\infty G^{p-1}(x)f(x)dx.$$

Applied the reverse Hölder integral inequality on the right hand side of the last inequality, we get

$$\left(\int_0^\infty G^p(x)\right)^{\frac{1}{p}} \geq K\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}},$$

since $p < 0$, thus

$$\int_0^\infty G^p(x)dx \leq K^p\int_0^\infty f^p(x)dx,$$

this gives us that

$$\int_0^\infty \left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)^p dx \leq K^p\int_0^\infty f^p(x)dx. \tag{14}$$

let $\varphi^p = \phi$,as a result $\varphi'' = p\phi^{\frac{1}{p}-2}\left(\phi\phi'' - (1-\frac{1}{p})(\phi')^2\right)$. Since the assumption (12) thus $\varphi'' \leq 0$ where $\varphi(x) > 0$, hence φ is concave. Then by Jensens inequality

$$\varphi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right) \geq \frac{1}{\Lambda(x)}\int_0^x \lambda(t)\varphi(f(t))dt,$$

then

$$\phi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)^{\frac{1}{p}} \geq \frac{1}{\Lambda(x)}\int_0^x \lambda(t)\phi^{\frac{1}{p}}(f(t))dt,$$

since $p < 0$, we get

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)f(t)dt\right)dx \leq \int_0^\infty \left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)\phi^{\frac{1}{p}}(f(t))dt\right)^p dx, \tag{15}$$

on the another hand applying (14) to $\phi^{\frac{1}{p}}(f)$ instead of f, we get

$$\int_0^\infty \left(\frac{1}{\Lambda(x)}\int_0^x \lambda(t)\phi^{\frac{1}{p}}(f(t))dt\right)^p dx \leq K^p \int_0^\infty \phi(f(x))dx. \tag{16}$$

Using (15) and (16) to complete the proof. □

Theorem 2.4. *Suppose $f(x) > 0$, $\phi(x) > 0$ is a convex function and $\Lambda \in \mathcal{C}^2(0.\infty)$ such that $\Lambda' = \lambda$ and $\lambda(\infty) = \infty$. Let $p < 0$, $K > 0$, if*

$$0 < \frac{\lambda'\Lambda}{\lambda^2} + p - 1 \leq \frac{-p}{K}, \tag{17}$$

and

$$\phi\phi'' \geq (1 - \frac{1}{p})(\phi')^2, \tag{18}$$

then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(x)}\int_x^\infty \lambda(t)f(t)dt\right) dx \leq K^p \int_0^\infty \phi(f(x))dx. \tag{19}$$

Proof. The proof is similar to the proof of the theorem 2.3
□

- **Levinson's inequality for Hardy operators**

Theorem 2.5. *Suppose $f(x) \geq 0$, $\lambda(x) > 0$ is absolutely continuous function. Let $0 < p < 1$, $K > 0$ and*

$$J(x) = \frac{\lambda(x)}{x}\int_x^\infty \frac{f(t)}{\lambda(t)}dt.$$

If

$$0 < \frac{1-p}{p} + \frac{x\lambda'}{\lambda} \leq \frac{1}{K}, \tag{20}$$

then

$$\int_0^\infty J^p(x)dx \geq K^p \int_0^\infty f^p(x)dx. \tag{21}$$

Proof. Let

$$G(x) = \lambda(x)\int_x^\infty \frac{f(t)}{\lambda(t)}dt,$$

we have

$$G'(x) = \frac{\lambda'(x)}{\lambda(x)} G(x) - f(x),$$

integrating by part in the left hand side of (21), we get

$$\begin{aligned}
\int_0^\infty J^p(x)dx &= \int_0^\infty x^{-p} G^p(x)dx \\
&= \left[\frac{x^{1-p} G^p(x)}{1-p}\right]_0^\infty - \frac{p}{1-p}\int_0^\infty x^{1-p} G'(x) G^{p-1}(x)dx \\
&= \frac{p}{1-p}\int_0^\infty \left(x^{1-p} f(x) G^{p-1}(x) - \frac{x^{1-p}\lambda'(x)}{\lambda(x)} G^p(x)\right) dx \\
&= \frac{p}{1-p}\left(\int_0^\infty J^{p-1}(x) f(x)dx - \int_0^\infty \frac{x\,\lambda'(x)}{\lambda(x)} J^p(x)dx\right),
\end{aligned}$$

therefore

$$\int_0^\infty \left(\frac{p}{1-p}\frac{x\,\lambda'(x)}{\lambda(x)} + 1\right) J^p(x)dx = \frac{p}{1-p}\int_0^\infty J^{p-1}(x) f(x)dx.$$

Using the assumption (20), we get

$$\int_0^\infty \frac{1}{K} J^p(x)dx \geq \int_0^\infty \left(\frac{x\,\lambda'(x)}{\lambda(x)} + \frac{1-p}{p}\right) J^p(x)dx,$$

thus

$$\int_0^\infty J^p(x)dx \geq K\int_0^\infty J^{p-1}(x) f(x)dx.$$

Applied the reverse Hölder integral inequality on the right hand side of the last inequality for $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\int_0^\infty J^p(x)dx \geq K\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty J^p(x)\right)^{\frac{1}{q}},$$

hence

$$\int_0^\infty J^p(x)dx \geq K^p\int_0^\infty f^p(x)dx.$$

□

- If we take $\lambda(x) = x$ and $K = p$, we get

Corollary 2.6. *Let $0 < p < 1$ and f be a non-negative integrable function on $(0, \infty)$, then*

$$\int_0^{\infty} [\mathcal{H}^*(x)]^p dx \geq p^p \int_0^{\infty} f^p(x) dx. \tag{22}$$

the constant factor p^p is the best possible.

Proof. Let $0 < \theta < 1$, we take $f_\theta(x) = \begin{cases} x^{\frac{\theta-1}{p}}, & x \in (0,1] \\ 0, & x \in (1, \infty) \end{cases}$,
using the assumption on f_θ in L_1 the left hand side of (22), we have

$$\begin{aligned} L_1 &= \int_0^{\infty} \left(\int_x^{\infty} \frac{f_\theta(t)}{t} dt \right)^p dx \\ &= \int_0^1 \left(\int_x^{\infty} t^{\frac{\theta-1}{p}-1} dt \right)^p dx \\ &= \left(\frac{p}{1-\theta} \right)^p \int_0^1 x^{\theta-1} dx \\ &= \left(\frac{p}{\theta-1} \right)^p \frac{1}{\theta}. \end{aligned}$$

Let R_1 the right hand side of (29), we get

$$\begin{aligned} R_1 &= p^p \int_0^{\infty} f_\theta^p(x) dx \\ &= p^p \int_0^1 x^{\theta-1} dx \ = p^p \frac{1}{\theta}. \end{aligned}$$

For $\theta \longrightarrow 0$, we get the constant factor p^p is the best possible in (29).
□

- If we take $\lambda(x) = 1$ and $K = \dfrac{p}{1-p}$, hence we get

Corollary 2.7. *Let $0 < p < 1$ and f be a non-negative integrable function on $(0, \infty)$, then*

$$\int_0^{\infty} [\widetilde{\mathcal{H}}(x)]^p dx \geq \left(\frac{p}{1-p} \right)^p \int_0^{\infty} f^p(x) dx. \tag{23}$$

the constant factor $\left(\dfrac{p}{1-p} \right)^p$ is the best possible.

Proof. Now let $0 < \theta' < 1-p$, we put $f_{\theta'}(x) = \begin{cases} x^{\frac{\theta'-1}{p}}, & x \in (0,1] \\ 0, & x \in (1,\infty) \end{cases}$,

$$\begin{aligned}\int_0^\infty [\widetilde{\mathcal{H}}(x)]^p dx &= \int_0^\infty \left(\frac{1}{x}\int_x^\infty f_{\theta'}(t)dt\right)^p dx \\ &= \int_0^1 \left(\frac{1}{x}\int_x^\infty t^{\frac{\theta'-1+p}{p}-1}dt\right)^p dx \\ &= \left(\frac{p}{1-p-\theta'}\right)^p \int_0^1 x^{\theta'-1}dx \\ &= \left(\frac{p}{1-p-\theta'}\right)^p \int_0^\infty f^p(x)dx,\end{aligned}$$

For $\theta' \longrightarrow 0$, we get the constant factor $\left(\frac{p}{1-p}\right)^p$ is the best possible in (23). □

Theorem 2.8. *Suppose $f(x) \geq 0$, $\lambda(x) > 0$ is absolutely continuous funcion. Let $0 < p < 1$, $K > 0$ and*

$$\widetilde{H}(x) = \frac{1}{x\,\lambda(x)}\int_x^\infty \lambda(t)f(t)dt.$$

If

$$0 < \frac{1-p}{p} - \frac{x\lambda'}{\lambda} \leq \frac{1}{K}, \tag{24}$$

then

$$\int_0^\infty \widetilde{H}^p(x)dx \geq K^p \int_0^\infty f^p(x)dx. \tag{25}$$

Proof. Let

$$S(x) = \frac{1}{\lambda(x)}\int_x^\infty \lambda(t)f(t)dt,$$

we have

$$S'(x) = -\frac{\lambda'(x)}{\lambda(x)}S(x) - f(x),$$

integrating by part in the left hand side of (25), we get

$$\begin{aligned}\int_0^\infty \widetilde{H}^p(x)dx &= \int_0^\infty x^{-p}S^p(x)dx \\ &= \left[\frac{x^{1-p}S^p(x)}{1-p}\right]_0^\infty - \frac{p}{1-p}\int_0^\infty x^{1-p}S'(x)S^{p-1}(x)dx \\ &= \frac{p}{1-p}\left(\int_0^\infty \widetilde{H}^{p-1}(x)f(x)dx + \int_0^\infty \frac{x\,\lambda'(x)}{\lambda(x)}\widetilde{H}^p(x)dx\right).\end{aligned}$$

Using the assumption (24), we get

$$\int_0^\infty \frac{1}{K}\widetilde{H}^p(x)dx \geq \int_0^\infty \left(\frac{1-p}{p} - \frac{x\,\lambda'(x)}{\lambda(x)}\right)\widetilde{H}^p(x)dx,$$

therefore

$$\int_0^\infty \widetilde{H}^p(x)dx \geq K\int_0^\infty \widetilde{H}^{p-1}(x)f(x)dx,$$

and applied the reverse Hölder integral inequality, we get

$$\int_0^\infty \widetilde{H}^p(x)dx \geq K^p\int_0^\infty f^p(x)dx.$$

□

Remark 2.9. If we take $\lambda(x) = \frac{1}{x}$ and $K = p$, we get the inequality (22).

Theorem 2.10. *Suppose $f, \lambda > 0$, λ is absolutely continuous and $\lambda(\infty) = 0^+$. Let $p < 0$, $K > 0$ and*

$$H(x) = \frac{1}{x\,\lambda(x)}\int_0^x \lambda(t)f(t)dt.$$

If

$$0 < \frac{p-1}{p} + \frac{x\,\lambda'}{\lambda} \leq \frac{1}{K}, \tag{26}$$

then

$$\int_0^\infty H^p(x)dx \leq K^p\int_0^\infty f^p(x)dx. \tag{27}$$

Proof. Let
$$S_1(x) = \frac{1}{\lambda(x)} \int_0^x \lambda(t) f(t) dt,$$
we have
$$S_1'(x) = -\frac{\lambda'(x)}{\lambda(x)} S_1(x) + f(x).$$
Integrating by part in the left hand side of (27), we get
$$\begin{aligned}\int_0^\infty H^p(x)dx &= \int_0^\infty x^{-p} S_1^p(x)dx \\ &= \left[\frac{x^{1-p} S_1^p(x)}{1-p}\right]_0^\infty + \frac{p}{p-1} \int_0^\infty x^{1-p} S_1'(x) S_1^{p-1}(x)dx,\end{aligned}$$
since $p < 0$ and $\lambda(\infty) = 0^+$, then
$$\int_0^\infty H^p(x)dx = \frac{p}{p-1} \left(\int_0^\infty H^{p-1}(x) f(x)dx - \int_0^\infty \frac{x\,\lambda'(x)}{\lambda(x)} H^p(x)dx \right).$$
Using the assumption (26), we deduce that
$$\int_0^\infty \frac{1}{K} H^p(x)dx \geq \int_0^\infty \left(\frac{p-1}{p} + \frac{x\,\lambda'(x)}{\lambda(x)} \right) H^p(x)dx,$$
as a result
$$\int_0^\infty H^p(x)dx \geq K \int_0^\infty H^{p-1}(x) f(x)dx. \tag{28}$$
Using the reverse Hölder integral inequality on the right hand side of (28), we have
$$\left(\int_0^\infty H^p(x)dx \right)^{\frac{1}{P}} \geq K \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{P}},$$
since $p < 0$, then
$$\int_0^\infty H^p(x)dx \leq K^p \int_0^\infty f^p(x)dx.$$
□

If we take $\lambda(x) = \frac{1}{x}$ and $K = -p$, we get the following corollary

Corollary 2.11. *Let f be non-negative integrable function on $(0,\infty)$ and $p<0$, then*

$$\int_0^\infty \left(\int_0^x \frac{f(t)}{t}dt\right)^P dx \le (-p)^p \int_0^\infty f^p(x)dx, \tag{29}$$

the constant factor $(-p)^p$ is the best possible.

Proof. For $0<\theta<1$, we take $f_\theta(x) = \begin{cases} x^{\frac{\theta-1}{p}}, & x\in(0,1] \\ 0, & x\in(1,\infty) \end{cases}$,

we have

$$\int_0^\infty \left(\int_0^x \frac{f(t)}{t}dt\right)^P dx = \int_0^\infty \left(\int_0^x \frac{f_\theta(t)}{t}dt\right)^p dx$$

$$= \left(\frac{p}{\theta-1}\right)^p \int_0^1 x^{\theta-1}dx$$

$$= \left(\frac{p}{\theta-1}\right)^p \int_0^\infty f^p(x)dx.$$

For $\theta \longrightarrow 0$, we get the constant factor $(-p)^p$ is the best possible in (29).
□

Theorem 2.12. *Suppose $f,\lambda>0$, λ is absolutely continuous and $\lambda(\infty)=\infty$. Let $p<0$, $K>0$ and*

$$\widetilde{J}(x) = \frac{\lambda(x)}{x}\int_0^x \frac{f(t)}{\lambda(t)}dt.$$

If

$$0 < \frac{p-1}{p} - \frac{x\,\lambda'}{\lambda} \le \frac{1}{K}, \tag{30}$$

then

$$\int_0^\infty \widetilde{J}^p(x)dx \le K^p \int_0^\infty f^p(x)dx. \tag{31}$$

Proof. Let

$$G_1(x) = \lambda(x) \int_0^x \frac{f(t)}{\lambda(t)} dt,$$

we have

$$G_1'(x) = \frac{\lambda'(x)}{\lambda(x)} G(x) + f(x),$$

then

$$\begin{aligned} \int_0^\infty \widetilde{J}^p(x)dx &= \int_0^\infty x^{-p} G_1^p(x)dx \\ &= \left[\frac{x^{1-p}G_1^p(x)}{1-p}\right]_0^\infty + \frac{p}{p-1} \int_0^\infty x^{1-p} G_1'(x) G_1^{p-1}(x)dx \\ &= \frac{p}{1-p} \int_0^\infty \left(x^{1-p} f(x) G_1^{p-1}(x) + \frac{x^{1-p}\lambda'(x)}{\lambda(x)} G_1^p(x) \right) dx \\ &= \frac{p}{1-p} \left(\int_0^\infty \widetilde{J}^{p-1}(x) f(x)dx + \int_0^\infty \frac{x\,\lambda'(x)}{\lambda(x)} \widetilde{J}^p(x)dx \right). \end{aligned}$$

From the assumption (30), we get

$$\int_0^\infty \frac{1}{K} \widetilde{J}^p(x)dx \geq \int_0^\infty \left(\frac{x\,\lambda'(x)}{\lambda(x)} + \frac{1-p}{p} \right) \widetilde{J}^p(x)dx,$$

thus

$$\int_0^\infty \widetilde{J}^p(x)dx \geq K \int_0^\infty \widetilde{J}^{p-1}(x) f(x)dx. \tag{32}$$

Using the reverse Hölder integral inequality on the right hand side of (32), we get

$$\left(\int_0^\infty \widetilde{J}^p(x)dx \right)^{\frac{1}{P}} \geq K \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{P}},$$

since $p < 0$, we get

$$\int_0^\infty \widetilde{J}^p(x)dx \leq K^p \int_0^\infty f^p(x)dx.$$

□

Remark 2.13. If we take $\lambda(x) = x$ and $K = -p$, we get the inequality (29).

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