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# Some results of fixed points on $C^*$-algebra valued metric spaces

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**Abstract.** In this paper we present some fixed point theorems for mappings satisfying contractive conditions on $C^*$-algebra-valued b-metric spaces and some nonunique fixed point theorems in $C^*$-algebra-valued b-metric spaces. Specifically we extend some fixed point results on basic metric space to $C^*$-algebra valued case and prove the related fixed point theorems. Also several theorems in existence of n-periodic points are given.

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## 1 Introduction

Forty years ago, Bogdan Rzepecki [1],presented a generalized metric $d_E$ on a set $X$ in a way that $d_E : X \times X \to S$ where $E$ is a Banach space

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and $S$ is a normal cone with partial order $\preceq$ . In that article, the writer generalized the fixed point theorems of Maia type [2]. Seven years later, Shy-Der Lin [3] presented the notion -metric spaces by substituting real numbers with cone in the metric function, that is, $d_E : X \times\ X \to K$. In that writing, some conclusions of Khan and Imdad [4] on fixed point theorems were considered for $K$-metric spaces. In 2007, the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space was declared by Huang and Zhang [5]. In that article, they also argued some properties of convergence of sequences and verified the fixed point theorems of contractive mapping for cone metric spaces. In 2013 Liu and Xu [6] came to some new conclusions by substituting the usual real contraction constant with a vector constant and scalar multiplication with vector multiplication and confirmed that their results were different from those in cone metric spaces. Lately, many results on fixed point theorems have been developed in cone metric spaces (see e.g. [7-10]). To make it clear, now we remind some fundamental definitions, notations, and conclusions of $C^*$-algebras. The descriptions of $C^*$-algebras can be attained in [11-15].

An *algebra* is a vector space $\mathbb{E}$ together with a bilinear map

$$\mathbb{E}^2 \to \mathbb{E}, \quad (e,g) \to eg,$$

such that, $\quad e(gh) = (eg)h \quad (e,g,h \in \mathbb{E})$.

A *subalgebra* of $\mathbb{E}$ is a vector subspace $\mathbb{G}$ such that $g, g' \in \mathbb{G}$ then $gg' \in \mathbb{G}$. Endowed with the multiplication got by restriction, $\mathbb{G}$ is itself an algebra. A norm $\|.\|$ on $\mathbb{E}$ is said to be *submultiplicative* if

$$\|eg\| \leq \|e\|\|g\| \quad (e,g \in \mathbb{E}).$$

In this case the pair $(\mathbb{E}, \|.\|)$ is called a *normed algebra*. A complete normed algebra is called a *Banach algebra*. An *involution* on an algebra $\mathbb{E}$ is a conjugate-linear map $e \to e^*$ on $\mathbb{E}$, such that $e^{**} = e$ and $(eg)^* = g^*e^*$ for all $e, g \in \mathbb{E}$. The pair $(\mathbb{E}, *)$ is called an *involutive* algebra, or a $* - algebra$. A *Banach* $* - algebra$ is a $* - algebra$ $\mathbb{E}$ together with a complete submultiplicative norm such that $\|e^*\| = \|e\|$ $(e \in \mathbb{E})$. If, in addition, $\mathbb{E}$ has a unit such that $\|1\| = 1$, we call $\mathbb{E}$ a *unital Banach* $* - algebra$. A $C^* - algebra$ is unital Banach $*$-algebra such that, $\quad \|e^*e\| = \|e\|^2 \quad (e \in \mathbb{E})$.

It is clear that under the norm topology, $L(H)$, the set of all bounded linear operators on a Hilbert space $H$, is a $C^*$-algebra. Furthermore, given a $C^*$-algebra $\mathbb{E}$, there exists a Hilbert space $H$ and a faithfully $*$-representation $(\pi, H)$ of $\mathbb{E}$ such that $\pi\mathbb{E}$ can be made a closed $C^*$-subalgebra of $L(H)$ [13].

An element a of a $C^*$-algebra $\mathbb{E}$ is positive if $e = e^*$ and its spectrum $\sigma(e) \subset I\!R^+$. We write $e \succeq 0$ to mean that $e$ is positive, and denote by $\mathbb{E}^+$ the set of positive elements of $\mathbb{E}$.

**Theorem 1.1.** [13] *Let $\mathbb{E}$ be a $C^*$-algebra and $e \in \mathbb{E}^+$. Then there exists a unique element $g \in \mathbb{E}^+$ such that $g^2 = e$.*

If $\mathbb{E}$ is a $C^*$-algebra, we make $\mathbb{E}_h = \{t \in \mathbb{E} : t = t^*\}$ a poset by defining $e \preceq g$ to mean $g - e \in \mathbb{E}^+$. The relation $\preceq$ is translation-invariant; that is, $e \preceq g \Rightarrow e + k \preceq g + k$ for all $e, g, k \in \mathbb{E}_h$. Also, $e \preceq g \Rightarrow re \preceq rg$ for all $r \in I\!R^+$, and $e \preceq g \Leftrightarrow -e \succeq -g$. Let $\mathbb{E}' = \{e \in \mathbb{E} : eg = ge, \quad \forall g \in \mathbb{E}\}$ , and $\mathbb{E}'_+ = \mathbb{E}^+ \cup \mathbb{E}'$.

**Theorem 1.2.** [10] *Let $\mathbb{E}$ be a $C^*$-algebra.*

1. *The set $\mathbb{E}^+$ is equal to $\{e^*e \mid e \in \mathbb{E}\}$.*
2. *If $e, g \in \mathbb{E}_h$ and $k \in \mathbb{E}$, then $e \preceq g \Rightarrow k^*ek \preceq k^*gk$.*
3. *If $0_{\mathbb{E}} \preceq e \preceq g$, then $\|e\| \leq \|g\|$.*
4. *If $\mathbb{E}$, is unital and $e, g$ are positive invertible elements, then $e \preceq g \Rightarrow 0_{\mathbb{E}} \preceq g^{-1} \preceq e^{-1}$.*

Notice that in a $C^*$-algebra, one cannot conclude that $eg \succeq 0_{\mathbb{E}}$ wherever $e, g \succeq 0_{\mathbb{E}}$.

**Definition 1.3.** Let $\mathbb{E}$ be a $C^*$-algebra, and $X$ be a nonempty set. Let $\mathbb{E}'_+$ be such that $\|b\| \geq 1$. A mapping $D =: X \times X \to \mathbb{E}^+$ is said to be a $C^*$-algebra-valued $b$-metric on $X$ if the following conditions hold for all $e, g, k \in \mathbb{E}$ :

1. $D(e, g) = o_{\mathbb{E}}$ if and only if $e = g$.
2. $D(e, g) = D(g, e)$.
3. $D(e, g) \preceq b[D(e, g) + D(g, k)]$.

The triplet $(X, \mathbb{E}, D)$ is called a $C^* - algebra - valuedb - metricspace$

with coefficient b. A $C^*$-algebra-valued metric space is $C^*$-algebra-valued $b$-metric space, but the converse is not true. If $b = I$, then the ordinary triangle inequality condition in a $C^*$-algebra-valued metric space is satisfied. Thus a $C^*$-algebra-valued $b$-metric space is an ordinary $C^*$-algebra-valued metric space. In particular, if $\mathbb{E} = \mathbb{C}$ and $b = 1$, the $C^*$-algebra-valued $b$-metric spaces are just the ordinary metric spaces. The following example illustrates that, in general, a $C^*$-algebra-valued metric space is not necessary a $C^*$-algebra-valued b-metric space. For some details of $C^*$-algebra-valued metric spaces, one can see [17].

**Example 1.4.** $P = l_p$ is set of sequences $\{e_n\}$ in $I\!R$ when $\sum_{n=1}^{\infty} |e_n|^p < \infty$ such that $0 < p < 1$. If $\mathbb{E} = M_2(I\!R)$ and $e = e_n, g = g_n \in l_p$, define $D_b : X \times X \to \mathbb{E}$ as follow:

$$D_b(e,g) = \begin{pmatrix} (\sum_{n=1}^{\infty} |e_n - g_n|^p)^{\frac{1}{p}} & 0 \\ 0 & (\sum_{n=1}^{\infty} |e_n - g_n|^p)^{\frac{1}{p}}. \end{pmatrix}$$

We have $D_n$ is a $C^*$-algebra-valued b-metric space with coefficient $b = \begin{pmatrix} 2^{\frac{1}{p}} & 0 \\ 0 & 2^{\frac{1}{p}} \end{pmatrix}$ such that $\|b\| = 2^{\frac{1}{2}}$. When that:

$$(\sum_{n=1}^{\infty} |e_n - k_n|^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}[(\sum_{n=1}^{\infty} |e_n - g_n|^p)^{\frac{1}{p}} + (\sum_{n=1}^{\infty} |g_n - k_n|^p)^{\frac{1}{p}}].$$

Thus $D_n$ is not a usual $C^*$-algebra-valued metric on $P$.

**Example 1.5.** Let $P = I\!R$ and $\mathbb{E} = M_n(I\!R)$. Define

$D(e,g) = \text{diag } (c_1|e-g|^t, c_2|e-g|^t, \ldots, c_n|e-g|^t)$

where diag denotes a diagonal matrix, and $e, g \in I\!R, c_i > 0$ $(i = 1, 2, , n)$ are constants and $t > 1$. It is clear that $D(.,.)$ is a complete $C^*$-algebra-valued $b$-metric. To see that condition (3) in Definition 1.3 is consistent, we have:

$$|e-g|^t \leq 2^t(|e-k|^t + |k-g|^t),$$

thus $D(e,g) \preceq b[D(e,k)+D(k,g)]$ for all $e, g, k \in P$, where $b = 2^t I \in E'$ and clearly $b \succ I$ since $2^t > 1$. But not that $|e-g|^t \leq |e-k|^t + |k-g|^t$ is impossible for all $e > k > g$ [14]. Then $(P, M_n(I\!R), D)$ is not a $C^*$-algebra-valued metric space.

**Definition 1.6.** Let $(P, \mathbb{E}, D)$ be a $C^*$-algebra-valued $b$-metric space, $t \in P$, and $\{t_n\}$ a sequence in $P$. Then:

1. $\{t_n\}$ converges to $t$ with respect to $\mathbb{E}$ whenever for any $\varepsilon > 0$, there is an $n \in N$ such that $\|(D(t_n, t)\| < \varepsilon$ for all $n > N$. We denote this by $Lim_{n\to\infty} t_n = t$ or $t_n \to t$.

2. $\{t_n\}$ is a Cauchy sequence with respect to $\mathbb{E}$ if for each $\varepsilon > 0$, there is an $N \in I\!N$ such that $\|(D(t_n, t_m)\| < \varepsilon$ for all $n, m > N$.

3. $(P, \mathbb{E}, D)$ is complete if every Cauchy sequence in $P$ is convergent with respect to $\mathbb{E}$.

**Definition 1.7.** Suppose that $(P, \mathbb{E}, D)$ is a $C^*$-algebra-valued $b$-metric space. We call a mapping $F : P \to P$ is a $C^*$-algebra-$b$-valued contractive mapping on $P$, if there exists an $\lambda \in \mathbb{E}$ with $\|\lambda\| < 1$ such that $D(Ft, Fs) \preceq \lambda^* D(t, s)\lambda$ for all $t, s \in P$.

**Theorem 1.8.** $[17, Theorem 2.1]$ *If $(P, \mathbb{E}, D)$ is a complete $C^*$-algebra-valued b-metric space and $F$ is a contractive mapping, then $F$ has a unique fixed point in $P$ for $F$.*

**Theorem 1.9.** $[17, Theorem 2.2]$ *Let $(P, \mathbb{E}, D)$ be a complete $C^*$-valued b-metric space. Suppose the mapping $F : P \to P$ satisfies for all $t, s \in P$*

$$d(Ft, Fs) \preceq \lambda^*[D(Ft, s) + D(Fs, t)]\lambda,$$

*where $\lambda \in \mathbb{E}_+^{'}$ and $\|\lambda\| < \frac{1}{2}$. Then there exists a unique fixed point in $P$.*

## 2 Unique fixed point theorems

In the last two decades, many researchers investigated fixed point theorems in $C^*$-algebra valued metric spaces. In this section, we will prove some fixed point theorems for mapping with different contractive conditions in the setting of this spaces.

**Definition 2.1.** Mapping $F$ over a $C^*$-algebra-valued metric space $(P, \mathbb{E}, D)$ is said to be orbitally continuous if $Lim_{i\to\infty} F^{n_i}(t) = z$ implies that

$Lim_{i\to\infty}F(F^{n_i})(t) = Fz$ for any $t$ in $P$, where $\{n_i\}_{i\geq 1} \subset I\!N$. The $C^*$-algebra-valued metric space$(P, \mathbb{E}, D)$ is named $F$ orbitally complete when that all of cauchy sequence of the form $\{F^{n_i}(t)\}$, $t \in P$, converges on $(P, \mathbb{E}, D)$.

It can be clearly if $F$ be orbital continuity then for any $m$ in $I\!N, T^m$ is orbital continuity.

**Theorem 2.2.** *If $F : P \to P$ be an orbitally continuous mapping over $C^*$-algebra-valued metric space $(P, \mathbb{E}, D)$. And $(P, \mathbb{E}, D)$ is $F$ orbitally complete such that*

$$U(t, s) - D(Ft, s) \preceq \lambda^* D(t, s)\lambda \tag{1}$$

*where $\lambda \in \mathbb{E}^+$ with $\|\lambda\| < 1$ and every $t, s \in P$, when*

$$U(t, s) \in \{D(t, Fs), D(Ft, Fs), D(Fs, s)\}.$$

*Then,*

$$\forall t \in P, \{F^n(t)\} \to z$$

*when $z$ is a fixed point of $F$.*

**Proof.** Fix $t_0 \in P$. For $n \geq 1$ set $t_1 = Ft_0$ and recursively $t_{n+1} = F(t_n) = F^{n+1}(t_0)$. It is clear that the sequence $\{t_n\}_n$ is Cauchy when the equation $t_{n+1} = t_n$ holds for some $n \in I\!N$. Consider the case $t_{n+1} \neq t_n$ for all $n \in I\!N$. By replacing $t$ and $s$ with $t_n$ and $t_{n-1}$, respectively, in (2.1), one can get

$$U(t_{n-1}, t_n) - D(Ft_{n-1}, t_n) \preceq \lambda^* D(t_{n-1}, t_n)\lambda$$

Since $U(t_{n-1}, t_n) - D(Ft_{n-1}, t_n) \in \{D(t_n, t_{n+1}), D(t_{n-1}, t_n)\}$.

But $\|\lambda\| < 1$ , so this case yields contradiction.

Thus, $D(t_n, t_{n+1}) \preceq \lambda^* D(t_{n-1}, t_n)\lambda$. By $B$ we denote the element $D(t_1, t_0)$ in $\mathbb{E}$. So we have

$$\begin{aligned} D(t_n, t_{n+1}) &\preceq \lambda^* D(t_n, t_{n-1})\lambda \preceq (\lambda^*)^2 D(t_{n-1}, t_{n-2})\lambda^2 \\ &\preceq \ldots \preceq (\lambda^*)^n D(t_1, t_0)\lambda^n = (\lambda^*)^n B\lambda^n. \end{aligned}$$

So for $n+1 > m$, we get

$$\begin{aligned} D(t_{n+1}, t_m) &\preceq D(t_{n+1}, t_n) + D(t_n, t_{n-1}) + \ldots + D(t_{m+1}, t_m) \\ &= \sum_{k=m}^{n} (\lambda^*)^k B \lambda^k = \sum_{k=m}^{n} (\lambda^*)^k B^{\frac{1}{2}} B^{\frac{1}{2}} \lambda^k \\ &= \sum_{k=m}^{n} (B^{\frac{1}{2}} \lambda^k)^* (B^{\frac{1}{2}} \lambda^k) \preceq \sum_{k=m}^{n} \|B^{\frac{1}{2}}\|^2 \|\lambda^k\|^2 I \\ &\preceq \|B^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|\lambda\|^{2k} I \preceq \|B^{\frac{1}{2}}\|^2 \frac{\|\lambda\|^{2k}}{1 - \|\lambda\|} \to 0 \end{aligned}$$

as $m \to \infty$.

Therefore $\{t_n\}$ is a Cauchy sequence in $(P, \mathbb{E}, D)$. By the completeness of $(P, \mathbb{E}, D)$ there exists $z \in P$ such that $lim_{n\to\infty} t_n = z$. Since $(P, \mathbb{E}, D)$ is $F$ orbitally complete,

$$Lim_{n\to\infty} t_n = Lim_{n\to\infty} F^n(t_0) = z.$$

Given that $F$ is orbital continuty, we have

$$F(z) = Lim_{n\to\infty} F(F^n(t_0)) = Lim_{n\to\infty} F(F^{n+1}(t_0)) = z.$$

Then, $z$ is a fixed point for $F$. □

**Example 2.3.** let $P = [-1, 1]$ and $\mathbb{E} = M_{2\times 2}(I\!R)$ with $\|E\| = max_{ij}|e_{ij}|$ where $e_{ij}$ are entries of the matrix $E \in M_{2\times 2}(I\!R)$. Then, $(P, \mathbb{E}, D)$ is a $C^*$-algebra-valued $b$-metric space with

$$b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

where the involution is given by $E^* = (\overline{E})^T$,

$$D(t, s) = \begin{bmatrix} |t-s|^2 & 0 \\ 0 & |t-s|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{E}$ is given as

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \preceq \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \Longleftrightarrow e_{ij} \leq g_{ij} \quad \forall i, j = 1, 2, 3, 4$$

Define a mapping $F : P \to P$ by $Ft = t^3$. $F$ satisfies contraction of Theorem 2.2 implying that $F$ has a fixed point, $Fix(F) = \{0, \pm 1\}$.

**Definition 2.4.** A point $z$ is said to be a periodic point of a function $F$ with period $m$ if $F^m(z) = z$, where $F^0(t) = t$ and $F^m(t)$ is defined recursively by $F^m(t) = F(F^{m-1}(t))$.

**Theorem 2.5.** *If $F$ be an orbitally continuous self-map on a $C^*$-algebra-valued metric space $(P, \mathbb{E}, D)$ and $\varepsilon \in \mathbb{E}^+$. Suppose that there exists a point $t_0 \in P$ such that $D(t_0, F(t_0)) \preceq \varepsilon$ for some natural numbers $n$ such that*

$$0 < D(t, s) \preceq \varepsilon \Longrightarrow U(t, s) \preceq \lambda^* D(t, s)\lambda \tag{2}$$

*for every $t, s \in P$ and some $\lambda \in E^+, \|\lambda \| < 1$, where*

$$U(t, s) \in \{D(t, F(t)), D(F(t), F(s)), D(F(s), s)\}.$$

*Then, $F$ has a periodic point.*

**Proof.** $M = \{n : D(t, F^n(t)) \preceq \varepsilon \text{ for some } t \in P\}$. It is clear that $M \neq \phi$. $m = min\ M$, fix $t \in P$ thus $D(t, F^m(t)) \preceq \varepsilon$ so $\varepsilon - D(t, F^m(t)) \in \mathbb{E}^+$. Put $m = 1$ and $s = F(t)$ in (2.2), we have

$$U(t, F(t)) \preceq \lambda^* D(t, F(t))\lambda$$

Regarding $\| \lambda\| < 1$the case $D(t, F(t)) \preceq \lambda^* D(t, F(t))\lambda$, can not be happen.
Thus, $D(F(t), F^2((t)) \preceq \lambda^* D(t, F(t))\lambda$. According to Theorem 2.2, we can get $t_{n+1} = F(t_n), t = t_0$ so $Fz = z$ for some $z \in P$, and $F$ has a periodic point of period 1. Now if $m \geq 2$. So for each $s \in P$, the condition

$$\varepsilon - D(s, Fs) \notin \mathbb{E}^+. \tag{3}$$

But $D(t, F^m(t)) \preceq \varepsilon$ and by (2.2), we have

$$U(t, F^m(t)) \preceq \lambda^* D(t, F^m(t))\lambda. \tag{4}$$

Suppose that $F^m(t) = z$. By (2.3),(2.4) turns into

$$D(t, F^{m+1}(t)) \preceq \lambda^* D(t, F^m(t))\lambda$$

By using the similar proof in theorem 2.2, we can find that $\{t_n\}$ so that is a Cauchy sequence in $P$. Thereform $(P, \mathbb{E}, D)$ is $F$ orbitally complete, and that is some $z \in P$ such that

$$Lim_{n\to\infty} F^n(t_0) = z.$$

Given that Definition 2.1, $F$ is orbital continuity. Thus

$$\begin{aligned} F^m(z) &= Lim_{n\to\infty} F^m(F^n(t_0)) \\ &= Lim_{n\to\infty} F^{(n+1)m}(t_0) = z. \end{aligned}$$

and complete proof, because $z$ is a periodic point of $F$. □

**Theorem 2.6.** *Suppose $F$ be an orbitally continuous self-map in the $C^*$-algebra-valued $b-$metric space $(P, \mathbb{E}, D)$. And exist $e_1, e_2, e_3, e_4$ in $\mathbb{E}$ such that self mapping $F$ satises the conditions*
$I_{\mathbb{E}} \preceq e_1 + e_3 < 2I_{\mathbb{E}}, \quad 0_{\mathbb{E}} \preceq e_2 - e_4.$

$$\begin{aligned} e_1 D(Ft, Fs) + (I - e_1)[D(t, Ftx) + D(s, Fs)] + e_2[D(s, Ft) + D(t, Fs)] \\ \preceq e_3 D(t, s) + e_4 D(t, F^2 t) \qquad \forall t, s \in P. \end{aligned} \tag{5}$$

*Then, $F$ has at least one fixed point.*

**Proof.** Let $t_0 \in P$ be arbitrary. Construct a sequence $\{t_n\}$as follows:
$t_{n+1} := Ft_n \quad n = 0, 1, 2, \ldots$
When we put $t = t_n$ and $s = t_{n+1}$ in (2.5), thus

$$\begin{aligned} e_1 D(Ft_n, Ft_{n+1}) + (I - e_1)[D(t_n, Ft_n) + D(t_{n+1}, Ft_{n+1})] + e_2[D(t_{n+1}, Ft_n) \\ + D(t_n, Ft_{n+1})] \preceq e_3 D(t_n, t_{n+1}) + e_4 D(t_n, F^2 t_n) \end{aligned}$$

for all $e_1, e_2, e_3, e_4$.

$$\begin{aligned} e_1 D(t_{n+1}, t_{n+2}) + (I - e_1)[D(t_n, t_{n+1}) + D(t_{n+1}, t_{n+2})] + e_2[D(t_{n+1}, t_{n+1}) \\ + D(t_n, t_{n+2})] \preceq e_3 D(t_n, t_{n+1}) + e_4 D(t_n, t_{n+2}). \end{aligned}$$

By a simple calculation, one can get
$D(t_{n+1}, t_{n+2}) + (e_2 - e_4) D(t_n, t_{n+2}) \preceq (e_3 + e_1 - I) D(t_n, t_{n+1})$ which implies

$$D(t_{n+1}, t_{n+2}) \preceq k D(t_n, t_{n+1}) \tag{6}$$

where $k = e_3 + e_2 - I$. We have $0_{\mathbb{E}} \preceq k < I_{\mathbb{E}}$. Taking account of (2.6), we have get inductively

$D(t_n, t_{n+1}) \preceq kD(t_{n-1}, t_n) \preceq k^2 D(t_{n-2}, t_{n-1}) \preceq \cdots \preceq k^n D(t_0, t_1)$.

In the following, we prove that $\{t_n\}_{n\in I\!N}$ will be a Cauchy sequence.

$$\begin{aligned} D(t_n, t_{n+p}) &\preceq bD(t_n, t_{n+1}) + b^2 D(t_{n+1}, t_{n+2}) + \ldots + \\ &+ b^{p-2} D(t_{n+p-3}, t_{n+p-2}) + b^{p-1} D(t_{n+p-2}, t_{n+p-1}) + \\ &+ b^p D(t_{n+p-1}, t_{n+p}) \\ &\preceq bk^n D(t_0, t_1) + b^2 k^{n+1} D(t_0, t_1) + \ldots + \\ &+ b^{p-2} k^{n+p-3} D(t_0, t_1) + b^{p-1} k^{n+p-2} D(t_0, t_1) + \\ &+ b^p k^{n+p-1} D(t_0, t_1) \\ &= \frac{1}{b^n k} [b^{n+1} k^{n+1} D(t_0, t_1) + \ldots + \\ &+ b^{n+p-1} k^{n+p-1} D(t_0, t_1) + b^{n+p} k^{n+p} D(t_0, t_1)] \\ &\preceq \frac{1}{b^n k} [b^{n+1} k^{n+1} D(t_0, t_1) + \ldots + b^{n+p} k^{n+p} D(t_0, t_1)] \\ &= \frac{1}{b^n k} \sum_{i=n+1}^{n+p} b^i k^i D(t_0, t_1) \prec \frac{1}{b^n k} \sum_{i=n+1}^{\infty} b^i k^i D(t_0, t_1). \end{aligned}$$

The precedent inequality is

$$\|D(t_n, t_{n+p})\| < \|\frac{1}{b^n.k}. \sum_{i=n+1}^{\infty} b^i.k^i.D(t_0, t_1)\| \to 0$$

as $n \to \infty$. Thus $\{t_n\}_{n\in I\!N}$is a Cauchy sequence. As in the proof of previous theorem, regarding the construction $t_n = F^n t_0$ together with the fact that $(P, \mathbb{E}, D)$ is $F$-orbitally complete, there is $z \in P$ such that $t_n \to z$. Again by the orbital continuity of $F$,we deduce that $t_n \to Fz$. Hence $z = Fz$. □

**Corollary 2.7.** *$F$ be an orbitally continuous self-map in the $C^*$-algebra valued metric space $(P, \mathbb{E}, D)$. And exist $e_1, e_2, e_3, e_4$ in $\mathbb{E}$, self mapping $F : P \to P$ satises the conditions*

$$I_{\mathbb{E}} \preceq e_1 + e_3 < 2I_{\mathbb{E}}, \quad 0_{\mathbb{E}} \preceq e_2 - e_4.$$

$$e_1D(Ft,Fs) + (I-e_1)[D(t,Ft)+D(s,Fs)] + e_2[D(s,Ft)+D(t,Fs)]$$
$$\preceq e_3D(t,s) + e_4D(t,F^2t)$$

*hold for all $t,s \in P$. Then, $F$ has at least one fixed point.*

**Example 2.8.** If $P$ be the $C^*$-algebra-valued $b-$metric spaces in Example 2.3, and $F : P \to P$ be a self map on $P$ by $Ft = \frac{1}{2}t + \frac{1}{3}$ and let $e_1 = 1, e_2 = e_4 = 0$ and $e_3 = \frac{1}{2}$. $F$ satisfies hypothesis of Theorem 2.6. Then $F$ has at least one fixed point,and Fix$(F) = \{\frac{2}{3}\}$.

## 3 Nonunique fixed point theorems and n-periodic points

Let $I\!N$ be the set of positive integer and $(P,\mathbb{E},D)$ be a $C^*$-algebra-valued metric space. If $F : P \to P$ is mapping, then an *$n-$periodic point* of $F$, is a point $t_0 \in P$ such that $t_0 = F^nt_0$ and $t_0 \neq F^kt_0$ for $k = 1,2,3,...,n-1$ for some $n \in I\!N$.

The *orbit* of $F$ at a point $t$ is defined by $O_F(P) = \{F^nt : n \geq 0\}$. Suppose that $\Phi$ is the set of all nondecreasing $\varphi : \mathbb{E}^+ \to \mathbb{E}^+$ where $\varphi(t) \prec t$ for all $t \in \mathbb{E}^+$.

Obviously $\varphi(t) = \gamma^*t\gamma \in \Phi$ for all $\gamma \in \mathbb{E}$, where $\|\gamma\| < 1$.

We discuss the existence of fixed points for above mappings $F$.

$$D(Ft,Fs) + D(Fs,Fz) \preceq \varphi(D(t,s) + D(s,z)) \tag{7}$$

where $t,s,z \in X$ with $t \neq s \neq z \neq t$, and $\varphi \in \Phi$; or

$$Sup\{D(Ft,Fs),D(Fs,Fz)\} \preceq \varphi(Sup\{D(t,s),D(s,z)\}) \tag{8}$$

where $t,s,z \in P$ with $t \neq s \neq z \neq t$, and $\varphi \in \Phi$. The definition of n-periodic point follows that:

**Lemma 3.1.** *Let $F : P \to P$ be a mapping on a $C^*$-algebra-valued metric space $(P,\mathbb{E},D)$. Then $F^it_0 \neq F^jt_0$ for any n-periodic point $t_0 \in P$ of $F$, where $0 \leq i \leq j \leq n-1$.*

**Theorem 3.2.** *Suppose that $F$ is a mapping from a complete $C^*$-algebra-valued metric space $(P, \mathbb{E}, D)$ into itself and satisfy* (3.1).
*Then*

(1) *There are at most two distinct fixed points for $F$ in $P$;*
(2) *The number of 2-periodic points for $F$, in $P$ is zero or two;*
(3) *If $n \geq 3$ then $F$ has any $n$-periodic points in $P$;*
(4) *If $F$ has an orbit without 2-periodic points then $F$ has a fixed point in $P$.*

**Proof.** Suppose that $F$ has three distinct fixed points $t, s, z$, in $P$. Then get that

$$D(t,z) + D(z,s) \preceq \varphi(D(t,z) + D(z,s)) \prec D(t,z) + D(z,s),$$

which is a contradiction.

If $s \in P$ is a 2-periodic point of $F$. Then $Fs$ is also a 2-periodic point of $F$ distinct from $s$. We assert that $F$ has the only two 2-periodic points $s$ and $Fs$.

Now suppose that $z \in P$ other 2-periodic point of $F$ such that $s \neq z \neq Fs$. Thus we have $Fs \neq Fz \neq F^2 s \neq Fs$. Then

$$\begin{aligned} D(s,z) + D(z,Fs) &= D(F^2 s, F^2 z) + D(F^2 z, F^3 s) \\ &\preceq \varphi^2(D(F^2 s, F^2 z) + D(F^2 z, Fs)) \prec D(s,z) + D(z,Fs), \end{aligned}$$

which is a contradiction. Thus $F$ has exactly two 2-periodic points. Now we show that $F$ has any n-periodic point for $n \geq 3$. If $t_0 \in P$ is an n-periodic point of $F$ for $n \geq 3$ then $t_k = F^k t_0, D_k = D(t_k, t_{k+1}) + D(t_{k+1}, t_{k+2})$ for all $0 \leq k \leq n$. By Lemma 3.1, we infer that

$$D_k \preceq \varphi(D(t_{k-1}, t_k) + D(t_k, t_{k+1})) \prec D_{k-1},$$

for all $1 \leq k \leq n$. We have that

$$D_0 = D_n \preceq \varphi(D_{n-1}) \prec D_{n-1} \preceq ... \prec D_0,$$

which is a contradiction.

Let, there exists a point $t_0 \in P$ which is $F$ has no 2-periodic points in $O_F(t_0)$. Set $t_n = F_n t_0, D_n = D(t_n, t_{n+1}) + D(t_{n+1}, t_{n+2})$ for any $n \geq 0$.

If there exists some $n \geq 0$ with $t_n = t_{n+1}$, then $t_n$ is a fixed point of $F$; if $t_n \neq t_{n+1}$ for any $n \geq 0$, we get

$$D_n \preceq \varphi(D_{n-1}) \preceq \varphi^2(D_{n-2}) \preceq ... \preceq \varphi^n(D_0).$$

For each $i, j, l \in I\!N$ such that $i > j \geq l$, by the triangular inequality, we have that

$$D(t_i, t_j) \preceq \sum_{n=l}^{i-1} D_n \preceq \sum_{n=l}^{i-1} \varphi^n(D_0) \preceq \sum_{n=1}^{\infty} (\gamma^*)^n D_0 \gamma^n$$

$$\|D(t_i, t_j)\| \leq \sum_{n=1}^{\infty} \|\gamma\|^{2n} \|D_0\|.$$

When $\|\gamma\| < 1$ ensures that $\{t_n\}_{n\geq 0}$ is a Cauchy sequence in $P$. It follows from completeness of $(P, \mathbb{E}, D)$ that there exists a point $a \in P$ such that $lim_{n\to\infty} t_n = a$. Obviously, there exists some integer $k \in I\!N$ with $t_n \neq a$ for all $n \geq k$. We obtain that

$$\begin{aligned} D(t_{n+1}, Fa) + D(Fa, Ft_{n+2}) &\preceq \varphi(D(t_n, a) + D(a, t_{n+2})) \\ &\prec D(t_n, a) + D(a, t_{n+2}) \to 0 \end{aligned}$$

as $n \to \infty$, which implies that $lim_{n\to\infty} t_n = Fa$. Hence $Fa = a$. This completes the proof. □

**Corollary 3.3.** *If $F : P \to P$ and $(P, \mathbb{E}, D)$ is a complete $C^*$-algebra-valued metric space such that satisfy*

$$D(Ft, Fs) + D(Fs, Fz) + D(Fz, Ft) \preceq \varphi(D(t, s) + D(s, z) + D(z, t))$$

*for all $t, s, z \in P$ with $t \neq s \neq z \neq t$, where $\varphi \in \Phi$. Then the conclusions of Theorem 3.2 hold.*

**Definition 3.4.** If $\mathbb{E}$ is a $C^*$-algebra-valued metric space then $\mathbb{E}$ is called *minihedral* if $Sup\{t, s\}$ exists for all $t, s \in \mathbb{E}$ and strongly minihedral if every subset of $\mathbb{E}$ which is bounded from above has a supremum.(equivalently, if every subset of $\mathbb{E}$ which is bounded from below has an infimum.)

**Theorem 3.5.** *Suppose that $F$ is a mapping from a strongly minihedral complete $C^*$-algebra-valued metric space $(P, \mathbb{E}, D)$ into itself. Satisfy* (3.2). *Then the conclusions of Theorem 3.2 hold.*

**Proof.** For an n-periodic point $t_0$ for $n \geq 3$ of $F$ set $t_k = F^k t_0, D_k = D(t_k, t_{k+1})$ for all $0 \leq k \leq n$. From Lemma 3.1, we have

$$\begin{aligned} Sup\{D_0, D_1\} &= Sup\{D_n, D_{n+1}\} \\ &= Sup\{D(Ft_{n-1}, Ft_n), D(Ft_n, Ft_{n+1})\} \\ &\preceq \varphi(Sup\{D(t_{n-1}, t_n), D(t_n, t_{n+1})\}) \\ &= \varphi(Sup\{D_{n-1}, D_n\}) \\ &\preceq \varphi^n(Sup\{D_0, D_1\}) \prec Sup\{D_0, D_1\}, \end{aligned}$$

which is a contradiction.

Now, we prove that $F$ has a fixed point in $P$ provided that $F$ has an orbit without 2-periodic points in $P$. Suppose $z_0 \in P$ such that $F$ has no 2-periodic points in $O_F(z_0)$.

Set $z_n = F^n z_0, D_n = D(z_n, z_{n+1})$ for all $n \geq 0$. We have two cases:

Case 1. There exists some $n \geq 0$ with $z_n = z_{n+1}$. Then $z_n$ is a fixed point of $F$ in $P$.

Case 2. Otherwise if $z_n \neq z_{n+1}$ for any $n \geq 0$ then $z_n \neq z_m$. We have

$$\begin{aligned} Sup\{D_n, D_{n+1}\} &= Sup\{D(Fz_{n-1}, Fz_n), D(Fz_n, Fz_{n+1})\} \\ &\preceq \varphi(Sup\{D(z_{n-1}, z_n), D(z_n, z_{n+1})\}) \\ &\preceq \varphi^2(Sup\{D_{n-2}, D_{n-1}\}) \preceq \ldots \preceq \varphi^n(Sup\{D_0, D_1\}). \end{aligned}$$

For each $n \in I\!N$ and $p \in I\!N$, using the triangular inequality, we get that

$$\begin{aligned} D(z_n, z_{n+p}) &\preceq \sum_{i=n}^{n+p-1} D_i \preceq \sum_{i=n}^{n+p-1} \varphi^i(Sup\{D_0, D_1\}) \\ &\preceq \sum_{1}^{n} \varphi^i(Sup\{D_0, D_1\}) \preceq \sum_{1}^{n} \varphi^i(\gamma^*)^i D\gamma^i \\ \|D(z_n, z_{n+p})\| &\leq \sum_{i=1}^{\infty} \|\gamma\|^{2i} \|D\| \end{aligned}$$

When $\|\gamma\| < 1$ show that $\{z_n\}_{n\geq 0}$ is a Cauchy sequence in $P$. By completeness of $(P, \mathbb{E}, D)$ that there exists a point $a \in P$ such that $lim_{n\to\infty} z_n = a$. Obviously, there exists some integer $k \in I\!N$ with $z_n \neq a$ for all $n \geq k$. We obtain that

$$\begin{aligned} Sup\{D(z_{n+1}, Fa), D(Fa, Fz_{n+2})\} &\preceq \varphi(Sup\{D(z_n, a) + D(a, z_{n+2})\}) \\ &\prec Sup\{D(z_n, a) + D(a, z_{n+2})\} \to 0 \end{aligned}$$

as $n \to \infty$, which implies that $lim_{n\to\infty} z_n = Fa$. Hence $Fa = a$. This completes the proof. □

**Corollary 3.6.** *Let $(P, \mathbb{E}, D)$ be a strongly minihedral complete $C^*$-algebra-valued metric space and $F : P \to P$ satisfy*

$$Sup\{D(Ft, Fs), D(Fs, Fz), D(Fz, Ft)\} \preceq \varphi(Sup\{(D(t, s), D(s, z), D(z, t))\})$$

*for all $t, s, z \in P$ with $t \neq s \neq z \neq t$, where $\varphi \in \Phi$. Then the conclusions of Theorem 3.2 hold.*

**Example 3.7.** Let $\mathbb{E} = M_{2\times 2}(I\!R)$ be endowed with the norm

$$\|E\| = max_{i,j}|a_{ij}|,$$

where $a_{ij}$ are entries of the matrix $E \in M_{2\times 2}(I\!R)$, and the involution given by $E^* = (\overline{E})^F = E^F$. Clearly, each matrix of type $E = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ belongs to $\mathbb{E}^+$ if $\alpha, \beta \geq o$. This implies

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \preceq \begin{bmatrix} \delta & 0 \\ 0 & \gamma \end{bmatrix} \Leftrightarrow \alpha \leq \delta, \quad \beta \leq \gamma.$$

Let $P = \{1, 2, 3, 4\}$ and $D : P \times P \to \mathbb{E}$ be a function defined by $D(1, 2) = I_{\mathbb{E}},\ D(2, 3) = 3I_{\mathbb{E}},\ D(1, 3) = 4I_{\mathbb{E}},\ D(2, 4) = 2I_{\mathbb{E}},\ D(1, 4) = \frac{5}{2}I_{\mathbb{E}},\ D(3, 4) = \frac{7}{2}I_{\mathbb{E}},\ D(x, x) = 0_{\mathbb{E}}$ and $D(t, s) = D(s, t)$ for all $t, s \in P$. Let $F : P \to P$ be a mapping defined by $F1 = 1,\ F2 = 2,\ F3 = 4,\ F4 = 2$. Take $\varphi(x) = \begin{bmatrix} \frac{3x}{4} & 0 \\ 0 & \frac{3x}{4} \end{bmatrix}$ for $t \geq 0$. It is easy to check that the conditions of Theorem 3.5 are satisfied, and $F$ has two fixed points in $P$. But the Banach contraction principle is not available and $F$ has no 2-periodic point in $P$.

It is clear that orbital continuity of $F$ implies orbital continuity of $F^m$ for any $m$ in $I\!N$ .

**Theorem 3.8.** *Let $(P, \mathbb{E}, D)$ be a strongly minihedral $C^*$-algebra-valued metric space and $F$ be an orbitally continuous self mapping on $P$, such that $(P, \mathbb{E}, D)$ is $F$ orbitally completed. Suppose that $\psi \in \Phi$*

$$U(t, s) - V(t, s) \preceq \psi(D(t, s)) \tag{9}$$

*for all $t, s \in P$ where*

$$U(t, s) = inf\{D(Ft, Fs), D(s, Fs)\},$$

$$V(t, s) = inf\{D(t, Ft)D(t, Fs), D(s, Fs)D(Ft, s)\}.$$

*Then, the sequence $\{F^n(t_0)\}_{n \in I\!N}$ converges to a fixed point of $F$ for any $t_0 \in P$.*

**Proof.** Let $t_0 \in P$ be an orbital element. Set $t_n = F^n t_0$. If $t_n = t_{n-1}$ for some $n \in I\!N$ then $t_{n-1}$ is a fixed point for $F$. Otherwise if $t_n \neq t_{n-1}$ for all $n \in I\!N$ be inequality 3.3, we have that

$$\begin{aligned} U(t_{n-1}, t_n) &= inf\{D(Ft_{n-1}, Ft_n), D(t_n, Ft_n)\} \\ &= D(t_n, Ft_n) \end{aligned}$$

$$V(t_{n-1}, t_n) = inf\{D(t_{n-1}, Ft_{n-1})D(t_{n-1}, Ft_n), D(t_n, Ft_n)D(t_n, t_n)\} = 0$$

$$\begin{aligned} U(t_{n-1}, t_n) - V(t_{n-1}, t_n) &= D(t_n, Ft_n) \\ &\preceq \psi D(t_{n-1}, t_n) \end{aligned}$$

$$\begin{aligned} D(t_n, Ft_n) = D(t_n, t_{n-1}) &\preceq \psi D(t_{n-1}, t_n) \preceq \psi^2 D(t_{n-2}, t_{n-1}) \\ &\preceq \cdots \preceq \psi^n D(t_0, t_1). \end{aligned}$$

The rest of the proof is alike in the proof of Theorem 3.5. $\square$

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