NOTES ON FUZZY FILTERS OF PSEUDO CI-FILTERS.

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ABSTRACT. In this paper, the concept of fuzzy pseudo CI-filter is introduced and we show the equivalents of this definition. With the upper level set, the relationship between the pseudo CI-filter and the fuzzy pseudo CI-filter is shown and this relationship is illustrated with an example.

Fuzzy pseudo q-CI-filters and fuzzy pseudo a-CI-filters in a pseudo CI-algebra are the next aim of this paper. It is shown that each fuzzy pseudo q-CI-filters (Or fuzzy pseudo a-CI-filters) is a fuzzy pseudo CI-filter, but conversely is not true. We put some conditions on the fuzzy pseudo CI-filter, such that the fuzzy pseudo CI-filter becomes fuzzy pseudo q-CI-filters (Or fuzzy pseudo a-CI-filters). Fuzzy pseudo CI-filters generated by a fuzzy set is another aim that will be studied in this article.

Finally, Noetherian and Artinian pseudo CI-algebras are introduced and their properties are stated. By a strictly ascending sequence of pseudo CI-filter of pseudo CI-algebra and any strictly decreasing sequence in $(0, 1)$, we construct a fuzzy pseudo CI-filter.

Keywords: Pseudo CI-algebra, pseudo fuzzy CI-filter, generated fuzzy pseudo CI-filters

1. Introduction and Prelimiaries

1.Introduction. In 1966, Y. Imai and K. Iséki [10] introduced the classes of abstract algebras, BCK-algebra. The BCI-algebras [10], BCH-algebras [9] and BE-algebras [11] was introduced as BCK-algebras generalization. Meng [14] defined the notion of CI-algebras as a generalization of BE-algebras.The notation of pseudo CI-algebras introduced by A. borumand. et.al. [15], they defined the class of pseudo CI-algebras and studied some of its subclasses.

The concept of filters plays an important role in studying logical algebras. Meng [13] gave a procedure to generate a filter by a subset in a transitive BE-algebra and gave some characterizations of Noetherian and Artinian BE-algebras and proved that in transitive BE-algebras, the notion of ideals is equivalent to one of filters. Furthermore, he discussed on relations between singular CI-algebras and Abelian groups [14, 13]. After the concept of fuzzy sets was introduced by Zadeh [21], several studies were conducted on the generalization of then notion of fuzzy sets. Fuzzy ideas have been applied to algebraic structures. From the logic point of view, the sets of provable formulas in corresponding systems can be described by (fuzzy) filters of those algebraic semantics. Furthermore, fuzzy filter ideas function well in studying of algebraic structures. In lattice implication algebras, fuzzy positive implicative filters was introduced by Y. Xu et al.[17]. Further studies on fuzzy filters in BL-algebra were conducted by L.Z. Liu and K.T. Li in 2005 [12]. Some types of filters in BE-algebras have been widely studied and many important results are obtained [3]. Dymek et al. [8] applied the fuzzy set theory to filters of BE-algebras and later S. S. Ahn et al. [2] introduced the notion of fuzzy (implicative, positive implicative, fantastic) filters. Wang and Xin solved an open problem in pseudo BL-algebras between fuzzy normal and fuzzy Boolean filter [20]. N. Shojaei. et.al. [16] introduced the notion of pseudo CI-filter in a pseudo CI-algebra.

Since fuzzy filters are a useful tool to obtain results on algebraic structures of logic algebras. So, in this paper the concept of fuzzy pseudo CI-filter is introduced and equivalent propositions are found for this definition. We study the pseudo hemomorphism of fuzzy pseudo CI-filters and the relationship between the image and the preimage of a fuzzy pseudo CI-filter by pseudo homomorphisms.

Fuzzy pseudo a-CI-filters and fuzzy pseudo q-CI-filters in a pseudo CI-algebra is the another aim of this paper. It is proved that each pseudo q-CI-filters (pseudo a-CI-filters) is also a pseudo CI-filter, but with an example shown the converse is not true, we find some conditions where the converse is also true. Fuzzy pseudo CI-filters generated by a fuzzy set are also studied in this paper.

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Finally, Noetherian and Artinian fuzzy pseudo CI-algebras are introduced. we show that every CI-algebra X is Noetherian, iff, for each fuzzy pseudo CI-filter μ of X, $Im(\mu) = \{\mu(x) : x \in X\}$ is a well-ordered subset of $[0, 1]$, and so, for every fuzzy pseudo CI-filter of X, if it image is a finite set, then X is Artinian but by an example we show that the converse is not true.

Definition 1.1. [16] An algebra $X = (X; \to, \rightsquigarrow, 1)$ of type $(2; 2; 0)$ is a $pseudoCI-algebra$ if it satisfies;
$(pCI1)$ $x \to x = x \rightsquigarrow x = 1$,
$(pCI2)$ $1 \to x = 1 \rightsquigarrow x = x$,
$(pCI3)$ $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
$(pCI4)$ $x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1$.

The relation "$\leq$" on X define by, $x \leq y$ iff $x \to y = x \rightsquigarrow y = 1$.

If $(X; \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra satisfying $x \to y = x \rightsquigarrow y$; for all $x, y \in X$, then $(X; \to, \rightsquigarrow, 1)$ is a CI-algebra.

Definition 1.2. [16] Let a be an element of a pseudo CI-algebra $X = (X; \to, \rightsquigarrow, 1)$. An element a is said to be an atom in X if for any $x \in X$, $a \to x = 1$ (or $a \rightsquigarrow x = 1$) implies $a = x$.

Example 1.3. [16] Let $X = \{1, a, b, c, d\}$. Define the operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

Then X is a pseudo CI-algebra.

Proposition 1.4. [15] *If $X = (X; \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra, then for all $x, y, z \in X$, we have:*
(1) $x \leq (x \to y) \rightsquigarrow y$, $\;x \leq (x \rightsquigarrow y) \to y$,
(2) $x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z$,
(3) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1)$, $\;(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1)$,
(4) $x \to 1 = x \rightsquigarrow 1$,
(5) $x \leq y$ *implies* $x \to 1 = y \to 1$.

Proposition 1.5. [16] *In a pseudo CI-algebra X, for all $x, y \in X$, the following holds:*
(1) *If* $y \leq 1$ *then* $x \to (y \rightsquigarrow x) = 1$ *and* $x \rightsquigarrow (y \to x) = 1$,
(2) *If* $y \leq 1$ *then* $x \rightsquigarrow (y \rightsquigarrow x) = 1$ *and* $x \to (y \to x) = 1$,
(3) $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$ *and* $x \to ((x \to y) \rightsquigarrow y) = 1$,
(4) $(x \to 1) \to 1 = (x \to 1) \rightsquigarrow 1)$ *and* $(x \rightsquigarrow 1) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to 1$,

Example 1.6. [18] Let $X = \{1, a, b, c, d, e\}$. Define the operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	b	e	d
b	b	d	1	e	a	c
c	d	b	e	1	c	a
d	c	e	a	d	1	b
e	e	c	d	a	b	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	d	e	b	c
b	b	c	1	a	e	d
c	d	e	a	1	c	b
d	c	b	e	d	1	a
e	e	d	c	b	a	1

Then $X = (X; \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra which is not a pseudo BE-algebra. Because $a \to 1 = a \neq 1$ and $a \rightsquigarrow 1 = a \neq 1$.

Proposition 1.7. *If X be a pseudo CI-algebra and $b, c \in X$ such that $b \to a = c \to a$ and $b \rightsquigarrow a = c \rightsquigarrow a$ for some $a \in X$ then;*

$$b \to 1 = c \to 1 \text{ and } b \rightsquigarrow 1 = c \rightsquigarrow 1.$$

Definition 1.8. A non-empty subset F of X is called a pseudo CI-filter of $X = (X; \to, \rightsquigarrow, 1)$ if it satisfies in the following axioms:

$(F1)$ $1 \in F$,
$(F2)$ If $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in X$.

A pseudo CI-filter F is proper if and only if $F \neq X$.
A pseudo CI-filter F is called closed pseudo CI-filter of X if $x \to 1, x \rightsquigarrow 1 \in F$ for all $x \in F$.

Example 1.9. Let $(G, \cdot)$ be a group with identity element e. Define the operations $\to$ and $\rightsquigarrow$ on G by $a \to b = a^{-1}b$ and $a \rightsquigarrow b = ba^{-1}$ for all $a, b \in G$. Then $(G, \to, \rightsquigarrow, e)$ ia a pseudo CI-algebra.

Definition 1.10. Let X be a set. A fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

For any fuzzy sets μ and ν in X, we define,
$(\mu \vee \nu)(x) = \mu(x) \vee \nu(x)$.
$(\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x)$.
The relation " $\leq$ " in X defined by: $\mu \leq \nu$, if and only if, $\mu(x) \leq \nu(x)$ for all $x \in X$. (This relation is an order relation in the set of fuzzy sets in X).

For $\Gamma \in [0, 1]$ we defined $\bigwedge \Gamma = inf\Gamma$ and $\bigvee \Gamma = sup\Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$.

Definition 1.11. Let X and Y be any two sets, μ be any fuzzy set in X and $f : X \longrightarrow Y$ be any function. Set $f^{-1}(y) = \{x \in X : f(x) = y\}$ for $y \in Y$. The fuzzy set ν in Y defined by;

$$\nu(x) = \begin{cases} \bigvee\{\mu(x) : x \in f^{-1}(y)\} & if \quad f^{-1}(y) \neq \phi, \\ 0 & otherwise. \end{cases}$$

for all $y \in Y$, is called the image of μ under f and is denoted by $f(\mu)$. Let X and Y be any two sets, $f : X \longrightarrow Y$ be any function and ν be any fuzzy set in $f(X)$. The fuzzy set μ in X defined by,

$$\mu(x) = \nu(f(x)), \ \ forall \ \ x \in X,$$

is called the preimage of ν under f and is denoted by $f^{-1}(\nu)$.

Lemma 1.12. *Let X be a transitive pseudo CI-algebra. Let n, m be natural numbers. If in X;*
(1) $b \to (a \to x) = 1$,
(2) $(...a_n \to (a_{n-1} \to ... \to (a_1 \to a))) = 1$,
(3) $(...b_m \to (b_{m-1} \to ... \to (b_1 \to b))) = 1$.
then, $(...b_m \to (b_{m-1} \to ... \to (b_1 \to a_n \to (a_{n-1} \to ... \to (a_1 \to x))))) = 1$.
Proof. *Since $b \to (a \to x) = 1$, by (pCI4),*

$$b \rightsquigarrow (a \to x) = 1.$$

From (pCI3) we have,

$$a \to (b \rightsquigarrow x) = 1,$$

that is,

$$a \leq (b \rightsquigarrow x).$$

Hence, by proposition 4.7 (16)*,*

$$(a_1 \to a) \leq (a_1 \to (b \rightsquigarrow x)).$$

Next, again by proposition 4.7 (16)*, we have;*

$$(a_2 \to (a_1 \to a)) \leq (a_2 \to (a_1 \to (b \rightsquigarrow x))).$$

Repeating the process we get;

$$1 = (...a_n \to (a_{n-1} \to ... \to (a_1 \to a))) \leq (...a_n \to (a_{n-1} \to ... \to (a_2 \to (a_1 \to (b \rightsquigarrow x))))).$$

So,

$$(...a_n \to (a_{n-1} \to ... \to (a_2 \to (a_1 \to (b \rightsquigarrow x))))) = 1.$$

From $(pCI3)$*,*

$$(b \rightsquigarrow ...(a_n \to (a_{n-1} \to ... \to (a_2 \to (a_1 \to x))))) = 1.$$

That is,

$$b \leq (...(a_n \to (a_{n-1} \to ... \to (a_2 \to (a_1 \to x))))).$$

Hence, by proposition 4.7 (16)*,*

$1 = (...b_m(\to (b_{m-1}) \to ... \to (b_1 \to b))) \le (...b_m \to (b_{m-1} \to ... \to (b_1 \to a_n \to (a_{n-1} \to ... \to (a_1 \to x))))).$

Therefore,

$$(...b_m \to (b_{m-1} \to ... \to (b_1 \to a_n \to (a_{n-1} \to ... \to (a_1 \to x))))) = 1.$$

2. **Fuzzy pseudo CI-filters**

In the following, let X denote a pseudo CI-algebra unless otherwise specified.

Definition 2.1. A fuzzy set μ in X is called a fuzzy pseudo CI-filter of X if for all $x, y, z \in X$, it satisfies the following conditions:

$(pFF1)$ $\mu(1) \ge \mu(x)$,

$(pFF2)$ if $x \le y \to z$, then $\mu(z) \ge \mu(x) \wedge \mu(y)$.

Let $pFF(X)$ denote the set of all fuzzy pseudo CI-filters of a pseudo CI-algebra X.

Example 2.2. Let $X = \{1, a, b, c, d\}$. Define the operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

Then X is a pseudo CI-algebra. Define a fuzzy set $\mu : X \longrightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 0.7 & if\ x = 1, \\ 0.6 & x \in \{b, c\}, \\ 0.5 & x \in \{a, d\}. \end{cases}$$

It is easily seen that μ is a fuzzy pseudo CI-filter of X.

Proposition 2.3. *Every fuzzy pseudo CI-filter μ of X satisfies the following assertions:*
if $x, y \in X$ and $x \le y$, then $\mu(x) \le \mu(y)$.

Proof. *Assume that $x \le y$, so, $x \rightsquigarrow y = 1$. By proposition* (1.4)

$$x \le (x \rightsquigarrow y) \to y.$$

By (pFF2) and (pFF1),

$$\mu(y) \ge \mu(x) \wedge \mu(x \rightsquigarrow y) = \mu(x) \wedge \mu(1) \ge \mu(x) \wedge \mu(x) = \mu(x).$$

Proposition 2.4. *A fuzzy set μ in X is a fuzzy pseudo CI-filter of X if and only if μ satisfies (pFF1) and one of the following conditions holds, for all $x, y \in X$:*

$(pFF3)$ $\mu(y) \ge \mu(x) \wedge \mu(x \to y)$

$(pFF4)$ $\mu(y) \ge \mu(x) \wedge \mu(x \rightsquigarrow y)$.

$(pFF5)$ *If $x \le y \rightsquigarrow z$, then $\mu(z) \ge \mu(x) \wedge \mu(y)$.*

Proof.

$(pFF3)$*: Let μ satisfy (pFF3) and $x \le y \to z$. By (pFF3) and Proposition* (2.3)*, we have;*

$$\mu(z) \ge \mu(y) \wedge \mu(y \to z) \ge \mu(y) \wedge \mu(x).$$

Thus, μ satisfies $(pFF2)$, hence μ is a fuzzy pseudo CI-filter of X.

Conversely, Assume that μ be a fuzzy pseudo CI-filter. Since $x \to y \le x \to y$, by (pFF2) we obtain;

$$\mu(y) \ge \mu(x \to y) \wedge \mu(x).$$

Thus, μ satisfies $(pFF3)$.

$(pFF4)$*: Is similar $(pFF3)$.*

$(pFF5)$*: Let μ be a fuzzy pseudo CI-filter and $x \le y \rightsquigarrow z$, then $x \to (y \rightsquigarrow z) = 1$, by $(pCI3)$,*

$$y \rightsquigarrow (x \to z) = 1.$$

Therefor,

$$y \le x \to z,$$

by $(pFF2)$, we obtain,

$$\mu(z) \ \geq \mu(x) \ \wedge \ \mu(y).$$

Conversely,Let μ satisfy (pFF5). Since $x \rightsquigarrow y \leq x \rightsquigarrow y$. By (pFF5),

$$\mu(y) \geq \mu(x) \wedge \mu(x \rightsquigarrow y).$$

Thus, μ satisfies $(pFF4)$, hence μ is a fuzzy pseudo CI-filter of X.

Corollary 2.5. *If $a_n \to (a_{n-1} \to ... \to (a_1 \to x)) = 1$, then $\mu(x) \geq (\mu(a_1) \wedge \mu(a_2) \wedge ... \wedge \mu(a_n))$.*

Proposition 2.6. *Let μ be a fuzzy pseudo CI-filter. Then:*

(2.1) $$\mu((x \to 1) \rightsquigarrow 1) = \mu((x \rightsquigarrow 1) \to 1) \geq \mu(x),$$

Proof. Let $x \in X$. From (1.4)(4) $x \to 1 = x \rightsquigarrow 1$. So,

$$\mu((x \rightsquigarrow 1) \to 1) = \mu((x \to 1) \rightsquigarrow 1).$$

It follows from (pFF4),(pCI3), (pCI1) and (pFF1) that,

$$\mu((x \rightsquigarrow 1) \to 1) \geq \mu(x \rightsquigarrow ((x \rightsquigarrow 1) \to 1)) \bigwedge \mu(x) = \mu((x \rightsquigarrow 1) \to (x \rightsquigarrow 1)) \bigwedge \mu(x) = \mu(1) \bigwedge \mu(x) = \mu(x).$$

This completes the proof.

Proposition 2.7. *Let μ be a fuzzy pseudo CI-filter of X. If μ^* defined in X by;*

(2.2) $$\mu^*(x) = \mu((x \rightsquigarrow 1) \to 1) \ \ for\ all \ \ x \in X,$$

then μ^* is a fuzzy pseudo CI-filter of X and $\mu \leq \mu^*$.

Proof. *For any $x \in X$, by using $(pFF1)$, $\mu^*(1) = \mu((1 \rightsquigarrow 1) \to 1) = \mu(1) \geq \mu((x \rightsquigarrow 1) \to 1) = \mu^*(x)$.*
Let $x, y \in X$. By (1.4)(4),(3);

(2.3) $$\mu^*(x \to y) = \mu(((x \to y) \rightsquigarrow 1) \to 1) = \mu(((x \to y) \to 1) \rightsquigarrow 1) = \mu(((x \rightsquigarrow 1) \rightsquigarrow (y \to 1)) \rightsquigarrow 1) = \mu(((x \ \rightsquigarrow 1) \to 1) \to ((y \to 1)) \rightsquigarrow 1)$$

Thus, from 2.1. and 2.3.

$$\mu^*(x \to y) \wedge \mu^*(x) = \mu((((x \ \rightsquigarrow 1) \to 1) \to ((y \to 1)) \rightsquigarrow 1)) \wedge (\mu((x \rightsquigarrow 1) \to 1)) \leq \mu((y \ \rightsquigarrow 1) \to 1) = \mu^*(y).$$

Therefore, μ^* is a pseudo fuzzy pseudo CI-filter.
By Proposition (2.6),

$$\mu^*(\ x\ ) = \mu((x \rightsquigarrow 1) \to 1) \geq \ \mu(x)$$

for all $x \in X$. So $\mu^* \geq \mu$.

Example 2.8. Let F be pseudo CI-filter of a pseudo CI-algebra X and let $a, b \in [0,1]$ with $a \geq b$. Define $\mu : X \to [0,1]$ as follows:

$$\mu(x) = \begin{cases} a & if\ x \in F, \\ b & if\ x \notin F. \end{cases}$$

Since $1 \in F$, then $\mu(1) = a \geq \mu(x)$ for all $x \in X$. To prove (pFF2), let $x, y \in X$. If $x \in F$, then $\mu(x) = a \geq \mu(x \to y) \wedge \mu(y)$. Suppose now that $x \notin F$. Since F is pseudo CI-filter so, $x \to y \notin F$ or $y \notin F$. Therefore, $\mu(x \to y) \wedge \mu(y) = b = \mu(x)$. Thus $\mu \in pFF(X)$.

In particular, the characteristic function χ_F of F define by:

$$\chi_F(x) = \begin{cases} 1 & if\ x \in F, \\ 0 & if\ x \notin F. \end{cases}$$

is a fuzzy pseudo CI-filter of X.

Example 2.9. (i) Assume the group $U(5)$ under multiplication modulo 5. By (1.9), $(G, \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra defined $a \to b = a^{-1}b$ and $a \rightsquigarrow b = ba^{-1}$ for all $a, b \in G$. Let $\mu(1) = 1, \mu(3) = 0.6$ and $\mu(2) = \mu(4) = 0.4$, then μ is fuzzy pseudo CI-filter on $U(5)$ and $\mu^*(x) = \mu((x \rightsquigarrow 1) \to 1) = \mu(x)$. So $\mu^* = \mu$

(ii) If μ be a fuzzy pseudo CI-filter on X and for all $x \in X$, $x \leq 1$, then $\mu^*(x) = \mu((x \to 1) \rightsquigarrow 1)) = \mu(1)$. (This shows that may $\mu^* \neq \mu$).

The set $U(\mu, \alpha) = \{x \in X : \mu(x) \geq \alpha\}$ and $L(\mu, \alpha) = \{x \in X : \mu(x) \leq \alpha\}$ which are called, respectively, upper $\alpha - level$ subset and lower $\alpha - level$ subset of X, for $\alpha \in [0,1]$.

Proposition 2.10. *A fuzzy set μ in X is a fuzzy pseudo CI-filter of X if and only if its nonempty upper $\alpha - level$ subset $U(\mu, \alpha)$ is a pseudo CI-filter of X, for all $\alpha \in [0, 1]$.*

Proof. *Let μ be a fuzzy pseudo CI-filter of X, and let $\alpha \in [0, 1]$. Assume $U(\mu, \alpha) \neq \emptyset$. Then there exists $a \in X$ such that $\mu(a) \geq \alpha$. Since μ is a fuzzy pseudo CI-filter, by (pFF1) $\mu(1) \geq \mu(a) \geq \alpha$, we have $1 \in U(\mu, \alpha)$. Let $x, x \to y \in U(\mu, \alpha)$. Therefore $\mu(x) \geq \alpha$ and $\mu(x \to y) \geq \alpha$. It follows from $(pFF2)$ that $\mu(y) \geq \mu(x) \wedge \mu(x \to y) \geq \alpha$. So, $y \in U(\mu, \alpha)$. Hence $U(\mu, \alpha)$ is a pseudo CI-filter of X.*

Conversely, suppose that for each $\alpha \in [0, 1]$, $U(\mu, \alpha) = \emptyset$ or $U(\mu, \alpha)$ be a pseudo CI-filter of X . If (pFF1) does not hold, then there exists $a \in X$ such that $\mu(1) < \mu(a) = \beta$. Then $a \in U(\mu, \beta) \neq \emptyset$ and by assumption, $U(\mu, \beta)$ is a pseudo CI-filter of X. Hence $1 \in U(\mu, \beta)$ so, $\mu(1) \geq \beta$. This is a contradiction, and so (pFF1) holds.

Now, assume that (pFF2) is not satisfied. Then there are $a, b \in X$ such that $\mu(b) < \mu(a) \wedge \mu(a \to b)$. Taking $\beta = \frac{1}{2}(\mu(b) + (\mu(a) \wedge \mu(a \to b))$, we get $\mu(b) < \beta < \mu(a) \wedge \mu(a \to b)$.Thus $\beta < \mu(a)$ and $\beta < \mu(a \to b)$. Therefore, $a, a \to b \in U(\mu, \beta)$ but $b \notin U(\mu, \beta)$. This is impossible, and so μ is a fuzzy pseudo CI-filter.

Corollary 2.11. *If μ is a fuzzy pseudo CI-filter of X then:*
(a) *The set $X_a = \{x \in X : \mu(x) \geq \mu(a)\}$ is a pseudo CI-filter of X for all $a \in X$.*
(b) *The set $X_\mu = \{x \in X : \mu(x) = \mu(1)\}$ is a pseudo CI-filter of X.*

The following example shows that the converse of Corollary (2.11)(b) is not true in general.

Example 2.12. Let X be a pseudo CI- algebra. Define a fuzzy set μ in X by:

$$\mu(x) = \begin{cases} 0.1 & if\ x\ =\ 1 \\ 0.2 & otherwise. \end{cases}$$

Then $X_\mu = \{1\}$ and it is a pseudo CI-filter of X but μ is not a fuzzy pseudo CI-filter, since μ does not satisfy $(pFF1)$.

Corollary 2.13. *Let I be a nonempty set. Assume $\mu_i \in pFF(X)$ for $i \in I$. Then $\mu = \bigwedge\{\mu_i : i \in I\}$ is fuzzy pseudo CI-filters.*

Proof. *$\mu(1) = \bigwedge\{\mu_i(1) : i \in I\} \geq \bigwedge\{\mu_i(x) : i \in I\} = \mu(x)$, for all $x \in X$. Therefore $(pFF1)$ holds. Now, let $x, y \in X$. Since $\mu_i \in pFF(X)$, by $(pFF2)$, $\mu_i(y) \geq \mu_i(x \to y) \bigwedge \mu_i(x)$, for all $i \in I$. So, $\mu(x) = \bigwedge\{\mu_i(y) : i \in I\} \geq \bigwedge\{\mu_i(x \to y) \bigwedge \mu_i(x) :\ i \in I\} = \mu(x \to y) \bigwedge \mu(y)$. Consequently, $(pFF2)$ holds and therefore $\mu \in PFF(X)$.*

Example 2.14. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $\to$ " and" $\rightsquigarrow$ " on Xas follows:

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	c	d	1
b	b	a	1	1	1	e
c	c	a	1	1	1	e
d	d	a	1	1	1	e
e	e	a	d	d	d	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	b	c	d	1
b	b	a	1	1	1	e
c	c	a	1	1	1	e
d	d	a	1	1	1	e
e	e	a	c	c	d	1

By example (16)(4.11) $(X, \to, \rightsquigarrow, 1)$ ia a pseudo CI-algebra and;

$$F(X) = \{\{1\}, \{1, a\}, \{1, b\}, \{1, c\}, \{1, d\}, \{1, e\}, \{1, b, c, d\}, X\}$$

are pseudo CI-filter of X.

Let μ be a fuzzy set in X such that;
$\mu(1) = \alpha_1,\ \mu(a) = \alpha_2,\ \mu(b) = \alpha_3,\ \mu(c) = \mu(d) = \mu(e) = \alpha_4$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$ and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$.
Observe that μ is a fuzzy pseudo CI-filter of X. It is easy to check that for all $\alpha \in [0; 1]$, we have

$$U(\mu,\ \alpha) = \begin{cases} \phi & if\ \ \alpha >\ \alpha_1 \\ \{1\} & if\ \ \alpha_2 < \alpha \leq \alpha_1 \\ \{1, a\} & if\ \ \alpha_3 < \alpha \leq \alpha_2 \\ \{1, a, b, c\} & if\ \ \alpha_4 < \alpha \leq \alpha_3 \\ X & if\ \ \alpha \leq \alpha_4 \end{cases}$$

Since $\{1\}$, $\{1, a\}$, $\{1, a, b, c\}$ and X are pseudo CI-filter of X, from Theorem (2.10) conclude that μ is a fuzzy pseudo CI-filter of X.

From now, assume $X = (X; \rightarrow, \rightsquigarrow, 1)$ and $Y = (Y; \rightarrow^{'}, \rightsquigarrow^{'}, 1^{'})$ be pseudo CI-algebras.
The following theorems give the homomorphic properties of pseudo fuzzy CI-algebra.

Proposition 2.15. *Let $f : X \longrightarrow Y$ be a surjective homomorphism and $\mu \in pFF(Y)$. Then $f^{-1}(\mu) \in pFF(X)$.*

Proof. *Let $x \in X$. Since $f(x) \in Y$ and $\mu \in pFF(Y)$, we have $\mu(1) \geq \mu(f(x)) = (f^{-1}(\mu))(x)$, but $\mu(1) = \mu(f(1)) = (f^{-1}(\mu))(1)$. Thus we get $(f^{-1}(\mu))(1) \geq (f^{-1}(\mu))(x)$ for any $x \in X$, that is, $f^{-1}(\mu)$ satisfies (pFF1). Now let $x, y \in X$. Since $\mu \in pFF(Y)$, we have $\mu(f(y)) \geq \mu(f(x) \rightarrow f(y)) \wedge \mu(f(x)) = \mu(f(x \rightarrow y)) \wedge \mu(f(x))$ and hence $f^{-1}(\mu)(y) \geq f^{-1}(\mu)(x \rightarrow y) \wedge f^{-1}(\mu)(x)$. Consequently, $f^{-1}(\mu) \in pFF(X)$.*

Proposition 2.16. *Let $f : X \longrightarrow Y$ be a homomorphism and $\mu \in pFF(Y)$. If μ is constant on $Kerf = f^{-1}(1)$, then $f^{-1}(f(\mu)) = \mu$.*

Proof. *Let $x \in X$ and $f(x) = y$. Hence $(f^{-1}(f(\mu)))(x) = (f(\mu))(f(x)) = (f(\mu))(y) = \bigvee\{\mu(a) : a \in f^{-1}(y)\}$. For all $a \in f^{-1}(y)$, we have $f(a) = f(x)$. Then by $(pCI1)$, $f(a \rightarrow x) = f(a) \rightarrow f(x) = 1$. That is, $a \rightarrow x \in Kerf$. Thus $\mu(a \rightarrow x) = \mu(1)$. Therefore, $\mu(x) \geq \mu(a \rightarrow x) \wedge \mu(a) = \mu(1) \wedge \mu(a) = \mu(a)$. Similarly, $\mu(a) \geq \mu(x)$. Hence $\mu(x) = \mu(a)$. Thus $f^{-1}(f(\mu))(x) = \bigvee\{\mu(a) : a \in f^{-1}(y)\} = \mu(x)$, i.e., $f^{-1}(f(\mu)) = \mu$.*

Proposition 2.17. *Let X and Y be pseudo CI-algebras and let $f : X \longrightarrow Y$ be a homomorphism and $\mu \in pFF(Y)$ be such that $X_\mu \supseteq Kerf$. Then $f(\mu) \in pFF(Y)$.*

Proof. *Since μ is a fuzzy pseudo CI-filter of X and $1 \in f^{-1}(1)$, we have $(f(\mu))(1) = \bigvee\{\mu(a) : a \in f^{-1}(1)\} = \mu(1) \geq \mu(x)$ for any $x \in X$. Hence $(f(\mu))(1) \geq \bigvee\{\mu(x) : x \in f^{-1}(y)\} = (f(\mu))(y)$ for any $y \in Y$. Thus $f(\mu)$ satisfies (pFF1). Suppose that $f(\mu)(d) < f(\mu)(c \rightarrow d) \wedge f(\mu)(c)$ for some $c, d \in Y$. Since f is surjective, there are $a, b \in X$ such that $f(a) = c$ and $f(b) = d$. Hence $f(\mu)(f(b)) < f(\mu)(f(a \rightarrow b)) \wedge f(\mu)(f(a))$. Therefore $f^{-1}(f(\mu))(b) < f^{-1}(f(\mu))(a \rightarrow b) \wedge f^{-1}(f(\mu))(a))$. Since $X_\mu \supseteq Kerf$, μ is constant on $Kerf$. Hence, by Lemma (2.16), we get $\mu(b) < \mu(a \rightarrow b) \wedge \mu(a)$, which is a contradiction with the fact that μ is a fuzzy pseudo CI-filter. Thus $f(\mu) \in pFF(Y)$.*

3. Some fuzzy filter results

Definition 3.1. Let ν be a fuzzy set in X. A fuzzy pseudo CI-filter μ of X is said to be generated by ν if $\nu \leq \mu$ and for any pseudo fuzzy CI-filter ω of X, $\nu \leq \omega$ implies $\mu \leq \omega$.

The fuzzy pseudo CI-filter generated by ν will be denoted by $(\nu]$. The fuzzy pseudo CI-filter ν we can define equivalently as follows:

$$(\nu] = \bigwedge\{\omega : \omega \in pFF(X) \text{ and } \nu \leq \omega\}.$$

Lemma 3.2. *Let μ and ν be fuzzy sets of X. The following properties hold:*

(i) $\mu \leq \nu$ implies $(\mu] \leq (\nu]$,

(ii) If μ be a fuzzy pseudo CI-filter, then $(\mu] = \mu$.

proof. *(i) Since $\mu \leq \nu$, hence $\{\omega : \omega \in pFF(X), \mu \leq \omega\} \subseteq \{\omega : \omega \in pFF(X), \nu \leq \omega\}$. So $(\mu] = \bigwedge\{\omega : \omega \in pFF(X), \mu \leq \omega\} \leq \bigwedge\{\omega : \omega \in pFF(X), \nu \leq \omega\} = (\nu]$.*
(ii) From $\mu \in \{\omega : \omega \in pFF(X), \mu \leq \omega\}$,it follows $(\mu] = \mu$.

Proposition 3.3. *Let μ be a fuzzy set of a pseudo CI-algebra X. Then $(\mu](x) = \bigvee\{\mu(a_1) \wedge ... \wedge \mu(a_n) : (a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x))) = 1 \text{ where } a_1, ..., a_n \in X\}$.*

Proof. *Let $\nu(x) = \bigvee\{\mu(a_1) \wedge ... \wedge \mu(a_n) : (a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x))) = 1 \text{ and } a_1, ..., a_n \in X\}$.It is easy to see that $\nu(1) \geq \nu(x)$ for all $x \in X$. Now we prove that ν satisfies $(FF3)$. Suppose that $b \leq (a \rightarrow x)$, so $b \rightarrow (a \rightarrow x) = 1$, where $a, b, x \in X$. For every $k \in N$. By the definition of ν, we can select $a_1, ..., a_n, b_1, ..., b_m \in X$ such that,*

$(a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow a))) = 1$, and $\mu(a_1) \wedge ... \wedge \mu(a_n) > \nu(a) - \frac{1}{k}$,

$(b_m \rightarrow (b_{m-1} \rightarrow ... \rightarrow (b_1 \rightarrow b))) = 1$, and $\mu(b_1) \wedge ... \wedge \mu(b_n) > \nu(b) - \frac{1}{k}$.

From Lemma (1.12) it follows that $(b_m \rightarrow (b_{m-1} \rightarrow ... \rightarrow (b_1 \rightarrow a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x))))) = 1$. Therefor $\nu(x) \geq \mu(a_1) \wedge ... \wedge \mu(a_n \wedge \mu(b_1) \wedge ... \wedge \mu(b_m) > (\nu(a) - \frac{1}{k}) \wedge (\nu(b) - \frac{1}{k})$. Hence $\nu(x) \geq \nu(a) \wedge \nu(b)$. So ν is a fuzzy pseudo CI-filter.
Since $x \rightarrow x = 1$ for all $x \in X$. From this we see that $\mu(x) \leq \nu(x)$ for all $x \in X$. Thus $\mu \leq \nu$. By (3.2) $(\mu] \leq \nu$. Now, suppose ω be a fuzzy pseudo CI-filter of X such that $(\mu] \leq \omega$. Then for any $x \in X$ we obtain $\nu(x) = \bigvee\{\mu(a_1) \wedge ... \wedge \mu(a_n) : (a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x))) = 1, a_1, ..., a_n \in X\} \leq \bigvee\{\omega(a_1) \wedge ... \wedge \omega(a_n) : (a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x)) = 1, a_1, ..., a_n \in X\}$. If $a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x)) = 1$, by note in definition (2.1) $\omega(x) \geq (\omega(a_1) \wedge \omega(a_2) \wedge ... \wedge \omega(a_n))$, therefor $\bigvee\{\omega(a_1) \wedge ... \wedge \omega(a_n) : (a_n \rightarrow (a_{n-1} \rightarrow ... \rightarrow (a_1 \rightarrow x))) = 1 \text{ and } a_1, ..., a_n \in X\} \leq \omega(x)$ for all $x \in X$. Hence $\nu(x) \leq \omega(x)$. So ν is a fuzzy pseudo filter generated by μ, $(\mu] = \nu$.

In the sequel we need the notion of fuzzy points.

Definition 3.4. Let $a \in X$ and $s \in [0,\ 1]$. Define a_s be a fuzzy set in X as follows:

$$a_s(x) = \begin{cases} s & if\ x = a, \\ 0 & if\ x \neq a. \end{cases}$$

a_s is called a fuzzy point in X with value s at a.

Proposition 3.5. *Let μ be a fuzzy pseudo CI-filter in X. If $s, t \in [0,1]$ satisfies $s \geq \mu(a),\ t \geq \mu(b),\ s \wedge t \leq \mu(a \wedge b)$ where $a, b \in X$ then;*

$$(\mu \vee a_s] \wedge (\mu \vee b_t] = \mu$$

Proof. *It is obvious that $(\mu \vee a_s] \wedge (\mu \vee b_t] \geq \mu$, so we need to prove the converse inequality. Consider any fixed $x \in X$ and an arbitrary small number $\epsilon > 0$, it is sufficient to consider the following three cases.*
Case I. By (3.3) There are $a_1, ..., a_n \in X \setminus \{a\}$ such that,
$a_n \to (a_{n-1} \to ... \to (a_1 \to x))) = 1$, and, $(\mu \vee a_s](x) - \epsilon < \{(\mu \vee a_s](a_1) \wedge ... \wedge (\mu \vee a_s](a_n)$.
Since $(\mu \vee a_s](a_i) = \mu(a_i)$, for $i = 1, 2, ..., n$. we obtain $(\mu \vee a_s](x) < \mu(a_1) \wedge ... \wedge (\mu(a_n) + \epsilon \leq \mu(x) + \epsilon$. Hence $((\mu \vee a_s] \wedge (\mu \vee b_t])(x) < \mu(x) + \epsilon$. Therefor $((\mu \vee a_s] \wedge (\mu \vee b_t])(x) \leq \mu(x)$.

Case II. Again, by (3.3) There are $b_1, ..., b_m \in X \setminus \{b\}$ such that,

$$b_n \to (b_{n-1} \to ... \to (b_1 \to x))) = 1 \text{ and } (\mu \vee b_t](x) - \epsilon < \{(\mu \vee b_t](b_1) \wedge ... \wedge (\mu \vee b_t](b_n).$$

Since $(\mu \vee b_t](b_i) = \mu(b_i)$, for $i = 1, 2, ..., n$. we obtain $(\mu \vee b_t](x) < \mu(b_1) \wedge ... \wedge (\mu(b_n) + \epsilon \leq \mu(x) + \epsilon$. Hence $((\mu \vee b_t] \wedge (\mu \vee b_t])(x) < \mu(x) + \epsilon$. Therefor $((\mu \vee b_t] \wedge (\mu \vee b_t])(x) \leq \mu(x)$.

Case III. There are $a_1, ..., a_n \in X \setminus \{a\}$ and $b_1, ..., b_m \in X \setminus \{b\}$, such that:
$a_n \to (a_{n-1} \to ... \to (a_1 \to (a \to x))) = 1$, $(\mu \vee a_s](x) - \epsilon < (\mu \vee a_s](a_1) \wedge ... \wedge (\mu \vee a_s](a_n) = \mu(a_1) \wedge ... \wedge (\mu(a_n) \wedge s$.
and,
$b_n \to (b_{n-1} \to ... \to (b_1 \to (b \to x))) = 1$, $(\mu \vee b_t](x) - \epsilon < (\mu \vee b_t](b_1) \wedge ... \wedge (\mu \vee b_t](b_m) = \mu(b_1) \wedge ... \wedge (\mu(b_n) \wedge t$.
We have $b_n \to (b_{n-1} \to ... \to (b_1 \to a_n \to (a_{n-1} \to ... \to (a_1 \to ((a \wedge b) \to x))))) = 1$ and
$\mu(x) + \epsilon \geq \mu(a_1) \wedge ... \wedge (\mu(a_n) \wedge mu(b_1) \wedge ... \wedge (\mu(b_n) \wedge \mu(a \wedge b) + 2\epsilon \geq \mu(a_1) \wedge ... \wedge (\mu(a_n) \wedge mu(b_1) \wedge ... \wedge (\mu(b_n) \wedge (s \wedge t) + 2\epsilon = [\mu(a_1) \wedge ... \wedge \mu(a_n) \wedge s + \epsilon] \wedge [mu(b_1) \wedge ... \wedge \mu(b_n) \wedge t + \epsilon] = [(\mu \vee a_s](a_1) \wedge ... \wedge (\mu \vee a_s](a_n) \wedge (\mu \vee a_s](a) + \epsilon] \wedge [(\mu \vee b_t](b_1) \wedge ... \wedge (\mu \vee b_t](b_m) \wedge (\mu \vee b_t(b) + \epsilon]] \geq (\mu \vee a_s](x) \wedge (\mu \vee b_t](x)$.
Thus $(\mu \vee a_s] \wedge (\mu \vee b_t] \leq \mu$. So $(\mu \vee a_s] \wedge (\mu \vee b_t] = \mu$.

In the continuation of this section, we will study the Noetherian and Artinian pseudo CI-algebras and its relation with fuzzy pseudo CI-fuzzy filter.

Definition 3.6. A pseudo CI-algebra X is called Noetherian if for every ascending sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ of pseudo CI-filters of X, there exists $k \in N$ such that $F_n = F_k$ for all $n \geq k$.
A pseudo CI-algebra X is called Artinian if for every descending sequence $F_1 \supseteq F_2 \supseteq F_2 \supseteq \cdots$ of pseudo CI-filters of X, there exists $k \in N$ such that $F_n = F_k$ for all $n \geq k$.

Example 3.7. Let $(X, \leq)$ be a poset with a greatest element 1. For $x, y \in X$, define by:

$$x \to y = x \rightsquigarrow y = \begin{cases} 1 & if\ x \leq y, \\ y & otherwise. \end{cases}$$

Then $(X, \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra. For every $x \in X$ the subset $[x, 1] = \{y \in X : x \leq y\}$ is a CI-filter of X. For $X = Q \wedge (-\infty, 1]$ (Q is rational numbers) with ordinary $\leq$, then $(X, \leq)$ ia a poset and $(X, \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra. Since $[0,1] \subset [-1,1] \subset \cdots \subset [-n,1] \subset \cdots$ and $[0,1] \supset [\frac{1}{2},1] \supset [\frac{1}{3},1] \cdots \supset [\frac{1}{n},1] \cdots$. Hence the pseudo CI-algebra $(X, \to, \rightsquigarrow, 1))$ is not Noetherian and also not Artinian.

Lemma 3.8. *Let $F_1 \subset F_2 \subset F_3 \subset \cdots$ be a strictly ascending sequence of pseudo CI-filter of X and t_n be a strictly decreasing sequence in $(0,1)$. Let μ be the fuzzy set of X defined by;*

$$\mu(x) = \begin{cases} 0 & if\ x \notin F_n, \quad \forall n \in N, \\ t_n & if x \in F_n \setminus F_{n-1}, \quad n \in N. \end{cases}$$

($F_0 = \phi$). Then μ is a fuzzy CI-filter of X.

Proof. $F = \bigcup_{n\in} F_n$ *is a fuzzy CI-filter of* X. *Obviously,* $\mu(1) = t_1 \geq \mu(x)$ *for all* $x \in X$*,so , (FF1) holds. Now, let* $x, y \in X$. *We have two cases.*

Case 1: If $y \notin F$. *Then* $x \to y \notin F$ *or* $x \notin F$. *Therefore* $\mu(x \to y) \wedge \mu(x) = 0 = \mu(y)$.
Case 2: $y \in F_n \setminus F_{n-1}$ *for some* $n = 1, 2, ...$, *then* $y \notin F_{n-1}$, *therefore* $x \to y \notin F_{n-1}$ *or* $x \notin F_{n-1}$. *Hence* $t_n \geq \mu(x \to y)$ *or* $t_n \geq \mu(x)$. *Hence* $\mu(y) = t_n \geq \mu(x) \wedge \mu(x \to y)$. *Thus (pFF3) holds. Consequently* μ *is a fuzzy pseudo CI-filter of* X.

Proposition 3.9. *Let* X *be pseudo CI-algebra. The following statements are equivalent:*

(i) X *is Noetherian,*

(ii) *for each fuzzy pseudo CI-filter* μ *of* X, $Im(\mu) = \{\mu(x) : x \in X\}$ *is a well-ordered subset of* $[0, 1]$.

Proof.

$(i) \Rightarrow (ii)$ *Let* X *be Noetherian and* μ *a fuzzy pseudo CI-filter of* X *such that* $Im(\mu)$ *is not a well-ordered subset of* $[0, 1]$. *Then there exists a strictly decreasing sequence* $\{\mu(x_n)\}$, *where* $x_n \in X$. *Let* $U_n = U(\mu, \mu(x_n)) = \{x \in X : \mu(x) \geq \mu(x_n\}$ *for any* $n \in N$. *Then, by proposition* (2.10), U_n *is pseudo CI-filter of* X *for all* $n \in N$ *and* $U_1 \subset U_2 \subset U_3 \subset ...$ *is a strictly ascending sequence of pseudo CI-filter of* X. *This contradicts the assumption that* X *is Noetherian. Therefore* $Im(\mu)$ *is a well-ordered set for each fuzzy pseudo CI-filter* μ.

$(ii) \Rightarrow (i)$ *Assume that for each fuzzy pseudo CI-filter* μ *of* X, $Im(\mu) = \{\mu(x) : x \in X\}$ *is a well-ordered subset of* $[0, 1]$. *Suppose that* X *is not Noetherian. There exists a strictly ascending sequence* $F_1 \subset F_2 \subset F_3 \subset \cdots$ *of pseudo CI-filters of* X. *Let* μ *be a fuzzy set of* X *defined by;*

$$\mu(x) = \begin{cases} 0 & if \quad x \notin F_n, \quad \forall n \in N, \\ \frac{1}{n} & if x \in F_n \setminus F_{n-1}, \quad n \in N. \end{cases}$$

($F_0 = \phi$*). By Lemma* (3.8)*,*μ *is a fuzzy pseudo CI-filter, but* $Im(\mu)$ *is not a well-ordered set of* $(0, 1)$, *which is a contradiction.*

Corollary 3.10. *Let* X *be a pseudo CI-algebra. If for every fuzzy pseudo CI-filter* μ *of* X, $Im(\mu)$ *is a finite set, then* X *is Noetherian.*

Proposition 3.11. *Let* X *be a pseudo CI-algebra and let* $S = \{s_1, s_2, ...\} \cup \{0\}$, *where* s_n *is a strictly decreasing sequence in* $(0, 1)$. *Then the following conditions are equivalent:*

(i) *is Noetherian,*

(ii) *for each fuzzy pseudo CI-filter of* X, *if* $Im(\mu) \subseteq S$, *then there exists* $k \in N$ *such that* $Im(\mu) \subseteq \{s_1, s_2, ..., s_k\} \cup \{1\}$.

Proof. $(i) \Rightarrow (ii)$. *Let* X *be Noetherian. Assume that* μ *be fuzzy pseudo CI-filter of* X, *if* $Im(\mu) \subseteq S$*,. From Proposition* (3.9)*,* $Im(\mu)$ *is a well-ordered subset of [0,1]. Thus there exists* $k \in N$ *such that* $Im(\mu) \subseteq \{s_1, s_2, ..., s_k\} \cup \{1\}$.
$(ii) \Rightarrow (i)$. *Let* (ii) *be true. Assume that* X *is not Noetherian. Then there exists a strictly ascending sequence* $F_1 \subset F_2 \subset F_3 \subset$ *of pseudo CI-filters of* X. *Define a fuzzy set* μ *of* X *by,*

$$\mu(x) = \begin{cases} 0 & if\; x \notin F_n, \quad \forall n \in N, \\ \frac{1}{n} & if n \in F_n \setminus F_{n-1} \quad n \in N. \end{cases}$$

($F_0 = \phi$*). By Lemma* (3.8)*,*μ *is a fuzzy pseudo CI-filter, but* $Im(\mu)$ *is not a well-ordered set of* $(0, 1)$, *which is a contradiction. Thus* X *is Noetherian.*

Proposition 3.12. *Let* X *be a pseudo CI-algebra and let* $T = \{t_1, t_2, ...\} \cup \{0, 1\}$, *where* (t_n) *is a strictly increasing sequence in* $(0; 1)$. *Then the following conditions are equivalent:*
(i) X *is Artinian,*
(ii) *for each fuzzy pseudo CI-filter* μ *of* X, *if* $Im(\mu) \subseteq T$, *then there exists* $k \in N$ *such that* $Im(\mu) \subseteq \{t_1, t_2, ..., t_k\} \cup \{0, 1\}$.
Proof. $(i) \Rightarrow (ii)$, *Suppose that* $t_{k_1} < t_{k_2} < t_{k_3} < \cdot . \cdot \cdot \cdot$ *. is a strictly increasing sequence of elements of* $Im(\mu)$. *Let* $U_m = U(\mu, t_{k_m})$ *for* $m \in N$. *It is immediately seen that* $U_1 \supset U_2 \supset U_3 \supset \cdot . \cdot \cdot \cdot$ *. is a strictly descending sequence of pseudo CI-filters of* X. *This contradicts the assumption that* X *is Artinian, thus there is not a strictly increasing sequence of elements of* $Im(\mu)$.

$(ii) \Rightarrow (i)$, *Assume that* (ii) *is true. Suppose that* X *is not Artinian. Then there exists a strictly descending sequence* $F_1 \supset F_2 \supset F_3 \supset \cdots$ *of pseudo CI-filters of* X. *Let* μ *in* X *defined by;*

$$\mu(x) = \begin{cases} 0 & if \quad x \notin F_1, \\ t_n & if n \in F_n \setminus F_{n+1}, \quad n \in N, \\ 1 & if \quad x \in \bigcap\{F_n : n \in N\}. \end{cases}$$

.

μ is a fuzzy set of X and $\mu(1) = 1 \geq \mu(x)$ for every $x \in X$ so, $(pFF1)$ holds. Now, let $x, y \in X$. We have three cases.
Case 1. If $y \notin F_1$. Then $x \to y \notin F_1$ or $x \notin F_1$. Therefore $\mu(x \to y) \wedge \mu(x) = 0 = \mu(y)$.
Case 2. If $y \in F_n \setminus F_{n+1}$ for some $n = 1, 2, ...$, then $y \notin F_{n+1}$, therefore $x \to y \notin F_{n+1}$ or $x \notin F_{n+1}$. Hence $t_n \geq \mu(x \to y)$ or $t_n \geq \mu(x)$. Hence $\mu(y) = t_n \geq \mu(x) \wedge \mu(x \to y)$. Thus $(pFF3)$ holds.
case3. If $y \in \bigcap\{F_n : n \in N\}$, then $\mu(y) = 1 \geq \mu(x) \wedge \mu(x \to y)$. Consequently μ is a fuzzy pseudo CI-filter of X.

Corollary 3.13. *Let X be a pseudo CI-algebra. If, for every fuzzy pseudo CI-filter μ of X, $Im(\mu)$ is a finite set, then X is Artinian.*

The following example shows that the converse of Corollary (3.13) does not hold.

Example 3.14. Let p be a prime number. Set $X = \{z \in \mathcal{C} : z^{p^n} = 1 \ \ forsome \quad n \geq 0\}$. It is known that $(X, \cdot, 1)$ is the p-quasicyclic group. Define $x \to y = x^{-1}y$ and $x \rightsquigarrow y = xy^{-1}$ for all $x, y \in X$. By (1.9) $X = (X, \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra. Let $F_n = \{z \in \mathcal{C} : z^{p^n} = 1\}$ for $n \in N \cup \{0\}$. It follows easily seen that F is CI-filter of X if and only if $F = X$ or $F = F_n$ for some $n \geq 0$. We have $F_0 = \{1\} \subset F_1 \subset F_2 \subset .... \subset X$ and hence X is Artinian. Define μ by, $\mu(x) = \frac{1}{n+1}$ if $x \in F_n \setminus F_{n-1}$ for some $n \in N \cup \{0\}$. Where $F_{-1} = \phi$.

Since $X = \bigcup_{n \in N} F_n$, μ is a fuzzy set in X. By the proof of Proposition (3.8), μ is a fuzzy pseudo CI-filter. However, $Im(\mu) = \{\frac{1}{n} : n \in N\}$ is not a finite set.

4. Some fuzzy pseudo CI-filters

In this section some fuzzy pseudo CI-filters and the relationships between them and the fuzzy pseudo CI-filter are checked.

Definition 4.1. A fuzzy set μ in X is called a fuzzy pseudo q-CI-filter of X if it satisfies $(FF1)$ and for all $x, y, z \in X$.
$(qFF2)$ $\mu(x \to z) \geq \mu((x \rightsquigarrow y) \to z) \wedge \mu(y)$,
$(qFF3)$ $\mu(x \rightsquigarrow z) \geq \mu((x \to y) \rightsquigarrow z) \wedge \mu(y)$.

Example 4.2. Let $X = \{1, a, b, c\}$. Define the operations " $\to$ " and" $\rightsquigarrow$ " on Xas follows:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	1	1	b
c	1	1	1	1

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	a
c	1	1	1	1

$(X, \to, \rightsquigarrow, 1)$ ia a pseudo CI-algebra.

$$\mu(x) = \begin{cases} 0.7 & if\ x\ =\ 1 \\ 0.3 & x \in \{a, b, c\}. \end{cases}$$

is a fuzzy pseudo q-CI-filter of X.

Proposition 4.3. *Every fuzzy pseudo q-CI-filter μ of X for all $x, y \in X$ satisfies the following:*
$\mu(x \to z) \geq \mu((x \rightsquigarrow 1) \to z)$,
$\mu(x \rightsquigarrow z) \geq \mu((x \to 1) \rightsquigarrow z)$.

Proof. *Let $x, y \in X$. Putting $y := 1$ in $(qFF2)$ and using (FFl), we have,*

$$\mu(x \to z) \geq \mu((x \to 1) \rightsquigarrow z) \wedge \mu(1) = \mu((x \to 1) \rightsquigarrow z),$$

and if putting $y := 1$ in $(qFF3)$ by using (FFl), we have,

$$\mu(x \rightsquigarrow z) \geq \mu((x \rightsquigarrow 1) \to z) \wedge \mu(1) = \mu((x \rightsquigarrow 1) \to z).$$

This completes the proof.

Proposition 4.4. *Every fuzzy pseudo q-CI-filter of X is a fuzzy pseudo CI-filter of X.*
Proof. *Let μ be a fuzzy pseudo q-CI-filter of X . Taking $x = 1$ in $(qFF3)$ and using $(pCI2)$, we have,*

$$\mu(z) = \mu(1 \rightsquigarrow z) \geq \mu(1 \rightsquigarrow y) \to z)) \wedge \mu(y) = \mu(y \to z) \wedge \mu(y).$$

for all $x, y \in X$. Theefor μ is a fuzzy pseudo CI-filter of X.

In the example below, we show that the conversely of this proposition is not true

Example 4.5. In Example (2.2) μ is a fuzzy pseudo CI-filter. Since $\mu(a \to (b \rightsquigarrow 1) \wedge \mu(1) = \mu(1) \wedge \mu(1) = 0.7$, and $\mu(a \to b) = \mu(b) = 0.6$, so, $\mu(a \to b) = 0.6 < 0.7 = \mu(a \to (b \rightsquigarrow 1) \wedge \mu(1)$. Therefor μ is not fuzzy pseudo q-CI-filter.

We add a condition such that the conversely of (4.4) is true.

Proposition 4.6. *If a fuzzy pseudo CI-filter μ of X satisfies the following conditions:*
(i) $\mu(x \to (y \rightsquigarrow z)) \geq \mu((x \rightsquigarrow y) \to z)$,
(ii) $\mu(x \rightsquigarrow (y \to z)) \geq \mu((x \to y) \rightsquigarrow z)$.
for all $x, y, z \in X$, then μ is a fuzzy pseudo q-filter of X.

Proof. *Let μ be a fuzzy pseudo CI-filter of X that satisfies (i), (ii). For any $x, y, z \in X$, it follows from $(pFF3)$, $(pCI3)$ and (i) that $\mu(x \to z) \geq \mu(y \rightsquigarrow (x \to z)) \wedge \mu(y) = \mu(x \to (y \rightsquigarrow z)) \wedge \mu(y) \geq \mu((x \rightsquigarrow y) \to z) \wedge \mu(y)$. and $\mu(x \rightsquigarrow z) \geq \mu(y \to (x \rightsquigarrow z)) \wedge \mu(y) = \mu(x \rightsquigarrow (y \to z)) \wedge \mu(y) \geq \mu((x \to y) \rightsquigarrow z) \wedge \mu(y)$. Then μ is a fuzzy pseudo q-filter of X.*

Proposition 4.7. *Let μ be a fuzzy pseudo CI-filter of X which for all $x, y \in X$ satisfies:*

$$\mu(x \to y) \geq \mu(y) \quad and \quad \mu(x \rightsquigarrow y) \geq \mu(y). \qquad (*)$$

Then μ is a fuzzy pseudo q-CI-filter of X .
Proof. *For $x, y, z \in X$. Using $(pFF2)$ and $(*)$, we have,*

$$\mu(x \to z) \geq \mu(z) \geq \mu((x \rightsquigarrow y) \to z) \wedge \mu(x \rightsquigarrow y) \geq \mu((x \rightsquigarrow y) \to z) \wedge \mu(y),$$

and,

$$\mu(x \rightsquigarrow z) \geq \mu(z) \geq \mu((x \to y) \rightsquigarrow z) \wedge \mu(x \to y) \geq \mu((x \to y) \rightsquigarrow z) \wedge \mu(y).$$

that μ is a fuzzy pseudo q-CI-filter of X .

Definition 4.8. A fuzzy set μ is called fuzzy a-CI-filter of X if it satisfies $(pFF1)$ and for all $x, y, z \in X$;
$(aFF1)$ $\mu(x \to z) \geq \mu((z \rightsquigarrow 1) \to (y \rightsquigarrow x)) \wedge \mu(y)$,
$(aFF2)$ $\mu(x \rightsquigarrow z) \geq \mu((z \to 1) \rightsquigarrow (y \to x)) \wedge \mu(y)$.

Lemma 4.9. *Let μ be a fuzzy a-CI-filter of a X then for all $x, y, z \in X$ the following statements hold,*
(1) $\mu(x \to y) \geq \mu((y \rightsquigarrow 1) \to x)$, and $\mu(x \rightsquigarrow y) \geq \mu((y \to 1) \rightsquigarrow x)$.
(2) $\mu(x) = \mu(x \to 1) = \mu(x \rightsquigarrow 1)$.
(3) $\mu((x \to 1) \to x) = \mu((x \rightsquigarrow 1) \rightsquigarrow x) = \mu(1)$
proof. *(1) Put $z := y$ and $y := 1$ in definition* (4.8)*.*
(2) By definition (4.8)*, $(pCI2)$ and proposition* (16)*(2.8)(4) we have,*

$$\mu(x \to 1) \geq \mu((1 \rightsquigarrow 1) \to (1 \to x) \wedge \mu(1) = \mu(x) \wedge \mu(1) = \mu(x).$$

On the other hand,

$$\mu(x) = \mu(1 \rightsquigarrow x) \geq \mu((x \to 1) \rightsquigarrow (1 \to 1) \wedge \mu(1) = \mu((x \to 1) \rightsquigarrow 1) = \mu((x \to 1) \to 1) \geq \mu(x \to 1).$$

So $\mu(x) = \mu(x \to 1) = \mu(x \rightsquigarrow 1)$.
(3) From proposition (1.4) *(4) and $(aFF2)$,*

$$\mu((x \to 1) \to x) = \mu((x \rightsquigarrow 1) \to x) \geq \mu((x \rightsquigarrow 1) \to (x \rightsquigarrow 1)) = \mu(1).$$

By this and $(FF1)$, $\mu((x \to 1) \to x) = \mu(x)$. With similarly way $\mu((x \rightsquigarrow 1) \rightsquigarrow x) = \mu(1)$

Proposition 4.10. *Every fuzzy pseudo a-CI-filter of X is a fuzzy pseudo CI-filter of X.*
Proof. *Let μ be a fuzzy pseudo a-CI-filter of X, and let $x, y \in X$. Then by using lemma* (4.9)*(2) and definition* (4.8)*,*

$$\mu(x) = \mu(x \rightsquigarrow 1) \geq \mu((1 \to 1) \rightsquigarrow (y \to x)) \wedge \mu(y) = \mu((1) \rightsquigarrow (y \to x)) \wedge \mu(y) = \mu(y \to x)) \wedge \mu(y).$$

and

$$\mu(x) = \mu(x \to 1) \geq \mu((1 \rightsquigarrow 1) \to (y \rightsquigarrow x)) \wedge \mu(y) = \mu((1) \to (y \rightsquigarrow x)) \wedge \mu(y) = \mu(y \rightsquigarrow x)) \wedge \mu(y).$$

This means that μ is a fuzzy pseudo CI-filter of X.

Proposition 4.11. *Let X be transitive pseudo CI-algebra, μ be fuzzy pseudo CI-filter of X and for all $x, y, z \in X$;*

$$\mu(x \to y) \geq \mu((y \rightsquigarrow 1) \to x) \quad , \quad \mu(x \rightsquigarrow y) \geq \mu((y \to 1) \rightsquigarrow x).$$

Then μ is a fuzzy a-CI-filter of X. (See Proposition (4.9) *(1).)*

Proof. *Since X is transitive pseudo CI-algebra and by Proposition* (1.5) *(3),*

$$((y \to 1) \rightsquigarrow (z \to x)) \rightsquigarrow ((y \to 1) \rightsquigarrow x) \geq (z \to x) \rightsquigarrow x \geq z,$$
$$((y \rightsquigarrow 1) \to (z \rightsquigarrow x)) \to ((y \rightsquigarrow 1) \to x) \geq (z \rightsquigarrow x) \to x \geq z.$$

Since μ is a fuzzy CI-filter, by Definition (2.1)*;*

$$\mu((y \to 1) \rightsquigarrow x) \geq \mu((y \to 1) \rightsquigarrow (z \to x)) \wedge \mu(z),$$
$$\mu((y \rightsquigarrow 1) \to x) \geq \mu((y \rightsquigarrow 1) \to (z \rightsquigarrow x)) \wedge \mu(z).$$

By hypothesis,

$$\mu((x \rightsquigarrow y) \geq \mu((y \to 1) \rightsquigarrow x) \geq \mu((y \to 1) \rightsquigarrow (z \to x)) \wedge \mu(z),$$
$$\mu((x \to y) \geq \mu((y \rightsquigarrow 1) \to x) \geq \mu((y \rightsquigarrow 1) \to (z \rightsquigarrow x)) \wedge \mu(z).$$

Therefor μ is a fuzzy a-CI-filter.

Proposition 4.12. *If X be transitive pseudo CI-algebra and μ a fuzzy a-CI-filter of X then for all $x, y, z \in X$,*

$$\mu(y) \geq \mu(x) \wedge \mu(x \to y) \quad \textit{and} \quad \mu(y) \geq \mu(x) \wedge \mu(x \rightsquigarrow y).$$

Proof. *Since X is transitive pseudo CI-algebra therefor,*

$$(y \to 1) \leq ((x \to y) \to (x \to 1)).$$

From Proposition (2.3)*,*

$$\mu(y \to 1) \leq \mu((x \to y) \to (x \to 1)).$$

By Proposition (4.10) *μ is fuzzy pseudo CI-filter, hence by $pFF3$,*

$$\mu(x \to 1) \geq \mu((x \to y) \to (x \to 1)) \wedge \mu(x \to y) \geq \mu(y \to 1) \wedge \mu(x \to y).$$

Therefor $\mu(x) \geq \mu(y) \wedge \mu(x \to y)$.
In the same way the other relation is true.

Proposition 4.13. *Every fuzzy pseudo a-CI-filter μ of X satisfies the following assertions,*
(i) $\mu((x \rightsquigarrow z) \to y) \geq \mu((y \rightsquigarrow 1) \to (x \rightsquigarrow z))$,
(ii) $\mu((x \to z) \rightsquigarrow y) \geq \mu((y \to 1) \rightsquigarrow (x \to z))$.
for all $x, y, z \in X$.
Proof. *Let $x, y, z \in X$.*
(i) *It follows from $(aFF1)$ and $(FF1)$ that,*

$$\mu((x \rightsquigarrow z) \to y) \geq \mu((y \rightsquigarrow 1) \to (1 \to (x \rightsquigarrow z))) \wedge \mu(1) = \mu((y \rightsquigarrow 1) \to (x \rightsquigarrow z)).$$

(ii) *From $(aFF2)$ and $(FF1)$ it follows that,*

$$\mu((x \to z) \rightsquigarrow y) \geq \mu((y \to 1) \rightsquigarrow (1 \to (x \to z))) \wedge \mu(1) = \mu((y \to 1) \rightsquigarrow (x \to z)).$$

The following example shows that a fuzzy pseudo q-CI-filter may not be fuzzy pseudo a-CI-filter.

Example 4.14. In example (4.3) μ is a fuzzy pseudo q-CI-filter. $\mu((b \rightsquigarrow a) \to c) = \mu(1 \to c) = \mu(c) = 0.3$ and $\mu((c \rightsquigarrow 1) \to (b \to a)) = \mu(b \to a) = \mu(1) = 0.7$. This show that, $\mu((b \rightsquigarrow a) \to c) \ngeq \mu((c \rightsquigarrow 1) \to (b \to a))$, and Proposition (4.13) is not holds. Therefor μ is not a fuzzy pseudo a-CI-filter.

Proposition 4.15. *Let Let μ be a fuzzy q-CI-filter of a transitive pseudo CI-algebra X. If μ satisfy the condition $\mu((((x \to 1) \rightsquigarrow 1) \rightsquigarrow 1) \rightsquigarrow x) = \mu(1)$ for all $x \in X$. Then μ is a fuzzy a-CI-filter of X.*

Proof. *By Definition* (4.1) $(qFF2)$*, we have:*
$\mu(x \to y) \geq \mu((x \rightsquigarrow ((y \rightsquigarrow 1) \to x)) \to y) \wedge \mu((y \rightsquigarrow 1) \to x)$.
BY (1.1) $(PCI3)$ *and* $(pCI1)$*,*
$(x \rightsquigarrow ((y \rightsquigarrow 1) \to x)) \to y = ((y \rightsquigarrow 1) \to (x \to x)) \to y = ((y \rightsquigarrow 1) \to 1) \to y$.
By condition, $\mu((x \rightsquigarrow ((y \rightsquigarrow 1) \to x)) \to y) = \mu(((y \rightsquigarrow 1) \to 1) \to y) = \mu(1)$.
So, $\mu(x \to y) \geq \mu(1) \wedge \mu((y \rightsquigarrow 1) \to x) = \mu((y \rightsquigarrow 1) \to x)$. *By Proposition* (4.11) *μ is a fuzzy a-CI-filter of X.*

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