Density of Polynomials in Certain Weighted Dirichlet-type Spaces

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Abstract. We study weighted Dirichlet type spaces in the unit disk. We prove that analytic polynomials are dense in weighted Dirichlet type spaces if the (non-radial) weight function is super-biharmonic and satisfies a growth condition up to the boundary.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane, and let $\mathbb{T}$ denote its boundary; the set of all complex numbers whose absolute value is 1. Let X be a functional Banach space on the unit disk; it means that X is a closed linear subspace of the space of analytic functions on the unit disk, or $X \subseteq \text{Hol}(\mathbb{D})$. For example, X can be the Dirichlet space, the Hardy space, the Bergman space, the analytic Besov space, and so on. It is quite normal to assume that X has a norm (usually given by an integral with respect to the area measure on the unit disk) with respect to which the polynomials are included in X. It then

becomes a natural question in the operator theory of function spaces to ask whether the functions in X can be approximated by the analytic polynomials or not. It turns out that the answer to this question, in general, depends on the domain in question, as well as to the weight function if instead of the area measure $dA(z)$ one considers the measure $w(z)dA(z)$ for a positive real-valued integrable function w defined on the domain in question.

In this paper, we restrict our attention to the open unit disk $\mathbb{D}$, and assume that w is a weight function on the unit disk; it is a positive integrable function on the unit disk. Assume that X is the weighted Bergman space, consisting of functions in $\mathrm{Hol}(\mathbb{D})$ for which the integral

$$\int_{\mathbb{D}} |f(z)|^p w(z) dA(z)$$

is finite; here $dA(z) = \pi^{-1} dx dy$. If $w(z) \equiv 1$, we get the standard (unweighted) Bergman space. In most applications, the weight function

$$w(z) = (\alpha + 1)(1 - |z|^2)^{\alpha}, \quad \alpha > -1,$$

is considered. For $1 \le p < \infty$, the norm in the weighted Bergman space is given by

$$\|f\|_{A_w^p} = \left(\int_{\mathbb{D}} |f(z)|^p w(z) dA(z) \right)^{1/p}.$$

The history of norm approximation by polynomials begins with the Bergman spaces. It is well-known that if the weight function is radial, in the sense that $w(z) = w(|z|)$, then the polynomials are dense in the weighted Bergman space on the unit disk (see [9] for instance), but for arbitrary weight functions the problem is quite complicated. We mention is passing that for arbitrary bounded domains whose boundary is a Jordan curve, the problem can be traced back to T. Carleman's paper in 1923 who proved the result for Jordan regions [5]. This result was then extended by O. J. Farrell in 1934 [7] to Carathéodory regions [7]; see also the paper by L. I. Hedberg on approximation by polynomials on compact subsets of the plane [8], as well as the paper by S. N. Mergeljan [11] on the completeness of the system of polynomials.

In general, it is more difficult to treat non-radial weights; see the monograph authored by J. A. Peláez, and J. Rättya [13]. Some applications of non-radial weights in the theory of Bergman spaces were discussed in [2, 3].

In this paper we consider weights $w : \mathbb{D} \to (0, \infty)$ satisfying the following conditions:

$$\Delta^2 w(z) \geq 0; \tag{1}$$

that is, w is super-biharmonic, here Δ stands for the laplacian

$$\Delta = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Moreover, w satisfies the following growth condition:

$$\lim_{r \to 1} \int_T w(rz) d\sigma(z) = 0, \tag{2}$$

where $d\sigma$ is the normalised arc-length measure on $T = \partial \mathbb{D}$. A holomorphic function $f \in \mathrm{Hol}(\mathbb{D})$ is said to belong to the weighted Dirichlet type space $\mathcal{D}_w^p$ if f' belongs to the weighted Bergman space A_w^p; which means that

$$\|f\|_{\mathcal{D}_w^p} = \left(\int_{\mathbb{D}} |f'(z)|^p w(z) dA(z)\right)^{1/p} < \infty.$$

Indeed, this expression defines a semi-norm on the weighted Dirichlet type space. We aim to prove that the polynomials are dense in weighted Dirichlet type spaces. Our proof uses the fact that the dilations $f_r(z) = f(rz)$ for $0 \leq r < 1$, tend to f in the norm, or

$$\lim_{r \to 1} \|f_r - f\|_{\mathcal{D}_w^p} = 0.$$

This in turn uses some potential theory of the complex plane, including the bi-harmonic Green function for bi-laplatian Δ^2. The details of potential analysis will be given in the next section (§2).

2 Weighted Dirichlet-type Spaces

A close relative of the Bergman space is the Dirichlet space; the space of analytic functions whose derivative belongs to the Bergman space; that

is

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

The norm on the Dirichlet space is defined by

$$\|f\|_{\mathcal{D}^2} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z),$$

however, an equivalent norm for this space is

$$\|f\|_{\mathcal{D}^2} = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty,$$

where a_n's are the Taylor coefficients of f. The Dirichlet space is a Hilbert space of analytic functions whose reproducing kernel is given by

$$K_w(z) = k(\overline{w}z) = \frac{1}{\overline{w}z} \log\left(\frac{1}{1 - \overline{w}z}\right).$$

The weighted Dirichlet space consists of analytic functions in the unit disk for which

$$\int_{\mathbb{D}} |f'(z)|^2 w(z) dA(z) < \infty.$$

The norm is given by

$$\|f\|_{\mathcal{D}_w^2} = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 w(z) dA(z)\right)^{1/2}.$$

It is known that for radial weights like $w(|z|)$, the polynomials are dense in weighted Dirichlet spaces (see [12]).

We consider the weighted Dirichlet type spaces $\mathcal{D}_w^p$, $1 < p < \infty$, consisting of all analytic functions in the unit disk for which the following integral is finite:

$$\|f\|_{\mathcal{D}_w^p} = \left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p w(z) dA(z)\right)^{1/p}.$$

The weighted Dirichlet type spaces are particular cases of a scale of Banach spaces of analytic functions, namely, the weighted analytic Besov

spaces. Indeed, the weighted analytic Besov space $\mathcal{B}_w^p$ consists of analytic functions f in the unit disk for which the integral

$$\int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p w(z) dA(z)$$

is finite. If $p = 2$, then $\mathcal{B}_w^2$ coincides with the weighted Dirichlet space $\mathcal{D}_w^2$. For $w(z) \equiv 1$, this space was studied by K. Zhu (see [14], [15]).

3   Density Of Polynomials

This section is devoted to the study of density of polynomials in weighted Dirichlet type spaces. We should mention that the non-radial weights we consider should a priori satisfy some growth and smoothness condition to ensure that the polynomials belong to the space.

We begin by collecting some facts on the bi-harmonic Green function for the unit disk. The bi-harmonic Green function for the operator Δ^2 is the function

$$\Gamma(z, \zeta) = |z - \zeta|^2 \log \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|^2 + (1 - |z|^2)(1 - |\zeta|^2), \quad z \in \mathbb{D}, \zeta \in \mathbb{D}.$$

For fixed ζ, the function $\Gamma(z, \zeta)$ is the solution to the boundary value problem

$$\begin{cases} \Delta_z^2 \Gamma(z, \zeta) = \delta_\zeta(z), & z \in \mathbb{D} \\ \Gamma(z, \zeta) = 0, & z \in \mathbb{T} \\ \partial_{n(z)} \Gamma(z, \zeta) = 0, & z \in \mathbb{T} \end{cases}$$

where $\mathbb{T}$ is the unit circle, $\partial_{n(z)}$ is the inward normal derivative with respect to $z \in \mathbb{T}$, and $\delta_\zeta(z)$ is the Dirac point mass concentrated at $\zeta \in \mathbb{D}$.

Let w be a weight function satisfying the conditions mentioned above, then we can write (see [1], for details)

$$w(\zeta) = \int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2(z) dA(z) + \int_{\mathbb{T}} H(z, \zeta) d\mu(z), \quad \zeta \in \mathbb{D}, \qquad (3)$$

where

$$H(z, \zeta) = \frac{(1 - |\zeta|^2)^2}{|1 - \bar{\zeta} z|^2}, \quad z \in \mathbb{T}, \zeta \in \mathbb{D}.$$

We shall see that the formula (3) plays a prominent role in the following arguments.

The following properties of the bi-harmonic Green function is useful.

Proposition 3.1. *(see [3]). Let $\Gamma(z, \zeta)$ denote the bi-harmonic Green function for the unit disk. Then we have*

$$\frac{1}{2}(1 - |z|)^2(1 - |\zeta|^2)^2 \leq \Gamma(z, \zeta) \leq (1 + |z|)^2(1 - |\zeta|^2)^2, \quad z \in \mathbb{D}, \zeta \in \mathbb{D}.$$

We shall also appeal to the following statement from classical real analysis.

Proposition 3.2. *([9], page 66). Let μ be a finite positive measure on a measure space X, and let $\{f_n\}_n$ and f be measurable functions on X such that for $0 < p < \infty$,*

$$\limsup_{n \to \infty} \int_X |f_n|^p d\mu \leq \int_X |f|^p d\mu < \infty.$$

If $f_n \to f$ almost everywhere with respect to μ, then we have

$$\int_X |f_n - f|^p d\mu \to 0, \quad n \to \infty.$$

We now manage to prove that the polynomials are dense in weighted Dirichlet type spaces whose weights are super-biharmonic (in the sense of (1), and satisfy the growth condition (2). More precisely, we have the following theorem.

Theorem 3.3. *Let f be a function in the weighted Dirichlet type space $\mathcal{D}_w^p$. Then there is a sequence of analytic polynomials p_n such that*

$$\|f - p_n\|_{\mathcal{D}_w^p} \to 0, \quad n \to \infty.$$

Proof. Recall that

$$\|f\|_{\mathcal{D}_w^p}^p = \int_{\mathbb{D}} |f'(z)|^p w(z) dA(z),$$

and $f_r(z) = f(rz)$ for $0 \leq r < 1$. It is easy to see that each dilated function f_r can be approximated by polynomials since f_r is analytic in a neighborhood of the unit disk. Therefore, it suffices to verify that

$$\|f - f_r\|_{\mathcal{D}_w^p} \to 0, \quad r \to 1.$$

According to the arguments of the previous section, the weight function can be written as

$$w(\zeta) = \int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(z) dA(z) + \int_T H(z, \zeta) d\mu(z),$$

where μ is a positive Borel measure on T, and

$$H(z, \zeta) = \frac{(1 - |\zeta|^2)^2}{|1 - z\overline{\zeta}|^2}, \quad z \in T, \zeta \in \mathbb{D}.$$

This implies that (using Fubini's theorem)

$$\|f_r\|_{\mathcal{D}_w^p}^p = \int_{\mathbb{D}} \int_{\mathbb{D}} |f_r'(z)|^p \Gamma(z, \zeta) dA(z) \Delta^2 w(\zeta) dA(\zeta) \qquad (4)$$
$$+ \int_T \int_{\mathbb{D}} |f_r'(z)|^p H(\zeta, z) dA(z) d\mu(\zeta).$$

We now replace z by z/r in the second term of (4) to obtain

$$\int_T \int_{\mathbb{D}} |f_r'(z)|^p H(\zeta, z) dA(z) d\mu(\zeta)$$
$$= r^{p-3} \int_T \int_{|z| < r} |f'(z)|^p r H(\zeta, z/r) dA(z) d\mu(\zeta). \quad (5)$$

Note that for $|\zeta| < r < 1$, we have

$$rH(z, \zeta/r) = \frac{(r^2 - |\zeta|^2)^2}{r|r - z\overline{\zeta}|^2} = \frac{(\frac{r+|\zeta|}{\sqrt{r}})^2}{|\frac{r - z\overline{\zeta}}{r - |\zeta|}|^2}, \quad z \in T, \zeta \in \mathbb{D}. \qquad (6)$$

But
$$\frac{d}{dr}\left(\frac{r + |\zeta|}{\sqrt{r}}\right) = \frac{1}{\sqrt{r}}\left[1 - \frac{r + |\zeta|}{2r}\right] > 0,$$

which means that the numerator of (6) is increasing in r as long as $|\zeta| < r < 1$. As for the denominator of (6), we note that

$$\left|\frac{r - z\overline{\zeta}}{r - |\zeta|}\right|^2 = 1 + \frac{||\zeta| - \overline{\zeta}z|^2}{(r - |\zeta|)^2} + \frac{2Re(|\zeta| - \overline{\zeta}z)}{r - |\zeta|},$$

which shows that the denominator of (6) is a decreasing functions of r provided that r is greater than $|\zeta|$. These statements prove that (6) is an increasing function of r. Now we use (5) and (6) (by switching the roles of z and ζ) to obtain

$$\lim_{r \to 1} \int_T \int_{\mathbb{D}} |f_r'(z)|^p H(\zeta, z) dA(z) d\mu(\zeta) = \int_T \int_{\mathbb{D}} |f'(z)|^p H(\zeta, z) dA(z) d\mu(\zeta). \tag{7}$$

As for the limit of the first term in (4), we note that

$$\frac{1}{2}\tilde{\Gamma}(z, \zeta) \le \Gamma(z, \zeta) \le \tilde{\Gamma}(z, \zeta), \tag{8}$$

where

$$\tilde{\Gamma}(z, \zeta) = \frac{(1 - |z|^2)^2 (1 - |\zeta|^2)^2}{|1 - z\overline{\zeta}|^2}, \quad z \in \mathbb{D}, \zeta \in \mathbb{D}.$$

According to [4], for $|z| < r < 1$, the function $r\tilde{\Gamma}(z/r, \zeta)$ is increasing in r; so that by assuming $1/2 \le r < 1$, and using (8) we have

$$\int_{\mathbb{D}} |f_r'(z)|^p \Gamma(z, \zeta) dA(z) \Delta^2 w(\zeta)$$

$$\le r^{p-3} \int_{|z|<r} |f'(z)|^p r\tilde{\Gamma}(z/r, \zeta) dA(z) \Delta^2 w(\zeta)$$

$$\le 2r^{p-3} \int_{\mathbb{D}} |f'(z)|^p \Gamma(z, \zeta) dA(z) \Delta^2 w(\zeta)$$

$$\le C \int_{\mathbb{D}} |f'(z)|^p \Gamma(z, \zeta) dA(z) \Delta^2 w(\zeta),$$

where C is a constant depending on p. This inequality shows that the dominated convergence theorem can be applied to the first integral in (4). Therefore, it suffices to verify that the inner integral tends to the right limit; that is,

$$\lim_{r \to 1} \int_{\mathbb{D}} |f_r'(z)|^p \Gamma(z, \zeta) dA(z) = \int_{\mathbb{D}} |f'(z)|^p \Gamma(z, \zeta) dA(z). \tag{9}$$

To verify (9), we note that

$$\int_{\mathbb{D}} |f_r'(z)|^p (1 - |z|^2)^2 dA(z) = r^{p-6} \int_{|z|<r} |f'(z)|^p (r^2 - |z|^2)^2 dA(z),$$

from which it follows (using the monotone convergence theorem) that

$$\lim_{r \to 1} \int_{\mathbb{D}} |f_r'(z)|^p (1 - |z|^2)^2 dA(z) = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^2 dA(z).$$

Since

$$\lim_{r \to 1} F_r(z) := |f_r'(z)|^p (1 - |z|^2)^2 = |f'(z)|^p (1 - |z|^2)^2 := F(z),$$

it follows from Proposition 3.2 that

$$\lim_{r \to 1} \|F_r - F\|_{L^1(\mathbb{D}, dA)} = 0,$$

or

$$\lim_{r \to 1} \int_{\mathbb{D}} \big| \, |f_r'(z)|^p - |f'(z)|^p \, \big| (1 - |z|^2)^2 dA(z) = 0.$$

We now invoke Proposition 3.1 to obtain

$$0 \leq \int_{\mathbb{D}} \big| \, |f_r'(z)|^p - |f'(z)|^p \, \big| \Gamma(z, \zeta) dA(z)$$
$$\leq (1 + |\zeta|)^2 \int_{\mathbb{D}} \big| \, |f_r'(z)|^p - |f'(z)|^p \, \big| (1 - |z|^2)^2 dA(z) \to 0, \quad r \to 1.$$

This entails

$$\left| \int_{\mathbb{D}} \left(|f_r'(z)|^p - |f'(z)|^p \right) \Gamma(z, \zeta) dA(z) \right|$$
$$\leq \int_{\mathbb{D}} \big| \, |f_r'(z)|^p - |f'(z)|^p \, \big| \Gamma(z, \zeta) dA(z) \to 0, \quad r \to 1. \quad (10)$$

Using the dominated convergence theorem together with (10), we obtain

$$\lim_{r \to 1} \int_{\mathbb{D}} \int_{\mathbb{D}} |f_r'(z)|^p \, \Gamma(z, \zeta) dA(z) \Delta^2 w(\zeta) dA(\zeta)$$
$$= \int_{\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \, \Gamma(z, \zeta) dA(z) \Delta^2 w(\zeta) dA(\zeta). \quad (11)$$

It now follows from (4), (7), and (11) that

$$\|f_r\|_{\mathcal{D}_w^p}^p \to \|f\|_{\mathcal{D}_w^p}^p, \quad r \to 1,$$

from which, using the Proposition (3.2), we obtain

$$\|f_r - f\|^p_{\mathcal{D}^p_w} \to 0, \quad r \to 1.$$

This argument proves the theorem. $\square$

Concluding Remark. The proof of our main result uses a slight modification of the technique we already applied for the weighted Bergman spaces in [3]. It is interesting to know that if the same method can be applied to other situations; in particular, to the weighted Fock-Sobolev spaces of entire functions on the whole complex plane.

References

[1] A. Abkar, H. Hedenmalm, A Riesz representation formula for super-biharmonic functions, *Ann. Acad. Sci. Fenn. Math.*, 26 (2001), 305–324

[2] A. Abkar, Application of a Riesz-type formula to weighted Bergman spaces, *Proc. Amer. Math. Soc.*, 131 (2003), 155–164

[3] A. Abkar, Norm approximation by polynomials in some weighted Bergman spaces, *J. Funct. Anal.*, 191 (2002), 224–240

[4] A. Aleman, S. Richter, and C. Sundberg, Beurling's theorem for the Bergman space, *Acta Math.*, 177 (1997), 275–310

[5] T. Carleman, Über die approximation analytischer funktionen durch lineare Aggregate von vorgegebenen Potenzen, *Ark. Mat. Astr. Fys.*, 17 (1923), 1–30

[6] O. Constantin, J. A. Peláez, Integral operators, embedding theorems, and a Littlewood-Paley formula on weighted Fock spaces, *J. Geom. Anal.*, 26 (2016), no. 2, 1109–1154

[7] O. J. Farrell, On approximation to an analytic function by polynomials, *Bull. Amer. Math. Soc.*, 40 (1934), 908–914

[8] L. I. Hedberg, Approximation in the mean by analytic functions, *Trans. Amer. Math. Soc.*, 40 (1934), 157–171

[9] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, 199, Springer–Verlag, New York, (2000)

[10] Z. Hu, X. Lv, Positive Toeplitz operators between different doubling Fock spaces, *Taiwanese J. Math.*, 21 (2017), no. 2, 467–487

[11] S. N. Mergelyan, On the completeness of systems of analytic functions, *Amer. Math. Soc. Transl.*, 19 (1962), 109–136; (*Uspehi Mat. Nauk* 8 (1953), 3–63)

[12] J. Pau, P. A. Pérez, Composition operators acting on weighted Dirichlet spaces, *J. Math. Anal. Appl.*, 401 (2013), 682–694

[13] J. A. Peláez, J. Rättya, Weighted Bergman spaces induced by rapidly increasing weights, *Memoirs of the American Mathematical Society*, Vol. 227, no. 1066, (2014)

[14] K. Zhu, Analytic Besov spaces, *J. Math. Anal. Appl.*, 157 (1991), 318–336

[15] K. Zhu, *Operator Theory in Function Spaces*, Mathematical Surveys and Monographs, Vol. 138, American Mathematical Society, (2007)

[16] K. Zhu, *Analysis on Fock Spaces*, Graduate Texts in Mathematics, 263, Springer-Verlag, New York, (2012)

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