\(\delta(2)\)-Ideal Euclidean Hypersurfaces of Null \(L_1\)-2-Type

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Abstract. We say that an isometric immersion hypersurface \(x : M^n \to \mathbb{E}^{n+1}\) is of null \(L_k\)-2-type if \(x = x_1 + x_2, x_1, x_2 : M^n \to \mathbb{E}^{n+1}\) are smooth maps and \(L_k x_1 = 0, L_k x_2 = \lambda x_2\), \(\lambda\) is non-zero real number, \(L_k\) is the linearized operator of the \((k+1)\)th mean curvature of the hypersurface, i.e., \(L_k f = \text{tr}(P_k \circ \text{Hessian} f)\) for \(f \in C^\infty(M)\), where \(P_k\) is the \(k\)th Newton transformation, \(L_k x = (L_k x_1, \ldots, L_k x_{n+1})\), \(x = (x_1, \ldots, x_{n+1})\). In this article, we classify \(\delta(2)\)-ideal Euclidean hypersurfaces of null \(L_1\)-2-type.

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1 Introduction

Let \(x : M^n \to \mathbb{E}^{n+1}\) be an isometrically immersed Euclidean hypersurface. It is well-known that the Laplacian of \(M\) is the first element of \(n\)-term sequence of operators \(L_0 = \Delta, L_1, \ldots, L_{n-1}\), where \(L_k\) is the linearized operator of the first variation of the \((k+1)\)th mean curvature (see
These operators are formulated by $L_k(f) = \text{tr}(P_k \circ \text{Hessian}f)$ for any $f \in C^\infty(M)$, where $P_k$ is the $k$th Newton transformation related to the second fundamental form of $M$. Motivated by this consideration, S.M.B. Kashani [6] developed the idea of finite type submanifold to $L_k$-finite type hypersurface in the Euclidean space.

The position vector $x$ and the $(k+1)$th mean curvature vector $\vec{H}_{k+1}$ of $M^n$ in $\mathbb{E}^{n+1}$ are related by generalized Beltrami’s formula:

$$L_k x = c_k \vec{H}_{k+1},$$

where $c_k = (n-k) \binom{n}{k}$ (see [1]). When $k = 0$, this turns into the classical Beltrami’s formula $\Delta x = n \vec{H}$. This states the well-known result: $M^n$ is a $k$-minimal hypersurface of $\mathbb{E}^{n+1}$ if and only if its coordinate functions are $L_k$-harmonic, i.e.,

$$L_k x = 0.$$

Particularly, $k$-minimal hypersurfaces of $\mathbb{E}^{n+1}$ are made by eigenfunctions of the operator $L_k$ with eigenvalue zero. There are many examples of $k$-minimal hypersurfaces in the space forms (see for instance [14]).

From [6], we see that a right spherical cylinder $\mathbb{R} \times S^{n-1}$ in $\mathbb{E}^{n+1}$ is composed of both $L_k$-harmonic function and eigenfunction of $L_k$ with a single nonzero eigenvalue, say $\lambda$, when $0 < k < n - 1$. Therefore, the position vector $x$ of a right spherical cylinder $\mathbb{R} \times S^{n-1}$ takes the following simple spectral decomposition

$$x = x_1 + x_2, \quad L_k x_1 = 0, \quad L_k x_2 = \lambda x_2,$$

for some non-constant smooth maps $x_1$ and $x_2$.

Based on $L_k$-finite type theory, a Euclidean hypersurface is said to be of null $L_k$-2-type if its position vector takes the spectral resolution (1). Similarly, a Euclidean hypersurface is said to be of $L_k$-1-type if its position vector satisfies the condition $L_k x = \lambda x$ (cf. [6, 7, 9, 8]). According to the generalized Takahashi’s theorem [7], a $L_k$-1-type hypersurface of a Euclidean space is either a $k$-minimal Euclidean hypersurface or an open part of a hypersphere. Specially, because of the simplicity of null $L_k$-2-type hypersurfaces, after the classification of the $L_k$-1-type hypersurfaces, it seems reasonable to propose the following problem.
Problem: Classify all null $L_k$-2-type hypersurfaces in Euclidean spaces.

Until now, only few results have been obtained concerning this problem. In [7], the first author and Kashani obtained the first results. They proved that there exists no null $L_{n-1}$-2-type Euclidean hypersurface, specially, there is no null $L_1$-2-type Euclidean surface. When $k \neq n - 1$, they also showed that every null $L_k$-2-type Euclidean hypersurface with at most two distinct principal curvatures is a right circular cylinder.

Now, assume that $M$ is a Riemannian $n$-manifold. Denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_p M$, $p \in M$. The scalar curvature $\tau$ at $p$ is defined by $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$, where $e_1, \ldots, e_n$ is an orthonormal basis of $T_p M$. By choosing $e_1, \ldots, e_r$, a $r$-orthonormal basis of the $r$-plane section $L^r$, the scalar curvature $\tau(L^r)$ is defined by

$$\tau(L^r) = \sum_{i<j} K(e_i \wedge e_j), \quad 1 \leq i, j \leq r.$$ 

For an integer $r \in [2, n - 1]$, the $\delta$-invariant $\delta(r)$ of $M$ is defined by

$$\delta(r)(p) = \tau(p) - \inf\{\tau(L^r)\},$$

where $L^r$ runs into all $r$-plane sections of $T_p M$.

For any $n$-dimensional submanifold $M$ in $\mathbb{E}^m$, Chen [3] proved that the $\delta$-invariant $\delta(2)$ satisfies the following inequality

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2. \quad (2)$$

The equality case of (2) is called the $\delta(2)$-equality. The classification of submanifolds in the Euclidean space $\mathbb{E}^m$ which satisfy the $\delta(2)$-equality condition is an interesting and important subject to research (see [4]). A submanifold $M^n$ in $\mathbb{E}^m$ is called $\delta(2)$-ideal if it satisfies the $\delta(2)$-equality.

Inspired by the above observation, it was proved in [5] that a null 2-type hypersurfaces in $\mathbb{E}^{n+1}$ is an open part of a spherical cylinder $S^{n-1} \times \mathbb{R}$ if and only if it is $\delta(2)$-ideal.

The main purpose of this paper is to extend this classification result to null $L_1$-2-type hypersurfaces as follows.
Theorem 1.1. A null $L_1$-2-type hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ with $n \geq 3$, is an open part of a spherical cylinder $S^{n-1} \times \mathbb{R}$ if and only if it is $\delta(2)$-ideal.

Note that from the before mentioned, if $n = 2$, there is no null $L_1$-2-type surface in $\mathbb{E}^3$.

2 Null $L_1$-2-type Hypersurfaces

Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion, with Gauss map $N$. Denote by $\nabla$ and $\nabla'$ the Levi-Cevita connections on $M^n$ and $\mathbb{E}^{n+1}$, respectively. The formulas of Gauss, Weingarten and Codazzi are given respectively by

$$\nabla_X Y = \nabla_X Y + \langle SX, Y \rangle N,$$

$$SX = -\nabla_X N,$$

$$(\nabla_X S)Y = (\nabla_Y S)X,$$

for $X, Y \in \mathfrak{X}(M^n)$, where $S : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ is the shape operator of $M^n$ arises from the Gauss map $N$.

The equation of Gauss is given by

$$R(X,Y)Z = \langle SY, Z \rangle SX - \langle SX, Z \rangle SY,$$

for $X,Y,Z \in \mathfrak{X}(M^n)$ where $R$ is the Riemann curvature tensor. The eigenvalues of $S$ are called the principal curvatures of $M^n$.

Let $\{k_1, \ldots, k_n\}$ be the $n$ principal curvatures of $M^n$. Associated to the principal curvatures, the 2th mean curvature $H_2$ of the hypersurface is defined by

$$(\binom{n}{2}) H_2 = \sum_{i_1 < i_2} \kappa_{i_1} \kappa_{i_2}.$$

$H_2$ defines an intrinsic invariant which is relevant to the scalar curvature of $M^n$.

Related to the shape operator $S$, the classical Newton transformation $P_1 : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ is defined by

$$P_1 = (\binom{n}{2}) H_2 I - S.$$
Now, consider the second-order linear differential operator

\[ L_1 : C^\infty(M) \to C^\infty(M), \]

which is given by

\[ L_1(f) = \text{tr}(P_1 \circ \text{Hessian} f). \]

For isometric immersion \( x : M^n \to \mathbb{E}^{n+1} \), it is well-known (see [1]),

\[ L_1x = n(n - 1)\vec{H}_2, \quad (4) \]

where \( \vec{H}_2 = H_2N \) defines the 2th mean curvature vector field. By formula in [1] page 122, we find

\[
L_1\vec{H}_2 = -\left(\frac{n}{2}\right)H_2\nabla H_2 - 2(S \circ P_1)(\nabla H_2) - (H_2 \text{tr}(S^2 \circ P_1) - L_1H_2)N. \quad (5)
\]

Suppose that \( M \) is of null \( L_1 \)-2-type hypersurface, from (4) we get

\[ L_1\vec{H}_2 = \lambda\vec{H}_2. \quad (6) \]

By combining (5) and (6) we obtain

\[
(S \circ P_1)\nabla H_2 = \frac{n(n - 1)}{4}H_2\nabla H_2, \quad (7)
\]

\[ L_1H_2 - H_2 \text{tr}(S^2 \circ P_1) = \lambda H_2. \quad (8) \]

Consequently, \( \nabla H_2 \) is an eigenvector of \( S \circ P_1 \) whenever \( \nabla H_2 \neq 0 \).

For the proof of our theorem we need the following lemma from [10].

**Lemma 2.1.** Let \( M \) be a hypersurface of a Euclidean space. Then the 2th mean curvature vector field \( \vec{H}_2 \) satisfies the condition \( L_1\vec{H}_2 = \lambda\vec{H}_2 \) for some constant \( \lambda \) if and only if \( M \) is one of the following hypersurfaces
(a) a \( L_1 \)-biharmonic hypersurface,
(b) a \( L_1 \)-1-type hypersurface,
(c) a null \( L_1 \)-2-type hypersurface.
3 Proof of Theorem 1.1

Suppose that $M$ is a null $L_1$-2-type hypersurface of $\mathbb{E}^{n+1}$, which is $\delta(2)$-ideal. Because $M$ is a $\delta(2)$-ideal, there is an orthonormal frame $\{e_1, ..., e_n\}$ such that the shape operator with respect to this frame takes the following form [Lemma 3.2 of [2]]

$$S = \begin{pmatrix}
\alpha & 0 & 0 & \cdots & 0 \\
0 & \beta & 0 & \cdots & 0 \\
0 & 0 & \alpha + \beta & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha + \beta
\end{pmatrix}.$$ \hspace{1cm} (9)

for some functions $\alpha$ and $\beta$ on $M$.

If $H_2$ is constant, then $M$ is non 1-minimal from lemma 2.1. So, (8) implies that $\text{tr}(S^2 \circ P_1)$ is constant. Since $H_2$ and $\text{tr}(S^2 \circ P_1)$ are constant, it follows from (9) that $M$ is isoparametric. According to a well-known result of Segre [13], any isoparametric hypersurface of $\mathbb{E}^{n+1}$ has $l$ distinct principal curvatures with $l \leq 2$. If $l = 0$, then $M$ is 1-minimal which is impossible by lemma 2.1. (9) shows easily that case $l = 1$ does not occur. So, $l = 2$. Therefore, from (9) we conclude that one of the principal curvatures is simple. Thus, $M$ is locally isometric to $S^{n-1} \times \mathbb{R}$ by Theorem 3.12 of [7].

From now on we assume that $H_2$ is non-constant. First, from (3) and (9) we see that the classical Newton transformation $P_1$ satisfies

$$P_1 = \begin{pmatrix}
(n-2)\alpha + (n-1)\beta & 0 & 0 & \cdots & 0 \\
0 & 0 & \beta((n-2)\beta + (n-1)\alpha) & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (n-2)(\alpha + \beta)
\end{pmatrix}.$$ \hspace{1cm} (10)

Therefore, we have

$$S \circ P_1 = \begin{pmatrix}
\alpha((n-2)\alpha + (n-1)\beta) & 0 & 0 & \cdots & 0 \\
0 & 0 & \beta((n-2)\beta + (n-1)\alpha) & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (n-2)(\alpha + \beta)^2
\end{pmatrix}.$$ \hspace{1cm} (11)

and

$$S^2 \circ P_1 = \begin{pmatrix}
\alpha^2((n-2)\alpha + (n-1)\beta) & 0 & 0 & \cdots & 0 \\
0 & \beta^2((n-2)\beta + (n-1)\alpha) & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (n-2)(\alpha + \beta)^3
\end{pmatrix}.$$ \hspace{1cm} (11)
Put $\nabla H_2 = \sum_{i=1}^{n} \lambda_i e_i$ for some functions $\lambda_1, \ldots, \lambda_n$ on $M$, then from (10) we have

$$(S \circ P_1) \nabla H_2 = \sum_{i=1}^{n} \lambda_i (S \circ P_1) e_i$$

$$= \alpha (\beta + (n-2)(\alpha + \beta)) \lambda_1 e_1 + \beta (\alpha + (n-2)(\alpha + \beta)) \lambda_2 e_2$$

$$+ \sum_{i=3}^{n} (n-2)(\alpha + \beta)^2 \lambda_i e_i$$

$$= ((3-n)\alpha \beta + (2-n)\beta^2) \lambda_1 e_1 + ((3-n)\alpha \beta$$

$$+ (2-n)\alpha^2) \lambda_2 e_2 + (n-2)(\alpha + \beta)^2 \nabla H_2.$$  

Thus, Eq. (7) yields

$$((n-2)(\alpha + \beta)^2 + \frac{1}{4} n(n-1) H_2) \nabla H_2 = ((n-3)\alpha \beta + (n-2)\beta^2) \lambda_1 e_1$$

$$+ ((n-3)\alpha \beta + (n-2)\alpha^2) \lambda_2 e_2.$$  

Hence, $\lambda_3 = \cdots = \lambda_n = 0$ and

$$[(n-2)(\alpha + \beta)^2 + \frac{1}{4} n(n-1) H_2 + (3-n)\alpha \beta + (2-n)\beta^2] \lambda_1 = 0, \quad (12)$$

$$[(n-2)(\alpha + \beta)^2 + \frac{1}{4} n(n-1) H_2 + (3-n)\alpha \beta + (2-n)\alpha^2] \lambda_2 = 0.$$  

Since $\nabla H_2$ is a nonzero, which implies at least one of $\lambda_1$ and $\lambda_2$ does not vanish. If both $\lambda_1$ and $\lambda_2$ do not vanish, then we find either $\alpha = \beta$ or $\alpha = -\beta$. If $\alpha = \beta$, then $M$ has at most two distinct principal curvatures, so from [7] we know that any null $L_1$-type hypersurfaces with at most two distinct principal curvatures have constant 2-th mean curvature, this is a contradiction. If $\alpha = -\beta$, then $H_2 = \frac{4\beta^2}{n(n-1)}$. Hence, we get

$$n(n-1) H_2 = \text{tr}(S \circ P_1) = -2\beta^2 = \frac{n(n-1)}{2} H_2,$$

which implies $H_2 = 0$, but this is impossible.

Therefore, we have either
(a) \( \lambda_1 \neq 0 \) and \( \lambda_2 = 0 \), or

(b) \( \lambda_2 \neq 0 \) and \( \lambda_1 = 0 \).

We only need to consider the case (a), case (b) can be done in a similar arguments as case (a).

First, from relation (12) we obtain that

\[
H_2 = \frac{4\alpha ((\alpha + \beta)n - 2\alpha - \beta)}{n(1-n)}.
\]  

(13)

On the other hand, since \( \text{tr}(S \circ P_1) = n(n - 1)H_2 \), by using (10) we can write

\[
H_2 = \frac{1}{n(n - 1)}[\alpha ((n - 2)\alpha + \beta (n - 1)) + \beta ((n - 2)\beta + \alpha (n - 1) + (n - 2)^2 (\alpha + \beta)^2].
\]  

(14)

Comparing (13) and (14), then after a straightforward computation, we find that there exist real numbers in terms of \( n \), say \( a_1, a_2 \), such that

\[
\left\{ \begin{array}{l}
\sqrt{H_2} \equiv a_1 \\
\sqrt{H_2} \equiv a_2
\end{array} \right.
\]

Note that since \( \nabla H_2 \) is a nonzero without loss of generality, we may assume that \( H_2 > 0 \).

Next, by taking \( e_1 \) in the direction of \( \nabla H_2 \), the shape operator satisfies

\[
S = \begin{pmatrix}
a_1\sqrt{H_2} & 0 & 0 & \cdots & 0 \\
0 & a_2\sqrt{H_2} & 0 & \cdots & 0 \\
0 & 0 & a_3\sqrt{H_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_3\sqrt{H_2}
\end{pmatrix},
\]

where \( a_3 = a_1 + a_2 \). Moreover, we have

\[
e_1(H_2) \neq 0, \quad e_k(H_2) = 0, \quad \forall k > 1.
\]  

(15)

We put \( \nabla_{e_i}e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k \), then using the equation of Codazzi for \( X = e_i \) and \( Y = e_j \) we get

\[
(\nabla_{e_i}S)e_j = \frac{a_j e_i(H_2)}{2\sqrt{H_2}} e_j + \sqrt{H_2} \sum_k (a_j - a_k) \omega_{ij}^k e_k.
\]

Then, we consider the special cases of \( i \) and \( j \).
For $i = 1, j = 2$, one obtains
\[ \frac{a_1^2}{2} e_1(H_2)e_2 + H_2 \sum_k (a_2 - a_k) \omega_{12}^k e_k = H_2 \sum_k (a_1 - a_k) \omega_{21}^k e_k. \]

Under the identification the coefficients corresponding to \{e_1, \ldots, e_n\}, we have the following
\begin{align*}
\omega_{12}^1 &= 0, \quad (16) \\
\omega_{12}^2 &= 0, \quad (17) \\
a_1 \omega_{12}^k &= a_2 \omega_{21}^k, \quad k \geq 3. \quad (18)
\end{align*}

Similarly, for $i = 1, j \geq 3$, we obtain the following
\begin{align*}
\omega_{1j}^1 &= 0, \quad j \geq 3, \quad (19) \\
\omega_{1j}^2 &= (1 - \frac{a_2}{a_1}) \omega_{1j}^2, \quad j \geq 3, \quad (20) \\
a_1 e_1(H_2) \delta_{jk} + 2a_2 H_2 \omega_{j1}^k &= 0, \quad j, k \geq 3. \quad (21)
\end{align*}

Finally, for $i = 2, j \geq 3$ we get
\begin{align*}
\omega_{2j}^1 &= (1 - \frac{a_1}{a_2}) \omega_{2j}^2, \quad j \geq 3, \quad (21) \\
\omega_{2j}^2 &= 0, \quad j \geq 3, \\
\omega_{jk}^k &= 0, \quad j, k \geq 3.
\end{align*}

From (15), we see easily $[e_2, e_j](H_2) = 0$. So, we have
\[ \sum_k (\omega_{2j}^k - \omega_{j2}^k) e_k(H_2) = 0. \]

Again, using (15), we get $\omega_{2j}^1 = \omega_{1j}^1$, for $j \geq 3$. Combining this with (21) yields
\[ \omega_{2j}^1 = \omega_{1j}^1 = 0. \quad (22) \]

Since $\{e_k\}_{k=1}^n$ is an orthonormal basis, we have
\begin{align*}
0 = e_i(e_j, e_k) &= \langle \nabla e_i e_j, e_k \rangle + \langle e_j, \nabla e_i e_k \rangle = \omega_{ij}^k + \omega_{ik}^j, \\
& \quad \forall i, j, k = 1, \ldots, n. \quad (23)
\end{align*}
By using (23), we derive that
\[ \omega_{11}^1 = 0, \quad \omega_{12}^2 = 0, \quad \omega_{1j}^j = 0, \quad j \geq 3, \] (24)
\[ \omega_{21}^1 = 0, \quad \omega_{22}^2 = 0, \quad \omega_{2j}^j = 0, \quad j \geq 3, \] (25)
\[ \omega_{k1}^1 = 0, \quad \omega_{k2}^2 = 0, \quad \omega_{kj}^j = 0, \quad j, k \geq 3. \] (26)

Combining (23) with (16), (17) and (22) we find that
\[ \omega_{11}^2 = 0, \quad \omega_{22}^1 = \left( \frac{a_2}{2(a_2 - a_1)} \right) e_{1}(H_2), \quad \omega_{j1}^j = 0, \quad j \geq 3. \] (27)

By applying (20) and (27) we obtain
\[ \omega_{1j}^2 = 0, \quad j \geq 3. \] (28)

Moreover, it follows from (23), (28) and (18) that
\[ \omega_{12}^j = 0, \quad \omega_{21}^j = 0, \quad j \geq 3. \]

In the same way, we derive that
\[ \omega_{11}^j = 0, \quad \omega_{22}^j = 0, \quad \omega_{jj}^1 = \left( \frac{a_1 + a_2}{2a_2} \right) \frac{e_{1}(H_2)}{H_2}, \quad j \geq 3. \] (29)

Now, it follows from the Codazzi’s equation that
\[ \sum_k (a_j - a_k) \omega_{ij}^k e_k = \sum_k (a_i - a_k) \omega_{ji}^k e_k, \quad i, j \geq 3. \]

Therefore, we get
\[ \omega_{ij}^1 = \omega_{ji}^1, \quad \omega_{ij}^2 = \omega_{ji}^2, \quad i, j \geq 3. \]

Then, (15), (16) and (25) imply that \([e_1, e_2](H_2) = 0\). Hence, we have
\[ e_2 e_1(H_2) = 0. \]

From (15), (19) and (26) we also have
\[ e_j e_1(H_2) = 0, \quad j \geq 3. \]
Applying Gauss’s equation to \( \langle R(e_1, e_2)e_1, e_2 \rangle, \langle R(e_1, e_j)e_1, e_j \rangle \) and \( \langle R(e_2, e_j)e_2, e_j \rangle \), we respectively obtain that

\[
e_1\left(\frac{e_1(H_2)}{H_2}\right) + \frac{a_2}{2(a_1 - a_2)} \left(\frac{e_1(H_2)}{H_2}\right)^2 + 2a_1(a_1 - a_2)H_2 = 0, \tag{30}
\]

\[
e_1\left(\frac{e_1(H_2)}{H_2}\right) - \frac{a_1 + a_2}{2a_2} \left(\frac{e_1(H_2)}{H_2}\right)^2 - 2a_1a_2H_2 = 0,
\]

\[
\left(\frac{e_1(H_2)}{H_2}\right)^2 - 4a_2(a_1 - a_2)H_2 = 0. \tag{31}
\]

On the other hand, from (17), (24), (29) and the definition of \( L_1 \), we find

\[
L_1H_2 = b \left[ e_1(e_1(H_2)) + \frac{c(e_1(H_2))^2}{H_2} \right] \sqrt{H_2}, \tag{32}
\]

for some real numbers \( b \) and \( c \).

Now, using (8), (11) and (32), we obtain that

\[
b \left[ e_1(e_1(H_2)) + \frac{c(e_1(H_2))^2}{H_2} \right] \sqrt{H_2} = dH_2^2 \sqrt{H_2} + \lambda H_2, \tag{33}
\]

for some real numbers \( b, c \) and \( d \).

By substituting (31) into (30), we get

\[
e_1(e_1(H_2)) = \left[ \frac{2a_1 - 3a_2}{2} (a_1 + a_2) \right] H_2^2. \tag{34}
\]

Also, substituting (31) into equation (33) gives

\[
e_1(e_1(H_2)) = \left[ \frac{bca_2(a_1 - a_2) - d}{b} - \frac{\lambda}{H_2 \sqrt{H_2}} \right] H_2^2. \tag{35}
\]

By comparing (34) and (35), we conclude that \( H_2 \) is constant, which is a contradiction.

This completes the proof of the theorem.
References


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