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Some Results on Pseudo CI-Filters

N. Shojaei*
Islamic Azad University, Kerman Branch

A. Borumand Saeid
Shahid Bahonar University of Kerman

Abstract. The aim of this paper is to introduce the notion of pseudo CI-filter in the pseudo CI-algebra and defined of congruence relation, transitive CI-algebra and pseudo CI-filters generated by a set. The Maximal pseudo-filter, prime pseudo CI-filter and closed pseudo CI-filter are defined, and we show that each Maximal pseudo CI-filter is a prime pseudo CI-filter. With an example, it shows that the inverse is not true. Define of homomorphism on CI-algebras is another aim of this article, and proves that every homomorphic image and preimage of a pseudo CI-filter is also a pseudo CI-filter. Furthermore, there are some relations between prime pseudo CI-filters, Maximal pseudo CI-filters and generated pseudo CI-filters. The notion of congruence relations on pseudo CI-algebras is one of the purposes of this article. There are some relations between pseudo CI-filters and pseudo congruence. Also, the transitive CI-algebra and generated pseudo CI-filter are introduced.

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Keywords and Phrases: Pseudo CI-algebra, pseudo CI-filter, congruence relation, transitive CI-algebra.

1 Introduction

In 1966, Y. Imai and K. Iséki [10] introduced the notion of a BCK-algebra. There exist several generalizations of BCK-algebras such as

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*Corresponding Author

BCI-algebras [11], BCH-algebras [9], BE-algebras [14], and CI-algebras [16]. J. Neggers, S. S. Ahn and H. S. Kim [18] introduced a new notion, called Q-algebra, which is a generalization of BCH, BCI, BCK-algebra, and generalized some theorems discussed in a BCI-algebra. D. Mundici [17] proved MV-algebras are categorically equivalent to bounded commutative BCK-algebra, and proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. The notion of BE-algebras was introduced by Kim and Kim [14]. Beside, BE-algebras were deeply studied by Ahn and So [1], Ahn et al. [14], Walendziak [21]. Also, Meng [16] gave a procedure to generate a filter by a subset in a transitive BE-algebra and gave some characterizations of Noetherian and Artinian BE-algebras. Meng [16] introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCK/BCI/BCH algebras and proved that in transitive BE-algebras, the notion of ideals is equivalent to one of filters. Furthermore, he discussed on relations between singular CI-algebras and Abelian groups [15, 16]. Y. B. Jun, H.S. Kim, J. Neggers [12] introduced the notion of a pseudo d-algebra as a generalization of the idea of a d-algebra. In 2001, G. Georgescu and A. Iorgulescu [8] introduced pseudo BCK-algebras as an extension of BCK-algebras. In 2008, W. A. Dudek and Y. B. Jun [6] defined pseudo BCI-algebras as a natural generalization of BCI-algebras and of pseudo BCI-algebras. G. Dymek studied p-semisimple pseudo BCI-algebras and periodic pseudo BCI-algebras [7]. R. A. Borzooei et al. [4] defined pseudo BE-algebras which are a generalization of BE-algebras. A. Walendziak introduced pseudo BCH-algebras and then investigated ideals in such algebras [22]. Borumand et al. [3, 4] introduced the notion of pseudo BE-algebras and defined distributive pseudo BE-algebras. Then, congruence relations to pseudo BE-algebras are studied in Rezaei and Borumand. et al. [20]. Recently, A. borumand. et.al. defined the class of pseudo CI-algebras and studied some of its subclasses [19].
In this paper, we introduce the notion of pseudo CI-filter in a pseudo CI-algebra. Some proposition equivalent to the definition of CI-filter were proved. The maximal pseudo CI-filter, prime pseudo CI-filter and closed pseudo CI-filter are defined and showed that each maximal pseudo CI-filter is a prime pseudo CI-filter and with an example it is shown that it conversely is not true. We introduce generated pseudo CI-filter and in a

transitive CI-algebra and show that for every set A of X, the generated pseudo CI-filter (A] is equal to $\{x \in X : \prod_{i=1}^{n}(a_i \to x) = 1, \exists a_1, , a_n \in A\}$. Also introduce a transitive CI-algebra is the purposes of this article. Furthermore, there are some relations between prime pseudo CI-filters Maximal pseudo CI-filters and generated pseudo CI-filters. Homomorphism between two pseudo CI-algebra is introduce and it is prove that every homomorphic image (under certain conditions) and preimage of a pseudo CI-filter is also a pseudo CI-filter. $A(x, y)$ as pseudo upper set of x and y is introduce and its relationship with the pseudo CI-filter is checked, and proved that every pseudo CI-filter is an union of pseudo upper sets.

The definition of congruence relations on pseudo CI-algebras is the another aim of this paper. There are some relations between pseudo CI-filters and congruence relation. In a transitive CI-algebra we created a congruence relation by a pseudo CI-filters, and apply the notion of congruence relations to pseudo CI– algebras and discuss on the quotient algebras via this congruence relations. With a congruence relation θ on a pseudo CI-algebra, the quotient set X/θ is defined and with two operation $\to$ and $\rightsquigarrow$, a pseudo CI-algebra is obtained.

2 Preliminaries

Definition 2.1. (See [18])*A Q-algebra is a nonempty set* X *with a constant* 0 *and a binary operation* ”$*$” *satisfying axioms:*

(i) $x * x = 0$,

(ii) $x * 0 = x$,

(iii) $(x * y) * z = (x * z) * y$ *for all* $x, y, z \in X$.

(See [15]) An algebra $X = (X; \to, 1)$ *of type* $(2, 0)$ *is called CI-algebra (dual Q-algebra), if for all* $x, y, z \in X$, *it satisfies the following axioms:*

$(CI1)$ $x \to x = 1$,

$(CI2)$ $1 \to x = x$,

$(CI3)$ $x \to (y \to z) = y \to (x \to z)$.

The relation ”$\leq$” *on* X *define by,* $x \leq y$ *if and only if* $x \to y = 1$.

Definition 2.2. (See [18])*An algebra $X = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ is called pseudo Q-algebra if, for all $x, y, z \in X$, it satisfies the following axioms:*

(psQ1) $x * x = x \diamond x = 0$,
(psQ2) $x * 0 = x \diamond 0 = x$,
(psQ3) $(x * y) \diamond z = (x \diamond z) * y$.

Example 2.3. See [18] Let $X = \{0, a, b, c\}$ be a set. Define the operations $*$ and $\diamond$ on X by the following tables:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	c	0	c
c	c	c	0	0

$\diamond$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	c	0	c
c	c	c	0	0

Then $(X; *, \diamond, 0)$ is a pseudo Q-algebra.

Definition 2.4. (see [13]) *An algebra $(X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called dual pseudo Q-algebra if, for all $x, y, z \in X$, it satisfies the following axioms:*

(dpsQ1) $x \to x = x \rightsquigarrow x = 1$,
(dpsQ2) $1 \to x = 1 \rightsquigarrow x = x$,
(dpsQ3) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.

A dual pseudo Q-algebra X is called a pseudo CI-algebra, if it satisfies the following condition:

(psCI4) $x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1$.

In a pseudo CI-algebra, we introduce a binary relation "$\leq$" by:

$$x \leq y \Leftrightarrow x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1, \text{ for all } x, y \in X.$$

Example 2.5. (See[13]) Let $X = \{1, a, b, c\}$ be a set. Define the operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	c	a
b	1	1	1	c
c	1	1	c	1

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	b	1	1	1
b	c	b	1	a
c	c	c	1	1

Then $X = (X; \to, \rightsquigarrow, 1)$ is a dual pseudo Q-algebra which is not a pseudo CI-algebra, because $b \to c = 1$ but $b \rightsquigarrow c = c \neq 1$.

A pseudo CI-algebra $(X; \to, \rightsquigarrow, 1)$ is said to be a pseudo BE-algebra, if for all $x \in X$;

$$x \to 1 = x \rightsquigarrow 1 = 1.$$

So,(See [19]) an algebra $X = (X; \to, \rightsquigarrow, 1)$ *of type* $(2; 2; 0)$ *is a pseudo CI-algebra if it satisfies;*

$(pCI1)$ $x \to x = x \rightsquigarrow x = 1,$

$(pCI2)$ $1 \to x = 1 \rightsquigarrow x = x,$

$(pCI3)$ $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z),$

$(pCI4)$ $x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1.$

If $(X; \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra satisfying $x \to y = x \rightsquigarrow y$; for all $x, y \in X$, then $(X; \to, \rightsquigarrow, 1)$ is a CI-algebra.

Definition 2.6. (See [19]) *Let a be an element of a pseudo CI-algebra* $X = (X; \to, \rightsquigarrow, 1)$. *An element a is said to be an atom in X if for any* $x \in X$, $a \to x = 1$ *(or* $a \rightsquigarrow x = 1$*) implies* $a = x$.

Example 2.7. Let $X = \{1, a, b, c, d\}$. Define the operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

Then X is a pseudo CI-algebra.

Proposition 2.8. *(See [19]) If* $X = (X; \to, \rightsquigarrow, 1)$ *is a pseudo CI-algebra, then for all* $x, y, z \in X$*, we have:*

(1) $x \leq (x \to y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \to y,$

(2) $x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z,$

(3) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1), \quad (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1),$

(4) $x \to 1 = x \rightsquigarrow 1$,
(5) $x \leq y$ *implies* $x \to 1 = y \to 1$.

Proposition 2.9. *In a pseudo CI-algebra X, for all $x, y \in X$, the following holds:*

(1) *If* $y \leq 1$ *then* $x \to (y \rightsquigarrow x) = 1$ *and* $x \rightsquigarrow (y \to x) = 1$,
(2) *If* $y \leq 1$ *then* $x \rightsquigarrow (y \rightsquigarrow x) = 1$ *and* $x \to (y \to x) = 1$,
(3) $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$ *and* $x \to ((x \to y) \rightsquigarrow y) = 1$,
(4) $(x \to 1) \to 1 = (x \to 1) \rightsquigarrow 1$ *and* $(x \rightsquigarrow 1) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to 1$.

Proof.

(1) *Let $x, y \in X$. By (pCI3), (pCI1), (pCI2) and $y \leq 1$ we have;*

$$x \to (y \rightsquigarrow x) = y \rightsquigarrow (x \to x) = y \rightsquigarrow 1 = 1,$$

and by similar way, we have;

$$x \rightsquigarrow (y \to x) = y \to (x \rightsquigarrow x) = y \to 1 = 1.$$

(2) *By (psCI4) and (1) the proof is clear.*

(3) *Let $x, y \in X$. By (pCI3) and (pCI1) we have;*

$$x \rightsquigarrow ((x \rightsquigarrow y) \to y) = (x \rightsquigarrow y) \to (x \rightsquigarrow y) = 1,$$

and

$$x \to ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow (x \to y) = 1.$$

(4) *By Proposition* (2.8),*(3) and* $(pCI1)$,

$$(x \to 1) \to 1 = (x \to 1) \rightsquigarrow (1 \rightsquigarrow 1) = (x \to 1) \rightsquigarrow 1,$$

and

$$(x \rightsquigarrow 1) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (1 \to 1) = (x \rightsquigarrow 1) \to 1.$$

□

Example 2.10. [19] Let $X = \{1, a, b, c, d, e\}$. Define the operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	b	e	d
b	b	d	1	e	a	c
c	d	b	e	1	c	a
d	c	e	a	d	1	b
e	e	c	d	a	b	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	d	e	b	c
b	b	c	1	a	e	d
c	d	e	a	1	c	b
d	c	b	e	d	1	a
e	e	d	c	b	a	1

Then $X = (X; \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra which is not a pseudo BE-algebra. Because $a \to 1 = a \neq 1$ and $a \rightsquigarrow 1 = a \neq 1$.

Proposition 2.11. *If X be a pseudo CI-algebra and $b, c \in X$ such that $b \to a = c \to a$ and $b \rightsquigarrow a = c \rightsquigarrow a$ for some $a \in X$ then;*

$$b \to 1 = c \to 1 \text{ and } b \rightsquigarrow 1 = c \rightsquigarrow 1.$$

Proof. *By (pCI3) and (pCI1) for any $a, b, c \in X$, we have;*

$$b \to 1 = b \to (a \rightsquigarrow a) = a \rightsquigarrow (b \to a) = a \rightsquigarrow (c \to a) = c \to (a \rightsquigarrow a) = c \to 1.$$

This complete the proof. □

Proposition 2.12. *(See [19]) Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo CI-algebra. If $a, b \in X$ are atoms in X, then the following are true:*
(1) $a = (a \to 1) \to 1$,
(2) *For any $x \in X$, $(a \to x) \rightsquigarrow x = a$, $(a \rightsquigarrow x) \to x = a$,*
(3) *For any $x \in X$, $(a \to x) \rightsquigarrow 1 = x \to a$, $(a \rightsquigarrow x) \to 1 = x \rightsquigarrow a$,*
(4) *For any $x \in X$, $x \to a = (a \to 1) \rightsquigarrow (x \to 1)$, $x \rightsquigarrow a = (a \rightsquigarrow 1) \to (x \rightsquigarrow 1)$.*

Example 2.13. Let $(G, \cdot)$ be a group with identity element e. Define the operations $\to$ and $\rightsquigarrow$ on G by, $a \to b = a^{-1}b$ and $a \rightsquigarrow b = ba^{-1}$ for all $a, b \in G$. Then $(G, \to, \rightsquigarrow, e)$ is a pseudo CI-algebra.

3 Main Results

In this section, we define the notion of pseudo CI-filter on pseudo CI-algebras. There are some relations between prime pseudo CI-filters, maximal pseudo CI-filters and generated pseudo CI-filters. We introduce $A(x, y)$ as pseudo upper set of x and y and we check its connection to the pseudo CI-filters.

In the following, let X denote a pseudo CI-algebra unless otherwise specified.

Definition 3.1. *A non-empty subset F of X is called a pseudo CI-filter of $X = (X; \rightarrow, \rightsquigarrow, 1)$ if it satisfies in the following axioms:*
$(F1)$ $1 \in F$,
$(F2)$ *If $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$, for all $x, y \in X$.*

A pseudo CI-filter F is proper if and only if $F \neq X$.

Proposition 3.2. *Let F be a pseudo CI-filter of X. If $x \in F$, $y \in X$ and $x \leq y$. Then $y \in F$.*
Proof. *Let $x \leq y$, so, by (F1) $x \rightarrow y = 1 \in F$. Since $x \in F$, by (F2) $y \in F$.* □

Proposition 3.3. *Let $F \subseteq X$ and $1 \in F$. F is a pseudo CI-filter if and only if,*
$(F3)$ *$x \in F$ and $x \rightsquigarrow y \in F$ implies $y \in F$, for all $x, y \in X$.*
Proof. *Let $x \in F$ and $x \rightsquigarrow y \in F$. By Proposition (2.8)(1), $x \leq ((x \rightsquigarrow y) \rightarrow y)$. Since $x \in F$, by (F2) $(x \rightsquigarrow y) \rightarrow y \in F$. By assumption $x \rightsquigarrow y \in F$. Hence $y \in F$.*
Conversely, let $x \in F$ and $x \rightarrow y \in F$. By Proposition (2.9)(3), $(x \rightsquigarrow ((x \rightarrow y) \rightsquigarrow y) = 1$. By assumption $1 \in F$, so $(x \rightsquigarrow ((x \rightarrow y) \rightsquigarrow y) \in F$. $x \in F$, by (F3) $(x \rightarrow y) \rightsquigarrow y) \in F$. By assumption $(x \rightarrow y) \in F$. Hence $y \in F$, therefore F is a pseudo CI-filter. □

Example 3.4. (i) $\{1\}$ and X are pseudo CI-filter of X, in which we call trivial pseudo CI-filter.

(ii) Let $X = (X; \rightarrow, \rightsquigarrow, 1)$ be pseudo CI-algebra in Example (2.7). Then $F_1 = \{1, a, c\}$, $F_2 = \{1, b, c\}$ and $F_3 = \{1, a\}$ are pseudo CI-filter on X. But $\{1, a, b\}$ is not pseudo CI-filter on X.

(iii) In example (2.13), let $G = (0, +\infty)$, and binary operation "$\cdot$" be the ordinary product on G. Then $(G, \cdot)$ is a group with identity 1. $(G, \rightarrow, \rightsquigarrow, 1)$ is pseudo CI-algebra. Let Q be rational numbers. Q is a pseudo CI-filter.

Definition 3.5. *A non-empty subset S of a pseudo CI-algebra X is said a subalgebra of X if $x \to y \in S$ and $x \rightsquigarrow y \in S$ for all $x, y \in S$.*

Proposition 3.6. *For any pseudo CI-algebra X, the set $K(X) = \{x \in X : x \leq 1\}$ is a pseudo subalgebra of X.*
Proof. *Let $x, y \in K(X)$. Then $x \leq 1$ and $y \leq 1$. Hence;*

$$x \to 1 = x \rightsquigarrow 1 = 1 \text{ and } y \to 1 = y \rightsquigarrow 1 = 1.$$

From, proposition (2.8),(3) we have;
$(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1) = 1 \rightsquigarrow 1 = 1$ *and* $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1) = 1 \to 1 = 1.$

So, $x \to y$, $x \rightsquigarrow y \in K(X)$. Thus $K(X)$ is a pseudo subalgebra of X. □

Note. Always any pseudo CI-filter of X is not a pseudo subalgebra. In example (2.10), $F = \{1, b, c\}$ ia a pseudo CI-filter in X. Since $brightarrowc = enotinF$, therefore F is not a pseudo subalgebra of X.

Proposition 3.7. *Let F be a subset of X containing 1. F is pseudo CI-filter of X if and only if:*
(i) $x \leq y \to z$ imply $z \in F$, for all $x, y \in F$ and $z \in X$.
(ii) $x \leq y \rightsquigarrow z$ imply $z \in F$, for all $x, y \in F$ and $z \in X$.
Proof.
(i) ($\Rightarrow$). Assume that F is a pseudo CI-filter of X and $x, y, z \in X$ such that $x, y \in F$ and $x \leq z \to y$. Then $x \rightsquigarrow (y \to z) = 1 \in F$, $x \in F$ so $y \to z \in F$. Since $y \in F$, therefore $z \in F$.

($\Leftarrow$). By assumption, $1 \in F$. Let $y \in X$ and $x, x \rightsquigarrow y \in F$. From proposition (2.9),(3) $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$. Hence $x \leq (x \rightsquigarrow y) \to y$. Since $x, x \rightsquigarrow y \in F$, therefore $y \in F$. By Proposition (3.3), F is a pseudo CI-filter.

(ii) It is proved in the same way (i). □

Proposition 3.8. *Let A be a pseudo CI-filter of a pseudo CI-algebra X. If B is a pseudo CI-filter of A, then B is a pseudo CI-filter of X.*
Proof. *Since B is a pseudo CI-filter of A, we have $1 \in B$. Let $x, x \to y \in B$ for some $y \in X$. If $y \in A$, then $y \in B$. Since B is a pseudo CI-filter of A. If $y \in X \setminus A$, since $x, x \to y \in B \subseteq A$ and, A is a pseudo CI-filter of X implies $x \in A$. This is impossible. Thus $y \in B$. Therefore B is a pseudo CI-filter of A.* □

Proposition 3.9. *If $\{F_i\}_{i\in I}$ be a family of pseudo CI-filters of X, then $\bigcap_{i\in I}\{F_i\}$ is a pseudo CI-filter of X, too.*

A Maximal pseudo CI-filter is a proper CI-filter such that it is not included in any other proper pseudo CI-filter. Denote by $F(X)$ the set of all pseudo CI-filters of X and $M(X)$ the set of all Maximal pseudo CI-filters of X.

Definition 3.10. *Let X be a pseudo CI-algebra. Then $P \in F(X)$ is called prime if and only if, for all $F_1, F_2 \in F(X)$such that $F_1 \cap F_2 \subseteq P$, implies $F_1 \subseteq P$ or $F_2 \subseteq P$.*
The set of all proper prime pseudo CI-filters of X is denoted by $P(X)$ and called the prime spectrum of X.

Example 3.11. Let $(X, \rightarrow, \rightsquigarrow, 1)$ be the pseudo CI-algebra from example (2.7). Then $F(X) = \{F_1 = \{1\},\ F_2 = \{1, a, c\},\ F_3 = \{1,\ b,\ c\},\ F_4 = \{1,\ \ a\},\ F_5 = X\}$ and $P(X) = M(X) = \{F_2,\ F_3\}$.

For element a of X, put $T(a) = \{x \in X : a \leq x\}$, which is called the terminal section of an element a. Since $a \in T(a)$, we can see that $T(a)$ is not an empty set.

Proposition 3.12. *For any element a of X, if, $1 \in T(a)$. Then $T(a)$ is a pseudo CI-filter if and only if one of the following implications hold:*
(i) For all $x, y, z \in X$, $z \leq x \rightarrow y$, and $z \leq x$ then $z \leq y$;
(ii) For all $x, y, z \in X$, $z \leq x \rightsquigarrow y$, and $z \leq x$ then $z \leq y$.

Proof.
(i) Let for each $a \in X$, $T(a)$ is a pseudo CI-filter of X. Assume that $x, y, z \in X$, such that $z \leq x \rightarrow y$ and $z \leq x$. Then $x \rightarrow y \in T(z)$ and $x \in T(z)$. By hypothesis $T(z)$ is pseudo CI-filter, so $y \in T(z)$. Therefore, $z \leq y$.
(ii) Is similarly way.
Conversely, let $a \in X$. By hypothesis $1 \in T(a)$. Let $x \rightarrow y \in T(a)$, and $x \in T(a)$, then, $a \leq x \rightarrow y$, and $a \leq x$. From hypothesis, $a \leq y$, thus $y \in T(a)$. Hence $T(a)$ is a pseudo CI-filter of X, for all $a \in X$.
□

Proposition 3.13. *Let F be a subalgebra of X. Then F is a pseudo CI-filter of X if and only if for all $x, y \in X$,*

$$x \in F \text{ and } y \in X \setminus F \text{ then } x \to y \in X \setminus F \text{ and } x \rightsquigarrow y \in X \setminus F. \quad (1)$$

Proof. *Let F be a pseudo CI-filter of X. We prove (1) by contradiction. Assume $x, y \in X$, $x \in F$ and $y \in X \setminus F$. If $x \to y \notin X \setminus F$, then $x \to y \in F$. $F \in F(X)$, so $y \in F$, which is a contradiction. Hence $x \to y \in X \setminus F$. Now, if $x \rightsquigarrow y \notin X \setminus F$, then $x \rightsquigarrow y \in F$, and so, $y \in F$, which is a contradiction. Hence $x \rightsquigarrow y \in X \setminus F$.*

Conversely, assume that (1) hold. Since $F \neq \emptyset$ is a subalgebra, there is $a \in F$. We have $a \to a = 1 \in F$. Let $x \in F$ and $x \to y \in F$. If $y \notin F$, then by (1) $x \to y \in X \setminus F$, which is a contradiction with $x \to y \in F$. Hence $y \in F$. Therefore, F is a pseudo CI-filter. □

Proposition 3.14. *Let F be a pseudo CI-filter of X. For all $x \in F$:*

$$(x \to 1) \rightsquigarrow 1 \in F \text{ and } (x \rightsquigarrow 1) \to 1 \in F.$$

Proposition 3.15. *Let F be a pseudo CI-filter of X and let $\tilde{F} = \{x \in X : (x \to 1) \rightsquigarrow 1, (x \rightsquigarrow 1) \to 1 \in F\}$. Then $\tilde{F}$ is a pseudo CI-filter of X and $F \subseteq \tilde{F}$.*

Proof. *Obviously, $1 \in \tilde{F}$. Let $x, y \in X$ such that $x, x \to y \in \tilde{F}$. By hypothesis*

$(x \to 1) \rightsquigarrow 1$, $(x \rightsquigarrow 1) \to 1 \in F$, and $((x \to y) \to 1)) \rightsquigarrow 1$, $((x \to y) \rightsquigarrow 1)) \to 1 \in F$.

Using Proposition (2.8),(3), we have;

$[((x \to 1) \rightsquigarrow 1)] \to [((y \rightsquigarrow 1) \to 1)] = ((x \to 1) \rightsquigarrow (y \rightsquigarrow 1)) \rightsquigarrow 1 = ((x \to y) \to 1) \rightsquigarrow 1 \in F$, and

$[((x \rightsquigarrow 1) \to 1)] \rightsquigarrow [((y \to 1) \rightsquigarrow 1)] = ((x \rightsquigarrow 1) \to (y \to 1)) \to 1 = ((x \rightsquigarrow y) \rightsquigarrow 1) \to 1 \in F$.

Since $(x \to 1) \rightsquigarrow 1$, $(x \rightsquigarrow 1) \to 1 \in F$. It follows from (F2) that $(y \to 1) \rightsquigarrow 1$, $(y \rightsquigarrow 1) \to 1 \in F$. Hence $y \in \tilde{F}$. Thus $\tilde{F}$ is a pseudo CI-filter of X. By Proposition (3.14), $F \subseteq \tilde{F}$. □

Definition 3.16. *Let $X = (X, \to, \rightsquigarrow, 1)$ and $Y = (Y, \to', \rightsquigarrow', 1')$ be pseudo CI-algebras. A mapping $f : X \longrightarrow Y$ is called a pseudo homomorphism if for all $x, y \in X$,*

$$f(x \to y) = f(x) \to' f(y) \quad and \quad f(x \rightsquigarrow y) = f(x) \rightsquigarrow' f(y).$$

From now, assume $X = (X; \to, \rightsquigarrow, 1)$ and $Y = (Y; \to', \rightsquigarrow', 1')$ be pseudo CI-algebras.
Let $f : X \to Y$ be a pseudo homomorphism from X to Y, we define $Ker(f) = \{x \in X : f(x) = 1'\}$. Since $f(1) = f(1 \to 1) = f(1) \to' f(1) = 1' \to' 1' = 1'$. So $1 \in Ker(f)$ and $Ker(f) \neq \phi$.

Proposition 3.17. *Let $f : X \longrightarrow Y$ be a pseudo homomorphism. Then,*
(i) If F is a pseudo CI-filter of Y, then $f^{-1}(F)$ is a pseudo CI-filter of X,
(ii) If f is surjective and G is a pseudo CI-filter of X containing $Ker(f)$, then $f(G)$ is a pseudo CI-filter of Y.
Proof.

(i) Assume that F is a pseudo CI-filter of X. From above $1 \in f^{-1}(F)$. Let x, $x \to y \in f^{-1}(F)$. It follows that $f(x)$, $f(x) \to f(y) = f(x \to y) \in F$. Since F is pseudo CI-filter and $f(x) \in F$, then $f(y) \in F$. Therefore, $y \in f^{-1}(F)$ and hence $f^{-1}(F)$ is a pseudo CI-filter of X.

(ii) From above and hypothesis $1' \in Ker(f) \subseteq f(G)$ so $1' \in f(G)$. Let $y \in Y$ and x, $x \to' y \in f(G)$. Since f is surjective, there exists $a \in X$ such that $f(a) = y$ and $\exists b, c \in G$, such that $f(b) = x$ and $f(c) = x \to' y$. We have;

$$f(c) = x \to' y = f(b) \to' f(a) = f(b \to a).$$

Thus $f(c \to (b \to a)) = 1'$ and hence $c \to (b \to a) \in Ker(f) \subseteq G$. Therefore, $c \to (b \to a) \in G$. Since $c \in G$, we see that $(b \to a) \in G$, since $b \in G$ so $a \in G$. Hence $y = f(a) \in f(G)$. Consequently, $f(G)$ is a pseudo CI-filter of Y. □

In the following example we show that if f is not surjective in proposition (3.16), (ii), $f(G)$ is not a pseudo CI-filter of Y.

Example 3.18. Let X be pseudo CI-algebra in Example (2.7), and $Y = \{1, a, b, c\}$. Define $\to$, $\rightsquigarrow$ as follow:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

Then $(Y;\to,\rightsquigarrow,1)$ is a pseudo CI-algebra. (Note, in the two pseudo CI-algebra are the operations with a symbol) Now, if we consider $f : Y \longrightarrow X$ the identity map, then f is a pseudo homomorphism and $f(Y) = Y$. Everyone can easily see that $Y = \{1, a, b, c\}$ is a trivial pseudo CI-filter of Y, but $f(Y)$ is not a pseudo CI-filter of X, because $a \to d = a \in f(Y)$ and $a \in f(Y)$ but $d \notin f(Y)$.

Let $f : X \longrightarrow Y$ be a pseudo homomorphism. Then $Im(f)$ is a pseudo subalgebra of Y. Obviously, $Im(f)$ is a non-empty set. If $y_1, y_2 \in Im(f)$, then there exist $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$, thus $y_1 \to y_2 = f(x_1) \to f(x_2) = f(x_1 \to x_2) \in Im(f)$. And $y_1 \rightsquigarrow y_2 = f(x_1) \rightsquigarrow f(x_2) = f(x_1 \rightsquigarrow x_2) \in Im(f)$. Consequently, $Im(f)$ is a pseudo subalgebra of Y.

This is an example that shows ”a pseudo subalgebra is may not be pseudo CI-filter”.

Corollary 3.19. *Let $f : X \longrightarrow Y$ be a pseudo homomorphism. Then $Ker(f)$ is a pseudo CI-filter of X.*

Definition 3.20. *Let $x, y \in X$. Define $A(x,\ y) = \{z \in X :\ x \to (y \rightsquigarrow z) = 1\}$. We call $A(x,\ y)$ a pseudo upper set of x and y.*

Proposition 3.21. *In a pseudo CI-algebra $X = (X;\to,\ \rightsquigarrow,\ 1)$, for all $x, y \in X$ we have:*
(i) If $y \leq 1$, then $A(x,\ 1) \subseteq A(x,\ y)$, for all $x \in X$,
(ii) if $A(x,\ 1)$ be a pseudo CI-filter of X and $y \in A(x,\ 1)$, then $A(x,\ y) \subseteq A(x,\ 1)$.

Proof.

(i) Let $z \in A(x,\ 1)$. Then $x \to (1 \rightsquigarrow z) = x \to z = 1$. Thus, by (pCI3) and $y \leq 1$, $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z) = y \rightsquigarrow 1 = 1$. Thus $z \in A(x,\ y)$ and so $A(x,\ 1) \subseteq A(x,\ y)$.
(ii) Let $A(x,\ 1)$ be a pseudo CI-filter, $y \in A(x,\ 1)$. Let $z \in A(x,\ y)$,

then $x \to (y \rightsquigarrow z) = 1$. *Since* $x \in A(x,\ 1)$ *and* $A(x,\ 1)$ *is a pseudo CI-filter, therefore* $y \rightsquigarrow z \in A(x,\ 1)$. *Now, since* $y \in A(x,\ 1)$ *then* $z \in A(x,\ 1)$. *Thus* $A(x,\ y) \subseteq A(x,\ 1)$. □

Note: In example (2.10).

(*i*) $A(x, 1) = \{z :\ x \to (1 \rightsquigarrow z = 1\} = \{z :\ x \to z = 1 =\} = \{1\}$. For all $x \in X$.

(*ii*) $A\,(a,\ b) = \{c\}$ and $A\,(b,\ a) = \{d\}$. This shows in general $A\,(a,\ b) \neq A\,(b,\ a)$ and $1 \notin A\,(a,\ b)$.

(*iii*) The set $A(x,\ y)$, where $x,\ y \in X$, is not a pseudo CI-filter of X, in general.

Proposition 3.22. *Let* $x \in X$, *set* $T = \{y : y \leq 1\}$. *Then* $A\,(x,\ 1) = \bigcap_{y \in T} A(x,\ y)$.

Proof. *By Proposition* (3.21), (*i*), *we have* $A(x,\ 1) \subseteq A(x,\ y)$, *for all* $y \in T$. *So* $A\,(x,\ 1) \subseteq \bigcap_{y\in T} A(x,\ y)$. *Let* $z \in \bigcap_{y\in T} A(x,\ y)$. *Then* $z \in A(x,\ y)$, *for all* $y \in T$, *and* $1 \in T$ *so* $z \in A(x,\ 1)$. *Hence* $A\,(x,\ 1) \subseteq \bigcap_{y\in T} A(x,\ y) \subseteq A(x,\ 1)$. □

Proposition 3.23. *Let* F *is a pseudo CI-filter of* X. *The following holds:*

(*i*) $A(x,\ y) \subseteq F$, *for all* $x, y \in F$,

(*ii*) $F = \bigcup_{x,y\in F} A(x,\ y)$,

(*iii*) $F = \bigcup_{x\in F} A(x,\ 1)$.

Proof. *Let* F *be a pseudo CI-filter of* X;

(*i*) *Assume* $x, y \in F$. *If* $z \in A(x,\ y)$, *then* $x \to (y \rightsquigarrow z) = 1 \in F$. *Since* F *is a pseudo CI-filter and* $x \in F$, *by using (F2),* $y \rightsquigarrow z \in F$. $y \in F$ *again by (F2),* $z \in F$. *Hence* $A(x,\ y) \subseteq F$.

(*ii*) *Let* $z \in F$. *Since* $z \to (1 \rightsquigarrow z) = z \to z = 1$, *we have* $z \in A(z,\ 1)$. *Hence* $F \subseteq \bigcup_{x,y\in F} A(x,\ y)$. *Let* $z \in \bigcup_{x,y\in F} A(x,\ y)$, *then there exist* $a, b \in F$ *such that* $z \in A(a,\ b)$. *By* (*i*), $A(a,\ b) \subseteq F$, *we have* $z \in F$. *This means that* $\bigcup_{x,y\in F} A(x,\ y) \subseteq\ F$.

(*iii*) *Since* $1 \in F$, *put* $y = 1$ *in equation* $F = \bigcup_{x,y\in F} A(x,\ y)$ *of* (*ii*) *it is proved.*

If $y \in F$. *Since* $y \to (1 \rightsquigarrow y) = y \to y = 1 \in F$, *we have,* $y \in A(y,\ 1)$. *Hence* $F \subseteq \bigcup_{x\in F} A(x,\ 1)$. *Now, let* $y \in \bigcup_{x\in F} A(x,\ 1)$, *then there exist an* $a \in F$ *such that* $y \in\ A(a,\ 1)$, *which means that*

$a \to y = a \to (1 \rightsquigarrow y) = 1 \in F$. Since F is a pseudo CI-filter of X and $a \in F$, we see that $y \in F$. This means that $\bigcup_{x\in F} A(x,\ 1) \subseteq\ F$. □

4 Congruences Relations

We define the notion of congruence relations on pseudo CI-algebras. There are some relations between pseudo CI-filters and pseudo congruence.

In this section, let X denote a pseudo CI-algebra unless otherwise specified.

Definition 4.1. *Let "θ" be an equivalence relation on X, "θ" is called:*

(i) Right congruence relation on X if $(x, y) \in \theta$ implies $(x \to z,\ y \to z) \in \theta$ and $(x \rightsquigarrow z,\ y \rightsquigarrow z) \in \theta$, for all $z \in X$.

(ii) Left congruence relation on X if $(x, y) \in \theta$ implies $(w \to x, w \to y) \in \theta$ and $(w \rightsquigarrow x, w \rightsquigarrow y) \in \theta$, for all $w \in X$.

(iii) Congruence relation on X if has the substitution property with respect to "$\to$" and "$\rightsquigarrow$",i.e., for any $(x, y), (z, w) \in \theta$ then $(x \to z,\ y \to w) \in \theta$ and $(x \rightsquigarrow z, y \rightsquigarrow w) \in \theta$.

Example 4.2. (i) It is obvious that $X\ \times X$ and $\triangle = \{(x, x) :\ x \in X\}$ is a congruence relation on X.
(ii) Let $X = \{1, a, b, c\}$ and operations "$\to$" and "$\rightsquigarrow$" defined as follows:

$\to$	1	a	b	c
1	1	a	b	c
a	a	1	c	1
b	a	1	1	1
c	1	a	b	1

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	a	1	a	1
b	a	1	1	1
c	1	a	a	1

Then $X = \{X; \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra (is not BL-algebra)

$\theta_1 = \{(1, 1), (a, a), (b, b), (c, c), (1, b), (b, 1), (b, c), (c, b)\}$ is a left congruence relation on X, but it is not right congruence relation because $(c, b) \in \theta_1$ but $(c \to a,\ b \to a) = (a, 1) \notin \theta_1$.

$\theta_2 = \{(1, 1), (a, a), (b, b), (c, c), (c, b), (b, c)\}$, then θ_2 is a right congruence relation. Since $(b, c) \in \theta_2$ and $a \in X$, but $(a, 1) = (a \to b, a \to c) \notin \theta_2$, it follows that θ_2 is not a left congruence neither a congruence relation.

In this section we assume that, θ be congruence relation on X. we will denote $C_x(\theta) = \{y \in X : \ y \sim_\theta x\}$, abbreviated by C_x and $X/\theta = \{C_x : x \in X\}$.

Proposition 4.3. *Let A be the set of all atom element of X. Then $C_1 = \{x \in A : \ x \sim_\theta 1\}$ is a pseudo CI-filter of X.*
Proof. *Since θ is a reflexive relation, we see that $(1,1) \in \theta$ and so $1 \sim_\theta 1$. Thus $1 \in C_1$. Now, let $x \in X$. Assume that $a \in C_1$, $a \to x \in C_1$. Then $a \sim_\theta 1$ and $a \to x \sim_\theta 1$. θ is congruence relation on X, we have $((a \to x) \rightsquigarrow x) \sim_\theta (1 \rightsquigarrow x)$. a is atom element, by proposition (2.12),(2) and (pCI2), we have $a = ((a \to x) \rightsquigarrow x) \sim_\theta (1 \rightsquigarrow x) = x$. Since $a \in C_1$, therefor $1 \sim_\theta a \sim_\theta x$ and so $x \in C_1$. This shows that C_1 is a pseudo CI-filter of X.* □

Definition 4.4. *A pseudo CI-algebra X is said to be transitive if for all $x, y, z \in X$,*
$(y \to z) \to [(x \to y) \to (x \to z)] = 1, (y \rightsquigarrow z) \rightsquigarrow [(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)] = 1.$

Proposition 4.5. *Let $X = \{X; \to, \rightsquigarrow, 1)$ be transitive pseudo CI-algebra and F be a pseudo CI-filter of X. Define $x \sim_F y$ if and only if $x \to y,\ y \to x \in F$ and $x \rightsquigarrow y,\ y \rightsquigarrow x \in F$. Then $\sim_F$ is congruence relation on X.*
Proof. *Since $1 \in F$, we have $x \to x = 1 \in F$, i.e., $x \sim_F x$. This means that" $\sim_F$ " is reflexive.*

It is obvious $\sim_F$ is symmetric.

Now, if $x \sim_F y$ and $y \sim_F z$, then $x \to y,\ y \to x,\ x \rightsquigarrow y,\ y \rightsquigarrow x \in F$ and $y \to z, z \to y,\ y \rightsquigarrow z,\ z \rightsquigarrow y \in F$. Since X is transitive CI-algebra, so $(y \to z) \leq (x \to y) \to (x \to z)$ and $(y \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$. By hypothesis $y \to z,\ y \rightsquigarrow z \in F$. Since F is a pseudo CI-filter, it follows that $(x \to y) \to (x \to z),\ (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \in F$. Because $(x \to y),\ (x \rightsquigarrow y) \in F$ and F is a pseudo CI-filter, so $x \to z,\ x \rightsquigarrow z \in F$. By a similar way we see that $z \to x,\ z \rightsquigarrow x \in F$. This shows that " $\sim_F$ " is transitive. Therefore, " $\sim_F$ " is an equivalence relation on X.

Now, let $x \in X$ and $u \sim_F v$. Then by definition (4.4), $v \to u \leq (x \to v) \to (x \to u)$. Now, since $v \to u \in F$ and F is a pseudo CI-filter, then $(x \to v) \to (x \to u) \in F$. By a similar way, $(x \to u) \to (x \to v) \in F$. Therefore, $x \to v \sim_F x \to u$. Also, $u \rightsquigarrow v \leq (x \rightsquigarrow u) \rightsquigarrow (x \rightsquigarrow v)$.

Since $u \rightsquigarrow v \in F$ *and* F *is a pseudo CI-filter, we have* $(x \rightsquigarrow u) \rightsquigarrow (x \rightsquigarrow v) \in F$. *By a similar way,* $(x \rightsquigarrow v) \rightsquigarrow (x \rightsquigarrow u) \in F$. *Therefore,* $x \rightsquigarrow v \sim_F x \rightsquigarrow u$. *Therefore,* $\sim_F$ *is congruence relation on* X. □

Example 4.6. Let $X = \{1, a, b, c, d\}$ and $(X, \rightarrow, \rightsquigarrow, 1)$ be pseudo CI-algebra of example (2.7), $F = \{1, a, d\}$ is a pseudo CI-filter. We have,
$\sim_F = \{(1, 1), (a, a), (b, b), (c, c), (d, d), (1, a), (a, 1), (d, 1), (1, d), (a, d), (d, a), (b, c), (c, b)\}$.

Therefore "$\sim_F$" is equivalence relation and congruence relation.

Proposition 4.7. *If CI-algebra* X *be transitive, then for all* $x, y, z \in X$,

(*i*) $y \leq z$ *implies* $x \rightarrow y \leq x \rightarrow z$,

(*ii*) $y \leq z$ *implies* $z \rightarrow x \leq y \rightarrow x$,

(*iii*) $1 \leq x$ *implies* $x = 1$.

Proof.

(*i*) *If* $y \leq z$ *(Or* $y \rightarrow z = 1$*), then by (pCI2) and transitive* X *we have;*
$(x \rightarrow y) \rightarrow (x \rightarrow z) = 1 \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = (y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
So, $x \rightarrow y \leq x \rightarrow z$.

(*ii*) *Let* $y \leq z$. *Since* $(z \rightarrow x) \rightarrow (y \rightarrow x) = (z \rightarrow x) \rightarrow [1 \rightarrow (y \rightarrow x)] = (z \rightarrow x) \rightarrow [(y \rightarrow z) \rightarrow (y \rightarrow x)] = 1$ *and* (*ii*) *holds.*

(*iii*) *Follows from (pCI2). (Note that,* (*iii*) *holds in every CI-algebra.)*
□

Definition 4.8. *Let* A *be a nonempty subset of* X. *If* F *is the least pseudo CI-filter containing* A *in* X, *then* F *is said to be the pseudo CI-filter generated by* A *and is denoted by* $(A]$.

If A *is a finite set of* X *then* $(A]$ *is said to be finitely generated.*

Since X is always a pseudo CI-filter of X containing any pseudo CI-filter, it follows from proposition (4.7), that definition (4.8), is well defined.

The pseudo CI-filter generated by finite set $\{a_1, , a_n\}$ is simply denoted by $(a_1, , a_n]$. For convenience, let $(\phi] = \{1\}$.

Proposition 4.9. *Suppose* A *and* B *are two subsets of* X. *Then the following hold:*

(i) $(1] = \{1\}, (X] = X$,
(ii) $A \subseteq B$ *implies* $(A] \subseteq (B]$,
(iii) If F *is a pseudo CI-filter of* X, *then* $(F] = F$.

Proposition 4.10. *it Let* A *be a nonempty subset of* X. *Then* $(A] = \{x \in X : \prod_{i=1}^{n}(a_i \to x) = 1, \exists a_1, , a_n \in A\}$.

Note: $(\prod_{i=1}^{n}(a_i \to x) = (a_1 \to ... \to (a_{n-1} \to (a_n \to x)))$.

Proof. *Let* $F = \{x \in X : \prod_{i=1}^{n}(a_i \to x) = 1, \exists a_1, , a_n \in A\}$. *Since* $a \to a = 1$ *for all* $a \in A$, *so* $A \subseteq F$. *By proposition* (4.8) *(ii), (iii),* $(A] \subseteq F$.

Now, by (pCI1) $1 \to 1 = 1$, *so* $1 \in F$. *Let* $x \to y \in F$ *and* $x \in F$, *then there are* $a_1, , a_n, b_1, , b_m \in A$ *such that,* $\prod_{i=1}^{n}(a_i \to (x \to y)) = 1$ *and* $\prod_{i=1}^{m}(b_i \to x) = 1$. *Hence* $x \leq \prod_{i=1}^{n}(a_i \to y)$. *Since* X *is a transitive CI-algebra, by using Proposition* (4.6)*,(1) we obtain* $1 = \prod_{i=1}^{n}(b_i \to x) \leq \prod_{i=1}^{m} b_i \to (\prod_{i=1}^{n}(a_i \to y))$. *By Proposition* (4.6)*,(3)*
$\prod_{i=1}^{m} b_i \to (\prod_{i=1}^{n}(a_i \to y)) = 1$. *So* $y \in F$. *Therefore* F *is a Pseudo CI-filter.*
□

Example 4.11. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $\to$ " and" $\rightsquigarrow$ " on Xas follows:

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	c	d	1
b	b	a	1	1	1	e
c	c	a	1	1	1	e
d	d	a	1	1	1	e
e	e	a	d	d	d	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	b	c	d	1
b	b	a	1	1	1	e
c	c	a	1	1	1	e
d	d	a	1	1	1	e
e	e	a	c	c	d	1

Then $(X, \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra and,

$$F(X) = \{\{1\}, \{1, a\}, \{1, b\}, \{1, c\}, \{1, d\}, \{1, e\}, \{1, b, c, d\}, X\}$$

is the set of pseudo CI-filter of X. Also, $(a, b] = X$ and $(b, c] = \{1, b, c, d\}$.

Proposition 4.12. *Let P be a proper filter of X. P is a prime pseudo CI-filter of X if and only if $(x] \cap (y] \subseteq P$ implies $x \in P$ or $y \in P$, for all $x, y \in X$.*

Proof. $(\Rightarrow)$ *Assume that P be a prime pseudo CI-filter of X and $x, y \in X$ such that $(x] \cap (y] \subseteq P$. Since P is a prime CI-fitter, we get $x \in (x] \subseteq P$ or $y \in (y] \subseteq P$.*

$(\Leftarrow)$ *Assume that F_1 and F_2 are pseudo CI-filter of X such that $F_1 \cap F_2 \subseteq P$. Let $x \in F_1$ and $y \in F_2$. Hence,$(x] \subseteq F_1$ and $(y] \subseteq F_2$. Then, $(x] \cap (y] \subseteq F_1 \cap F_2 \subseteq P$, and so $x \in P$ or $y \in P$. Thus, $F_1 \subseteq P$ or $F_2 \subseteq P$. Therefore, P is a prime pseudo CI-filter of X.* □

Proposition 4.13. *Let $F \in F(X)$ and $F \neq X$. Then the following statement are equivalent:*
(i) F is a Maximal pseudo CI-filter,
(ii) If $x \in X$, $x \notin F$ then $(F \cup \{x\}) = X$; for all $x \in X$.

Proof. *$(i) \Rightarrow (ii)$. We have $F \subseteq (F \cup \{x\}]$, since $x \notin X$, $F \neq (F \cup \{x\}]$. F is Maximal it follows that $((F \cup \{x\}] = X$.*
$(ii) \Rightarrow (i)$ Suppose that there is a proper pseudo CI-filter G of X such that $F \subseteq G$. Then there is $x \in G \setminus F$. By (ii), we have $(F \cup \{x\}] = X$. Since $(F \cup \{x\}] \subseteq G$, it follows that $G = X$. □

Proposition 4.14. *If M is Maximal pseudo CI-filter of X, then M is prime pseudo CI-filter of X.*

Proof. *Let M be Maximal pseudo CI-filter of X and $(x] \cap (y] \subseteq M$, for some $x, y \in X$. Suppose that $x \notin M$ and $y \notin M$. Then,$(M \cup \{x\}] = X$and $(M \cup \{y\}] = X$. Hence, $(M \cup \{x\}] \cap (M \cup \{y\}] = X$. So, $(x] \cap (y] \nsubseteq M$, which is a contradiction. Then, $x \in M$ or $y \in M$. Therefore, M is prime pseudo CI-filter of X.* □

The following example shows that any prime filter may not be a Maximal filter.

Example 4.15. Let X be pseudo CI-algebra in example (4.2), (ii). $F_1 = \{1, a\}$ is a prime pseudo CI-filter, who that not Maximal, because $F_2 = \{1, a, c\}$ is another pseudo CI-filter and $F_1 \subseteq F_2 \neq X$.

Proposition 4.16. *Let $f : X \longrightarrow Y$ be a homomorphism. Then f has the isotonic property, i.e., if $x \leq y$, then $f(x) \leq f(y)$, for all $x, y \in X$.*
Proof. *If $x \leq y$, Then $x \to y = x \rightsquigarrow y = 1$. So, $f(x) \to f(y) = f(x \to y) = f(1) = 1^{'}$, and $f(x) \rightsquigarrow f(y) = f(x \rightsquigarrow y) = f(1) = 1^{'}$. Hence $f(x) \leq f(y)$. Therefore, f has the isotonic property.* □

In the rest of this section, we assume that θ be a congruence relations on X.

Proposition 4.17. *Define operations "$\to$" and "$\rightsquigarrow$" on "X/θ" by $C_x \to C_y = C_{x\to y}$ and $C_x \rightsquigarrow C_y = C_{x\rightsquigarrow y}$. Then $(X/\theta; \to, \rightsquigarrow, C_1)$ is a pseudo CI-algebra.*
Proof. *If $C_x, C_y, C_z \in X/\theta$, then we have:*
(pCI1) $C_x \to C_x = C_{x\to x} = C_1$ and $C_x \rightsquigarrow C_x = C_{x\rightsquigarrow x} = C_1$,
(pCI2) $C_1 \to C_x = C_{1\to x} = C_x$ and $C_1 \rightsquigarrow C_x = C_{1\rightsquigarrow x} = C_x$,
(pCI3) $C_x \to (C_y \rightsquigarrow C_z) = C_x \to C_{y\rightsquigarrow z} = C_{x\to(y\rightsquigarrow z)} = C_{y\rightsquigarrow(x\to z)} = C_y \rightsquigarrow C_{x\to z} = C_y \rightsquigarrow (C_x \to C_z)$,
(pCI4) Since $x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1$, therefore,
$C_x \to C_y = C_{x\to y} = C_1 \Leftrightarrow C_x \rightsquigarrow C_y = C_{x\rightsquigarrow y} = C_1$. So, $(X/\theta; \to, \rightsquigarrow, C_1)$ is a pseudo CI–algebra. □

Proposition 4.18. *Let $f : X \longrightarrow Y$ be a homomorphism and $\theta = \{(x, y) : f(x) = f(y)\}$. Then:*
(i) θ is a congruence relation on X,
(ii) $X/\theta \cong f(X)$.
Proof. *(i). It is obvious θ is an equivalence relation on X.*
Assume that $(x, y),\ (u, v) \in \theta$. Then we have $f(x) = f(y)$ and $f(u) = f(v)$. Since f is a homomorphism So,

$$f(x \to u) = f(x) \to f(u) = f(y) \to f(v) = f(y \to v).$$

and

$$f(x \rightsquigarrow u) = f(x) \rightsquigarrow f(u) = f(y) \rightsquigarrow f(v) = f(y \rightsquigarrow v).$$

Therefore $(x \to u, y \to v), (x \rightsquigarrow u, y \rightsquigarrow v) \in \theta$.
In the same way $(u \to x, v \to y), (u \rightsquigarrow x, v \rightsquigarrow y) \in \theta$. Hence θ is a congruence relation on X.
(ii). Define $\psi : X/\theta \longrightarrow f(X)$ by $\psi(C_x) = f(x)$, for all $C_x \in X/\theta$.

Then ψ is well defined, because if $C_x = C_y$, for any $x, y \in X$, then $(x, y) \in \theta$. So, $f(x) = f(y)$ and this show $\psi(C_x) = \psi(C_y)$.
Since $Ker\psi = \{C_x : \psi(C_x) = f(x) = 1\} = \{C_x : f(x) = f(1)\} = \{C_x : (x, 1) \in \theta\} = C_1$. Then ψ is one to one.
If $y \in f(X)$ then there are $x \in X$ such that $y = f(x)$. Hence $\psi(C_x) = f(x) = y$ therefore ψ is onto.
Since $\psi(C_x \to C_y) = \psi(C_{x \to y}) = f(x \to y) = f(x) \to f(y) = \psi(C_x) \to \psi(C_y)$ and $\psi(C_x \rightsquigarrow C_y) = \psi(C_{x \rightsquigarrow y}) = f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y) = \psi(C_x) \rightsquigarrow \psi(C_y)$.

This shows ψ is homomorphism. Therefore, $X/\theta \cong f(X)$. □

Example 4.19. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $\to$ " and" $\rightsquigarrow$ " on Xas follows:

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	c	d	1
b	b	a	1	1	1	e
c	c	a	1	1	1	e
d	d	a	1	1	1	e
e	e	a	d	d	d	1

$\rightsquigarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	b	c	d	1
b	b	a	1	1	1	e
c	c	a	1	1	1	e
d	d	a	1	1	1	e
e	e	a	c	c	d	1

Then $(X, \to, \rightsquigarrow, 1)$ is a pseudo CI-algebra and,

$$\theta = \triangle \cup \{(b, c), (c, b), (b, d), (d, b), (c, d), (d, c)\}.$$

We can see that θ is a congruence relation on X and,

$$X/\theta = \{C_1 = \{1\}, C_a = \{a\}, C_b = C_c = C_d = \{b, c, d\}, C_e = \{e\}\}.$$

By following table $(X/\theta, \to, \rightsquigarrow, C_1)$ is a pseudo CI-algebra.

$\to$	C_1	C_a	C_a	C_a
C_1	C_1	C_a	C_b	C_e
C_a	C_a	C_1	C_b	C_1
C_b	C_b	C_a	C_1	C_e
C_e	C_e	C_a	C_b	C_1

$\rightsquigarrow$	C_1	C_a	C_a	C_a
C_1	C_1	C_a	C_b	C_e
C_a	C_a	C_1	C_b	C_1
C_b	C_b	C_a	C_1	C_e
C_e	C_e	C_a	C_b	C_1

Proposition 4.20. *Let F be a pseudo CI-filter in X. If we define $\hat{F} = \{C_x : x \in F\}$,then $\hat{F}$ is a pseudo CI-filter in X/θ.*
Proof. *(F1) Since $1 \in F$, so $C_1 \in \hat{F}$.*
(F2) If C_x and $C_x \to C_y \in \hat{F}$. Since $C_x \to C_y = C_{x\to y} \in \hat{F}$. Then $x, x \to y \in F$. F is a pseudo CI-filter, therefore $y \in F$ so $C_y \in \hat{F}$. □

Proposition 4.21. *The following statements hold:*
(i) if $\theta = X \times X$, then $X/\theta = \{C_1\}$,
(ii) if $\theta = \Delta_X$, then $X/\theta = \{X\}$,
(iii) if $x \leq y$, then $C_x \leq C_y$.
Proof. *(i). Let $C_x \in X/\theta$, for some $x \in X$. Since $\theta = X \times X$, we have $(x, y) \in \theta$ for all $y \in X$. Hence $C_x = C_y$. Putting $y = 1$, then $C_x = C_1$. Therefore, $X/\theta = \{C_1\}$.*

(ii). Let $C_x \in X/\theta$, for some $x \in X$. Since $\theta = \Delta_X$, we have $C_x = \{x\}$. Therefore, $X/\theta = \{X\}$.

(iii). Since $x \leq y$, we get that $x \to y = 1$. Hence $C_x \to C_y = C_{x\to y} = C_1$. Therefore, $C_x \leq C_y$. □

A pseudo CI-filter F is called closed pseudo CI-filter of X if $x \to 1, x \rightsquigarrow 1 \in F$ for all $x \in F$.
From now, for a pseudo CI-filter F, let $C_x = \{y \in X : y \sim_F x\}$. Denote $X/F = \{C_x : x \in X\}$ and define that $C_x \to C_y = C_{x\to y}$, and $C_x \rightsquigarrow C_y = C_{x\rightsquigarrow y}$. Since " $\sim_F$ " is a congruence relation on X, the operations " $\to$ " and " $\rightsquigarrow$ " are well defined.

Proposition 4.22. *Let F be a closed pseudo CI-filter of X. Then $F = C_1$.*
Proof. *If $x \in F$. F is a closed pseudo CI-filter, then $x \to 1 = 1 \in F$ and $1 \to x = x \in F$, i.e., $x \sim_F 1$. Hence $x \in C_1$. Conversely, let $x \in C_1$. Then $x = 1 \to x = 1 \rightsquigarrow x \in F$, and so $x \in F$. Hence $F = C_1$.*
□

Proposition 4.23. *Let F be a closed pseudo CI-filter of X. Then there is a canonical surjective homomorphism $\varphi : X \longrightarrow X/F$ by $\varphi(x) = C_x$, and $Ker\varphi = F$, where $Ker\varphi = \varphi^{-1}(C_1)$.*
Proof. *It is clear that φ is well-defined. Let $x, y \in X$. Then $\varphi(x \to y) = C_{x\to y} = C_x \to C_y = \varphi(x) \to \varphi(y)$ and $\varphi(x \rightsquigarrow y) = C_{x\rightsquigarrow y} = C_x \rightsquigarrow$*

$C_y = \varphi(x) \rightsquigarrow \varphi(y)$. *Hence* φ *is homomorphism.*
Clearly φ *is onto.*
Also, we have $Ker\varphi = \{x \in X : \varphi(x) = C_1\} = \{x \in X : C_x = C_1\} = \{x \in X : x \to 1, 1 \to x, x \rightsquigarrow 1, 1 \rightsquigarrow x \in F\} = \{x \in X : x \in F\} = F$.
□

Conclusion

In this paper, the notion of pseudo CI-filter is defined. The prime, Maximal pseudo CI-filters and generated pseudo CI-filters by one set is the are another purpose aim of this paper. In this article, the pseudo upper set of two elements is introduced and its relationship with pseudo CI-filters is examined and it is proved that each pseudo CI-filter is the union of the pseudo upper sets. Also, the congruence relation, transitive CI-algebra has been introduced and the relationships between pseudo CI-filters and compatibility relationship have been investigated. In a transitive CI-algebra we created a congruence relation with a pseudo CI-filters, and applied the concept of congruence relations to pseudo CI–algebras and discuss the quotient algebras via this congruence relations. We introduce homomorphism between two CI-algebra and prove that each homomorphic image (under certain conditions) and preimage of a pseudo CI-filter is also a pseudo CI-filter.

In the future work, We intend to define the concept of fuzzy pseudo CI-filters and intuitionistic fuzzy CI-filters on pseudo CI-algebras. in pseudo CI-algebras, and Other types of filters are defined in pseudo CI-algebras and the relationship between them is discussed.

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Neda Shojaei
Graduated Ph.D of Mathematics
Department of Mathematics
Islamic Azad University, Kerman Branch
Kerman, Iran
E-mail: n.shojaee64@yahoo.com

Arsham Borumand Saeid
Professor of Mathematics
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman
Kerman, Iran
E-mail: arsham@uk.ac.ir