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Original Research Paper

On Crossed Polysquare Version of Homotopy Kernels

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Abstract. In this paper, we introduce the notion of braiding crossed polymodule and Γ_equivariant braided crossed polymodule of polygroups and we give some of their properties. Further, we study the concept of fiber hyperproduct. Our results extend the classical results of crossed squares to crossed polysquares. Moreover, we study crossed polysquare version of homotopy kernels by using the fundamental relations.

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1 Introduction

Crossed modules and its applications play very important roles in category theory, homotopy theory, homology and cohomology of groups,

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Algebra, k-theory and etc. Crossed modules were initially defined by Whitehead [35] as a model for 2-types. Loday explored and gave the new direction to the category of crossed modules by defining equivalent category of cat^1-groups in his work [31]. Norrie gave a good example of crossed module such as actor crossed module in [32]. Conduché has defined a 2-crossed module as a model for 3-types [17]. His unpublished work determines that there exists an equivalence between the category of crossed squares of groups and that of 2-crossed modules of groups.

In [7], Arvasi and Porter showed how to go form a simplicial algebra to a 2-crossed module of algebras and back to a truncated form of simplicial algebra, and the link between simplicial algebras and crossed squares is explicitly given. The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many area, such as geometry, lattices, combinatorics and color scheme. There exists a rich bibliography: publications appeared within 2012 can be found in "Polygroup Theory and Related Systems" by Davvaz [24]. This book contains the principal definitions endowed with examples and the basic results of the theory. In [25], Dehghani, Davvaz and Alp defined crossed polysquare and some of its properties.

In this paper, we give a new application of crossed squares. This application is so important because we use the notion of polygroup to obtain crossed square. Therefore this application can be taught as a generalization of crossed square on groups. In the first two section of the paper, we review some basic facts about crossed squares and polygroups that underline the subsequent material. To define crossed polysquare, we need the notion of polygroup action. Finally we consider a crossed polysquare and by using the concept of fundamental relation, we obtain a crossed square.

2 Crossed Squares

As an algebraic model of connected 3-types, the notion of 2-crossed module was introduced by Conduché in [17], and these 2-crossed modules are equivalent to simplicial groups with Moore complex of length 2.

Crossed squares and quadratic modules are other algebraic models of connected 3-types defined by Loday and Guin-Walery [27] and Baues [10] respectively. Arvasi and Ulualan in [8] explored there relations among 2-crossed modules, quadratic modules, crossed squares and simplicial groups, and the homotopy equivalences between these structures.

Definition 2.1. *Let G be a group and Ω be a non-empty set. A binary operator $\tau : G \times \Omega \longrightarrow \Omega$ that satisfies the following axioms:*

1. $\tau(gh, \omega) = \tau(g, \tau(h, \omega))$, *for all $g, h \in G$ and $\omega \in \Omega$,*
2. $\tau(e, \omega) = \omega$, *for all $\omega \in \Omega$.*

For $\omega \in \Omega$ and $g \in G$, we write ${}^{g}\omega := \tau(g, \omega)$.

Definition 2.2. *A crossed module $\chi = (M, G, \partial, \tau)$ consists of groups M and G together with a homomorphism $\partial : M \longrightarrow G$ and a (left) action $\tau : G \times M \longrightarrow M$ on M, satisfying the conditions:*

1. $\partial({}^{g}m) = g\partial(m)g^{-1}$, *for all $m \in M$ and $g \in G$,*
2. ${}^{\partial(m)}m' = mm'm^{-1}$, *for all $m, m' \in M$.*

The standard examples of crossed modules are inclusion $M \longrightarrow G$ of a normal subgroup M of G, the zero homomorphism $M \longrightarrow G$ when M is a G-module, and any surjection $M \longrightarrow G$ with center central.

There is also an important topological example: if $F \longrightarrow E \longrightarrow B$ is a fibration sequence of pointed spaces, then the induced homomorphism $\pi_1 F \longrightarrow \pi_1 E$ of fundamental groups in naturally a crossed module [12]. To get more idea about category of crossed module we refer to read [1, 2, 3, 5, 14, 28].

In [27], Loday and Guin-Walery introduced the notion of crossed square as an algebraic model of connected 3-types.

Definition 2.3. *A crossed square is a commutative diagram of groups*

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ G_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

together with actions of the group Γ_0 on G_1, Γ_1 and G_0 (and hence actions of Γ_1 on G_1 and G_0 via ∂' and of G_0 on G_1 and Γ_1 via $\bar{p}_0$.) and a function $h : \Gamma_1 \times G_0 \longrightarrow G_1$, such that the following axioms are satisfied:

(i) the maps $\bar{p}_1$, ∂ preserve the actions of Γ_0. Furthermore with the given actions the maps ∂', $\bar{p}_0$ and $\partial\bar{p}_0 = \bar{p}_0\partial$ are crossed modules;

(ii) $\bar{p}_1 h(\beta, g) = \beta\, {}^{g}\beta^{-1}$, $\partial h(\beta, g) = {}^{\beta} g g^{-1}$;

(iii) $h(\bar{p}_1(\alpha, g)) = \alpha\, {}^{g}\alpha^{-1}$, $h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\alpha^{-1}$;

(iv) $h(\beta_1\beta_2, g) = {}^{\beta_1} h(\beta_2, g)h(\beta, g)$, $h(\beta, g_1g_2) = h(\beta, g_1)\, {}^{g_1}h(\beta, g_2)$;

(v) $h({}^{\sigma}\beta, {}^{\sigma}g) = {}^{\sigma} h(\beta, g)$;

for all $\alpha \in G_1$, $\beta, \beta_1, \beta_2 \in \Gamma_1$, $g, g_1, g_2 \in G_0$ and $\sigma \in \Gamma_0$.

Note that in these axioms a term such as ${}^{\beta}\alpha$ is α acted on by β, and so ${}^{\beta}\alpha = {}^{\partial'(\beta)}\alpha$. It is a consequence of (i) that ∂,$\bar{p}_1$ are crossed modules. Further, by (iv), h is normalized and by (iii), G_0 acts trivially on Ker$\bar{p}_1$ and Γ_1 acts trivially on Ker∂.

In [13, 31] exist some useful identities:

(a) ${}^{\beta}({}^{g}\alpha)h(\beta, g) = h(\beta, g)\, {}^{g}({}^{\beta}\alpha)$;

(b) ${}^{\beta_1}({}^{g_1}h(\beta_2, g_2))h(\beta_1, g_1) = h(\beta_1, g_1)\, {}^{g_1}({}^{\beta_1}h(\beta_2, g_2))$;

(c) $h(\bar{p}_1 h(\beta, g_1), g_2) = h(\beta, g_1)\, {}^{g_2}h(\beta, g_1)^{-1}$;

(d) $h(\beta_2, \partial h(\beta_1, g)) = {}^{\beta_2} h(\beta_1, g)h(\beta_1, g)^{-1}$;

(e) $h(\bar{p}_1(\alpha_1), \partial(\alpha_2)) = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}$;

(f) $h(\beta_1\, {}^{g_1}\beta_1^{-1}, {}^{\beta_2} g_2 g_2^{-1}) = h(\beta_1, g_1)h(\beta_2, g_2)h(\beta_1, g_1)^{-1}h(\beta_2, g_2)^{-1}$;

(g) ${}^{\beta}h(\beta^{-1}, g) = h(\beta, g)^{-1} = {}^{g} h(\beta, g^{-1})$;

(h) ${}^{\beta}({}^{g}h(\beta, g)) = h(\beta, g)$;

(i) $h(\bar{p}_1(\alpha_1)\beta_1, \partial(\alpha_2)g_2)\alpha_2\, {}^{g_2}\alpha_1 = \alpha_1\, {}^{\beta_1}\alpha_2 h(\beta_1, g_2)$;

for all $\alpha, \alpha_1, \alpha_2 \in G_1$ and $g, g_1, g_2 \in G_0$. The last three identities do not appear in any text and they are deducted from the axiom (iv).

Definition 2.4. *A morphism of crossed squares*

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ G_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array} \xrightarrow{\ \Phi\ } \begin{array}{ccc} G'_1 & \xrightarrow{\bar{p}'_1} & \Gamma'_1 \\ {\scriptstyle \bar{\partial}}\downarrow & & \downarrow{\scriptstyle \bar{\partial}'} \\ G'_0 & \xrightarrow[\bar{p}'_0]{} & \Gamma'_0 \end{array}$$

consists of four group homomorphisms $\Phi_{G_1} : G_1 \longrightarrow G'_1$, $\Phi_{G_0} : G_0 \longrightarrow G'_0$, $\Phi_{\Gamma_1} : \Gamma_1 \longrightarrow \Gamma'_1$ *and* $\Phi_{\Gamma_0} : \Gamma_0 \longrightarrow \Gamma'_0$ *such that the resulting cube of group homomorphisms is commutative;* $\Phi_{G_1}(h(\beta, g)) = h(\Phi_{\Gamma_1}(\beta), \Phi_{G_0}(g))$ *for every* $\beta \in \Gamma_1$, $g \in G_0$*; each of the homomorphisms* Φ_{G_1}, Φ_{G_0}, Φ_{Γ_1} *is* Φ_{Γ_0}*-equivariant.*

Example 2.5. (a) Given a pair of normal subgroups N_1, N_2 of a group G, we can form the following square:

$$\begin{array}{ccc} N_1 \cap N_2 & \longrightarrow & N_1 \\ \downarrow & & \downarrow \\ N_2 & \longrightarrow & G \end{array}$$

in which each morphism is an inclusion crossed module and there is a commutator map

$$h : N_1 \times N_2 \longrightarrow N_1 \cap N_2 \ .$$

$$(n_1, n_2) \longrightarrow [n_1, n_2]$$

This forms a crossed square of groups.

(b) [32] Let

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ G_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

be a crossed square with a function $h : \Gamma_1 \times G_0 \longrightarrow G_1$. Then $\langle \bar{p}_1, \bar{p}_0 \rangle$ is a morphism of crossed modules, and $\partial' : \Gamma_1 \longrightarrow \Gamma_0$ acts on $\partial : G_1 \longrightarrow G_0$.

(c) [15] Crossed squares can be seen as crossed modules in the category of crossed modules and they provide algebraic models of connected 3-types.

(d) [18] A 2-crossed module constructed from a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

as

$$L \xrightarrow{(\lambda^{-1}, \lambda')} M \rtimes N \xrightarrow{\mu\nu} P \qquad .$$

To get more idea about category of crossed square we refer to read [6, 9, 11, 13, 18, 33].

3 Polygroups and polygroup action

Suppose that H is a nonempty set and $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. Then, we can consider maps of the following type: $f_i : H \times H \longrightarrow \mathcal{P}^*(H)$, where $i \in \{1, 2, \ldots, n\}$ and n is a positive integer. The maps f_i are called *(binary) hyperoperations*. For all x, y of H, $f_i(x, y)$ is called a *(binary) hyperproduct* of x and y. An algebraic system $(H, f_1, \ldots, f_n)$ is called a *(binary) hyperstructure*. Usually $n = 1$ or $n = 2$. Under certain conditions, imposed to the maps f_i, we obtain the so-called semihypergroups, hypergroups, hyperrings or hyperfields. Sometimes, external hyperoperations are considered, which are maps of the following type: $h : R \times H \longrightarrow \mathcal{P}^*(H)$, where $R \neq H$. An example of a hyperstructure, endowed both with an internal hyperoperation and an external hyperoperation is the so-called hypermodule. Applications of hypergroups appear in special subclasses like polygroups, that they were studied by Comer [16], also see [20, 21, 22].

Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroups is a completely regular, reversible in itself multigroup.

Definition 3.1. [16] *A polygroup is a multi-valued system* $M = \langle P, \circ, e, {}^{-1}\rangle$, *with* $e \in P, {}^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, *where the following axioms hold for all* x, y, z *in* P*:*

1. $(x \circ y) \circ z = x \circ (y \circ z)$,
2. $e \circ x = x \circ e = x$,
3. $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ z$.

In the above definition, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and A, B are non-empty subsets of P, then $A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$. The following elementary facts about polygroups follow easily from the axiom: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$ and $(x^{-1})^{-1} = x$. For further discussion of polygroups, we refer to Davvaz's book [21]. Many important examples of polygroups are collected in [21] such as Double coset algebra,Prenowitz algebra, Conjugacy class polygroups, Character polygroups, Extension of polygroups, and Chromatic polygroups.

Example 3.2. Suppose that H is a subgroup of a group G. Define a system $G/\!/H = \langle \{HgH \mid \ g \in G\}, *, H, {}^{-1}\rangle$, where $(HgH)^{-1} = Hg^{-1}H$ and

$$(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H \mid \ h \in H\}.$$

The algebra of double cosets $G/\!/H$ is a polygroup.

Lemma 3.3. [21] *Every group is a polygroup.*

If K is a non-empty subset of P, then K is called a *subpolygroup* of P if $e \in K$ and $\langle K, \circ, e, {}^{-1}\rangle$ is a polygroup. The subpolygroup N of P is said to be *normal* in P if $a^{-1} \circ N \circ a \subseteq N$, for every $a \in P$. If N is a normal subpolygroup of P, then $\langle \frac{P}{N}, \bullet, N, {}^{-1}\rangle$ is a polygroup, where $N \circ a \bullet B \circ b = \{N \circ c \mid \ c \in N \circ a \bullet b\}$ and $(N \circ a)^{-1} = N \circ a^{-1}$ [21].

There are several kinds of homomorphisms between polygroups [21].

Definition 3.4. *Let $\langle P, \circ, e, ^{-1} \rangle$ and $\langle P', *, e, ^{-1} \rangle$ be two polygroups. Let Φ be a mapping from P into P' such that $\Phi(e) = e$. Then Φ is called*

1. *an inclusion homomorphism if $\Phi(a \circ b) \subseteq \Phi(a) * \Phi(b)$, for all $a, b \in P$,*
2. *a weak homomorphism if $\Phi(a \circ b) \cap \Phi(a) * \Phi(b) \neq \emptyset$, for all $a, b \in P$,*
3. *a strong homomorphism if $\Phi(a \circ b) = \Phi(\circ) * \Phi(b)$, for all $a, b \in P$.*

A strong homomorphism Φ is said to be an *isomorphism* if Φ is one to one and onto. Two polygroups P and P' are said to be *isomorphic* if there is an isomorphism from P onto P'.

For the definition of crossed polysquare, we need the notion of polygroup action.

Definition 3.5. [22] *Let $\mathcal{P} = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and Ω be a non-empty set. A map $\alpha : P \times \Omega \longrightarrow \mathcal{P}^*(\Omega)$, where $\alpha(p, \omega) :=^p \omega$ is called a (left) polygroupaction on Ω if the following axioms hold:*

1. ${}^e\omega = \omega$,
2. ${}^h({}^p\omega) =^{h \circ p} \omega$, *where* ${}^pA = \bigcup\limits_{a \in A} {}^pa$ *and* ${}^B\omega = \bigcup\limits_{b \in B} {}^b\omega$ *for all* $A \subseteq \Omega$ *and* $B \subseteq P$,
3. $\bigcup\limits_{\omega \in \Omega} {}^p\omega = \Omega$,
4. *for all* $p \in P$, $a \in {}^pb \Rightarrow b \in {}^{p^{-1}}a$.

Example 3.6. Suppose that $\langle P, \circ, e, ^{-1} \rangle$ is a polygroup. Then, P acts on itself by conjugation. Indeed if we consider the map $\alpha : P \times P \longrightarrow \mathcal{P}^*(P)$ by $\alpha(p, x) = {}^px := p \circ x \circ p^{-1}$, then

1. ${}^ex = x$,
2. ${}^h({}^px) =^h (p \circ x \circ p^{-1}) = h \circ p \circ x \circ p^{-1} \circ h^{-1} = (h \circ p) \circ x \circ (h \circ p)^{-1} = \bigcup\limits_{b \in h \circ p} (b \circ x \circ b^{-1}) = \bigcup\limits_{b \in h \circ p} {}^bx = {}^{h \circ p}x$,

3. $\bigcup_{x \in P} {}^{p}x = \bigcup_{x \in P} p \circ x \circ p^{-1} = P$,

4. if $a \in {}^{p}b = p \circ b \circ p^{-1}$, then $p \in a \circ p \circ b^{-1}$ and hence $b^{-1} \in p^{-1} \circ a^{-1} \circ p$. This implies that $b \in p^{-1} \circ a \circ p$.

Now, we give the notion of crossed polysquares.

Definition 3.7. *A crossed polysquares is a commutative diagram of polygroups*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

together with polyactions of the polygroup Γ_0 on P_1, Γ_1 and P_0 (and hence polyactions of Γ_1 on P_1 and P_0 via ∂' and of P_0 on P_1 and Γ_1 via $\bar{p}_0$) and a function $h : \Gamma_1 \times P_0 \longrightarrow \mathcal{P}^*(P_1)$, such that the following axioms are satiesfied:

1. *the maps $\bar{p}_1$, ∂ preserve the polyactions of Γ_0. Furthermore, with the given polyactions the maps ∂', $\bar{p}_0$ and $\partial'\bar{p}_1 = \bar{p}_0\partial$ are crossed polymodules;*

2. *$\bar{p}_1 h(\beta, p) = \beta^p \beta^{-1}$, $\partial h(\beta, p) = {}^{\beta}pp^{-1}$;*

3. *$h(\bar{p}_1(\alpha), p) = \alpha\, {}^{p}\alpha^{-1}$, $h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\alpha^{-1}$;*

4. *$h(\beta_1\beta_2, p) = {}^{\beta_1}h(\beta_2, p)h(\beta_1, p)$, $h(\beta, p_1p_2) = h(\beta, p_1)\, {}^{p_1}h(\beta, p_2)$;*

5. *$h({}^{\sigma}\beta, {}^{\sigma}p) = {}^{\sigma}h(\beta, p)$;*

for all $\alpha \in P_1$, $\beta, \beta_1, \beta_2 \in \Gamma_1$, $p, p_1, p_2 \in P_0$ and $\sigma \in \Gamma_0$.

It is a consequence of (1) that $\partial, \bar{p}_1$ are crossed polymodules. Further, by (4), h is normalized and by (3), P_0 acts trivially on Ker$\bar{p}_1$ and Γ_1 acts trivially on Ker∂.

We have some useful identities:

(a) ${}^{\beta}({}^{p}\alpha)h(\beta, p) = h(\beta, p)\, {}^{p}({}^{\beta}\alpha)$;

(b) ${}^{\beta_1}({}^{p_1}h(\beta_2, p_2))h(\beta_1, p_1) = h(\beta_1, p_1)\,{}^{p_1}({}^{\beta_1}h(\beta_2, p_2))$;

(c) $h(\bar{p}_1 h(\beta, p_1), p_2) = h(\beta, p_1)\,{}^{p_2}h(\beta, p_1)^{-1}$;

(d) $h(\beta_2, \partial h(\beta_1, p)) = {}^{\beta_2}h(\beta_1, p)h(\beta_1, p)^{-1}$;

(e) $h(\bar{p}_1(\alpha_1), \partial(\alpha_2)) = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}$;

(f) $h(\beta_1\,{}^{p_1}\beta_1^{-1}, {}^{\beta_2}p_2p_2^{-1}) = h(\beta_1, p_1)h(\beta_2, p_2)h(\beta_1, p_1)^{-1}h(\beta_2, p_2)^{-1}$;

(g) ${}^{\beta}h(\beta^{-1}, p) = h(\beta, p)^{-1} = {}^{p}h(\beta, p^{-1})$;

(h) ${}^{\beta}({}^{p}h(\beta, p)) = h(\beta, p)$;

(i) $h(\bar{p}_1(\alpha_1)\beta_1, \partial(\alpha_2)p_2)\alpha_2\,{}^{p_2}\alpha_1 = \alpha_1\,{}^{\beta_1}\alpha_2 h(\beta_1, p_2)$;

for all $\alpha, \alpha_1, \alpha_2 \in P_1$ and $p, p_1, p_2 \in P_0$.

Example 3.8. Given a pair of normal subpolygroups N_1, N_2 of a polygroup P, we can form the following square:

$$\begin{array}{ccc} N_1 \cap N_2 & \longrightarrow & N_1 \\ \downarrow & & \downarrow \\ N_2 & \xrightarrow[\bar{p}_0]{} & P \end{array}$$

in which each morphism is an inclusion crossed polymodule and there is a commutator map

$$h : N_1 \times N_2 \longrightarrow \mathcal{P}^*(N_1 \cap N_2) \;;$$

$$(n_1, n_2) \longrightarrow [n_1, n_2]$$

where $[x, y]$ is $\{z \mid\ z \in xyx^{-1}y^{-1}\}$. This forms a crossed polysquare of polygroups.

Example 3.9. If

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

be a crossed polysquare with function $h : \Gamma_1 \times P_0 \longrightarrow \mathcal{P}^*(P_1)$, then $\langle \bar{p}_1, \bar{p}_0 \rangle$ is a morphism of crossed polymodules, and $\partial' : \Gamma_1 \longrightarrow \Gamma_0$ acts on $\partial : P_1 \longrightarrow P_0$.

Example 3.10. Let

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

be a crossed polysquare with a function $h : \Gamma_1 \times P_0 \longrightarrow \mathcal{P}^*(P_1)$. Then we can construct the semi-direct crossed polymodule and other one, given by:

$$\langle \bar{p}_1, \bar{p}_0 \rangle : P_1 \rtimes P_0 \longrightarrow \Gamma_1 \rtimes \Gamma_0.$$

The polyactions of P_0 on P_1 and of Γ_0 on Γ_1 are the natural polyactions and the polyaction of $\Gamma_1 \rtimes \Gamma_0$ on $P_1 \rtimes P_0$ is defined by:

$${}^{(\beta,\sigma)}(\alpha, p) = \{(x, y) \mid \ x \in {}^{\partial'(\beta)\sigma} \alpha h(\beta, {}^{\sigma} p),\ y \in {}^{\sigma} p\}.$$

Theorem 3.11. *Every crossed square is a crossed polysquare.*

Proof. By using Lemma 3.3, the proof is straightforward. □

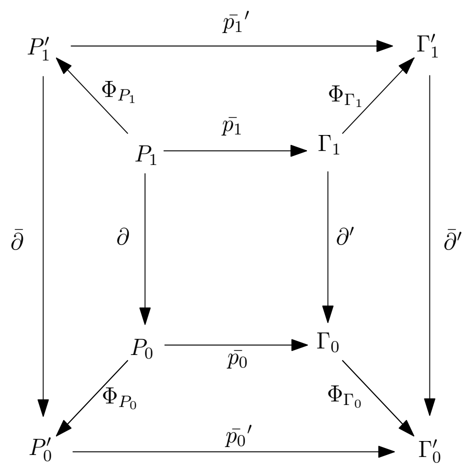
Definition 3.12. *A morphism of crossed polysquares*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array} \quad \xrightarrow{\Phi} \quad \begin{array}{ccc} P'_1 & \xrightarrow{\bar{p}'_1} & \Gamma'_1 \\ {\scriptstyle\bar{\partial}}\downarrow & & \downarrow{\scriptstyle\bar{\partial}'} \\ P'_0 & \xrightarrow[\bar{p}'_0]{} & \Gamma'_0 \end{array}$$

consists of four strong homomorphisms $\Phi = \langle \Phi_{P_1}, \Phi_{P_0}, \Phi_{\Gamma_1}, \Phi_{\Gamma_0} \rangle$,

$$\Phi_{P_1} : P_1 \longrightarrow P'_1, \quad \Phi_{P_0} : P_0 \longrightarrow P'_0, \quad \Phi_{\Gamma_1} : \Gamma_1 \longrightarrow \Gamma'_1, \quad \Phi_{\Gamma_0} : \Gamma_0 \longrightarrow \Gamma'_0$$

such that the resulting cube of polygroup strong homomorphisms is commutative; $\Phi_{P_1}(h(\beta, p)) = h(\Phi_{\Gamma_1}(\beta), \Phi_{P_0}(p))$ *for every* $\beta \in \Gamma_1$, $p \in P_0$*; each of the strong homomorphisms* Φ_{P_1}, Φ_{P_0}, Φ_{Γ_1} *is* Φ_{Γ_0}*-equivariant.*



We say that Φ is an *isomorphism* if Φ_{P_1}, Φ_{P_0}, Φ_{Γ_1} and Φ_{Γ_0} are isomorphisms. Similarly, we can defined *monomorphism*, *epimorphism* and *automorphism* of crossed polysquares.

Crossed polysquares and their morphisms form a category that will be denoted by CPS.

4 Crossed squares derived from crossed polysquares

In this section, we consider a crossed polysquare and by using the concept of fundamental relation, we obtain a crossed square.

Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. We define the relation β_P^* as the smallest equivalence relation on P such that the quotient $\frac{P}{\beta_P^*}$, the set of all equivalence classes, is a group. In this case β_P^* is called the *fundamental equivalence* on P and $\frac{P}{\beta_P^*}$ is called the *fundamental group*. The product $\odot$ in $\frac{P}{\beta_P^*}$ is defined as follows:

$$\beta_P^*(x) \odot \beta_P^*(y) = \beta_P^*(z), \quad \text{for all } z \in \beta_P^*(x) \circ \beta_P^*(y).$$

This relation is introduced by Koskas [29] and studied mainly by Corsin [19], Leoreanu-Fotea [30] and Freni [26] concerning hypergroups, Vougiouklis [34] concerning H_ν-groups, Davvaz concerning polygroups [23], and many others. We consider the relation β_P as follows:

$$x\beta_P y \Longleftrightarrow \text{ there exist } z_1,\ldots,z_n \text{ such that } \{x,y\} \subseteq \circ\prod_{i=1}^{n} z_i.$$

Freni in [26] proved that for hypergroups $\beta = \beta^*$. Since polygroups are certain subclass of hypergroups, we have $\beta_P^* = \beta_P$. The kernel of the *canonical map* $\phi_P : P \longrightarrow \frac{P}{\beta_P^*}$ is called the *core* of P and is denoted by w_P. Here we also denote by w_P the unite of $\frac{P}{\beta_P^*}$. It is easy to prove that the following statements: $w_P = \beta_P^*(e)$ and $\beta_P^*(x)^{-1} = \beta_P^*(x^{-1})$, for all $x \in P$.

Lemma 4.1. [4] *w_P is a subpolygroup of P.*

Lemma 4.2. [4] *For every $p \in P$, $p \circ p^{-1} \subseteq w_P$.*

Proposition 4.3. [4] *Let $\langle C, *, e, ^{-1}\rangle$ and $\langle P, \circ, e, ^{-1}\rangle$ be two polygroups and let $\partial : C \longrightarrow P$ be a strong homomorphism. Then, ∂ induces a group homomorphisms $\mathcal{D} : \frac{C}{\beta_c^*} \longrightarrow \frac{P}{\beta_P^*}$ be setting*

$$\mathcal{D}(\beta_C^*(c)) = \beta_P^*(\partial(c)), \quad \textit{for all } c \in C.$$

Definition 4.4. [4] *We say the action of P on C is productive, if for all $c \in C$ and $p \in P$ there exist $c_1,\ldots,c_n$ in C such that $c^p = c_1 * \cdots * c_n$.*

Example 4.5. Suppose that $\langle P, \circ, e, ^{-1}\rangle$ is a polygroup. Then, P acts on itself if we define ${}^g x := x \circ g^{-1}$ or ${}^g x := g \circ x$, for all $x, g \in P$ and the action is productive.

Example 4.6. Suppose that $\langle P, \circ, e, ^{-1}\rangle$ is a polygroup. Then, P acts on itself by conjugation and the action is productive.

According [4], let $\langle C, *, e, ^{-1}\rangle$ and $\langle P, \circ, e, ^{-1}\rangle$ be two polygroups and let $\alpha : P \times C \longrightarrow \mathcal{P}^*(C)$ be a productive action on C. We define the map $\Psi : \frac{P}{\beta_P^*} \times \frac{P}{\beta_C^*} \longrightarrow \mathcal{P}^*(\frac{P}{\beta_C^*})$ as usual manner:

$$\Psi(\beta_P^*(p), \beta_C^*(c)) = \{\beta_C^*(x) \mid\ x \in \bigcup_{\substack{y\,\in\,\beta_C^*(c)\\ x\,\in\,\beta_P^*(p)}} {}^z y\}.$$

By definition of β^*_C, since the action of P on C is productive, we conclude that $\Psi(\beta^*_P(p), \beta^*_C(c))$ is singleton, i.e., we have

$$\Psi : \frac{P}{\beta^*_P} \times \frac{P}{\beta^*_C} \longrightarrow \frac{P}{\beta^*_C}, \quad \Psi(\beta^*_P(p), \beta^*_C(c)) = \beta^*_C(x), \quad \text{for all } x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ x \in \beta^*_P(p)}} {}^z y$$

We denote $\Psi(\beta^*_P(p), \beta^*_C(c)) = {}^{[\beta^*_P(p)]}\,[\beta^*_C(c)]$.

Proposition 4.7. *[4] Let $\langle C, *, e, ^{-1} \rangle$ and $\langle P, \circ, e, ^{-1} \rangle$ be two polygroups and let $\alpha : P \times C \longrightarrow \mathcal{P}^*(C)$ be a productive action on C. Then, Ψ is an action of the group $\frac{P}{\beta^*_P}$ on the group $\frac{P}{\beta^*_C}$.*

Example 4.8. Suppose that $\langle P, \circ, e, ^{-1} \rangle$ is any polygroup. Then $\frac{P}{\beta^*_P}$ is a group. Suppose that $\mathrm{Aut}(\frac{\mathrm{P}}{\beta^*_{\mathrm{P}}})$ its a group of automorphisms. There is an obvious action α of $\mathrm{Aut}(\frac{\mathrm{P}}{\beta^*_{\mathrm{P}}})$ on $\frac{P}{\beta^*_P}$, and a group homomorphism $\partial : \frac{P}{\beta^*_P} \longrightarrow \mathrm{Aut}(\frac{\mathrm{P}}{\beta^*_{\mathrm{P}}})$ sending each $\beta^*_P(p)$ to the inner automorphism of conjugation by $\beta^*_P(p)$. These together a crossed module $(\frac{P}{\beta^*_P}, \mathrm{Aut}(\frac{\mathrm{P}}{\beta^*_{\mathrm{P}}}), \partial, \alpha)$.

Theorem 4.9. [25] *Let*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

Diagram (1)

be a crossed polysquare, such that the actions are productive. Then,

$$\begin{array}{ccc} \frac{P_1}{\beta^*_{P_1}} & \xrightarrow{\Psi} & \frac{\Gamma_1}{\beta^*_{\Gamma_1}} \\ {\scriptstyle\mathcal{D}}\downarrow & & \downarrow{\scriptstyle\mathcal{D}'} \\ \frac{P_0}{\beta^*_{P_0}} & \xrightarrow[\Psi']{} & \frac{\Gamma_0}{\beta^*_{\Gamma_0}} \end{array}$$

is a crossed square with actions and function $\bar{h} : \frac{\Gamma_1}{\beta^*_{\Gamma_1}} \times \frac{P_0}{\beta^*_{P_0}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$ defined as following;

(a) the action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_1}{\beta^*_{P_1}}$ *is induced by the polyaction of* Γ_0 *on* P_1.

(b) the action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{\Gamma_1}{\beta^*_{\Gamma_1}}$ *is induced by the polyaction of* Γ_0 *on* Γ_1.

(c) the action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_0}{\beta^*_{P_0}}$ *is induced by the polyaction of* Γ_0 *on* P_0.

(d) the map $\bar{h} : \frac{\Gamma_1}{\beta^*_{\Gamma_1}} \times \frac{P_0}{\beta^*_{P_0}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$ *is* $\bar{h}(\beta^*_{\Gamma_1}(\gamma_1), \beta^*_{P_0}(p_0)) = \beta^*_{P_1}(h(\gamma_1, p_0))$ *where the function* h *is given by the crossed polysquare structure up.*

Theorem 4.10. [25] *Let*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

be a crossed polysquare, Φ_{P_1}, Φ_{P_0}, Φ_{Γ_1} *and* Φ_{Γ_0} *be canonical maps. Then* $\Phi = \langle \Phi_{P_1}, \Phi_{P_0}, \Phi_{\Gamma_1}, \Phi_{\Gamma_0} \rangle$ *is a crossed polysquares morphism.*

5 Crossed polysquare version of homotopy kernels

There are two versions of the kernel of a morphisms of crossed polymodule,

$$\begin{array}{ccc} P_1 & \xrightarrow{\phi} & P'_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\Psi]{} & P'_0 \end{array}$$

The strict version is introduced by Davvaz and Alp[4]. In this approach, they considers crossed polymodule as the objects of a category CPM and the kernel of the morphism $\langle \phi, \Psi \rangle$ is $\partial|_{\text{Ker } \phi} : \text{Ker } \phi \longrightarrow \text{Ker } \Psi$.

The homotopical version is analyzed, where the crossed polymodules as the objects of a 2-category. The kernel is given by the homotopy fibre over the unit object of morphism of categorical polygroups $P(\partial) \longrightarrow P(\partial')$.

Definition 5.1. *(Fiber hyperproduct) Let P_1, P_2 and Q be polygroups, and let $\phi : P_1 \longrightarrow Q$ and $\psi : P_2 \longrightarrow Q$ be homomorphisms. The fiber hyperproduct of P_1 and P_2 over Q, also known as a pullback, is the following subpolygroup of $P_1 \times P_2$:*

$$P_1 \times_Q P_2 = \{(p_1, p_2) \mid (p_1, p_2) \in P_1 \times P_2,\ \phi(p_1) = \psi(p_2)\}.$$

If $\phi : P_1 \longrightarrow Q$ and $\psi : P_2 \longrightarrow Q$ are epimorphisms, then this is a subdirect product.

Example 5.2. Suppose that $P_1 = \{e, a, b, c, d, f, g, h\}$, $P_2 = \{e, x, y, z\}$ and $Q = \{e, t\}$ are three polygroups with the following hyperoperations:

$\bullet$	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	e	c	b	f	d	h	g
b	b	c	b	c	$\{e, b, d, g\}$	$\{a, c, f, h\}$	g	h
c	c	b	c	b	$\{a, c, f, h\}$	$\{e, b, d, g\}$	h	g
d	d	f	$\{e, h, d\}$	$\{a, c, f\}$	d	f	$\{d, g\}$	$\{f, h\}$
f	f	d	$\{a, c, f\}$	$\{e, b, d\}$	f	d	$\{f, h\}$	$\{d, g\}$
g	g	h	$\{b, g\}$	$\{c, h\}$	g	h	$\{e, b, d, g\}$	$\{a, c, f, h\}$
h	h	g	$\{c, h\}$	$\{b, g\}$	h	g	$\{a, c, f, h\}$	$\{e, b, d, g\}$

$\circ$	e	x	y	z
e	e	x	y	z
x	x	$\{e, y\}$	$\{x, y\}$	z
y	y	$\{x, y\}$	$\{e, x\}$	z
z	z	z	z	$\{e, x, y\}$

$*$	e	t
e	e	t
t	t	$\{e, t\}$

Suppose that

$$\begin{aligned} &\Phi : P_1 \longrightarrow Q \\ &\Phi(e) = \Phi(b) = \Phi(d) = \Phi(g) = e \\ &\Phi(a) = \Phi(c) = \Phi(f) = \Phi(h) = t \end{aligned}$$

and

$$\begin{aligned} &\phi : P_2 \longrightarrow Q \\ &\phi(e) = \phi(y) = e \\ &\phi(x) = \phi(z) = t \end{aligned}$$

Then the fiber hyperproduct of P_1 and P_2 over Q, is the following sub-polygroup of $P_1 \times P_2$:

$$\begin{aligned} P_1 \times_Q P_2 \;=\; & \{(e,e),(b,e),(b,y),(d,e),(d,y),(g,e),(g,y),(a,x),(a,z), \\ & (c,x),(c,z),(f,x),(f,z),(h,x),(h,z)\} \end{aligned}$$

Definition 5.3. *A braided crossed polymodule of polygroups $\partial : P_1 \longrightarrow P_0$ is a crossed polymodule with a braiding polyfunction $\{-,-\} : P_0 \times P_0 \longrightarrow \mathcal{P}^*(P_1)$ satisfying the following axioms:*

(i) $\{p_1, p_2p_3\} = \{p_1, p_2\}\,{}^{p_2}\{p_1, p_3\}$;

(ii) $\{p_1p_2, p_3\} = {}^{p_1}\{p_2, p_3\}\{p_1, p_3\}$;

(iii) $\partial\{p_1, p_2\} = p_1p_2p_1^{-1}p_2^{-1}$;

(iv) $\{\partial(\alpha), p\} = \alpha^p\alpha^{-1}$;

(v) $\{p, \partial(\alpha)\} = {}^p\alpha\alpha^{-1}$; *for all* $\alpha \in P_1$ *and* $p, p_1, p_2, p_3 \in P_0$.
If the braiding is symmetric, we also have:

(vi) $\{p_1, p_2\}\{p_2, p\} = 1$,

Then the crossed polymodule $\partial : P_1 \longrightarrow P_0$ is called symmetric crossed polymodule.

We denote such a braided crossed polymodule by (P_1, P_0, ∂).

Example 5.4. Any identity map of polygroups $\partial : P \longrightarrow P$ is a braided crossed polymodule with $\{a, b\} = ab$.

Example 5.5. Suppose P be a polygroup and P^2 be generated by $\{ab|\, a, b \in P\}$. $\partial : P^2 \longrightarrow P$ is a braided crossed polymodule with $\{a, b\} = ab$; for $a, b \in P$.

Example 5.6. Let (P_1, P_0, ∂_1) and (P_1', P_0', ∂_2) be two braided crossed polymodules, then $(P_1 \times P_1', P_2 \times P_2', \partial)$ is a braided crossed polymodule.

Example 5.7. Zero morphism $0 : P_1 \longrightarrow P_2$ is a braided crossed polymodule with $\{a, b\} = e$.

Proposition 5.8. *If $\partial : P_1 \longrightarrow P_0$ be a braided crossed polymodule, then*

(i) ${}^{p_1}\{p_1^{-1}, p_2\} = \{p_1, p_2\}^{-1} =^{p_2} \{p_1, p_2^{-1}\}$;

(ii) ${}^{p_1p_2}\{p_1^{-1}, p_2^{-1}\} = \{p_1, p_2\}$;

(iii) ${}^{p}\{p, p\} = \{p, p\}$;

(iv) $\{p_1p_2, p_3\} = \{p_1, p_2p_3p_2^{-1}\}\{p_2, p_3\}$;

(v) $\{p_1, p_2p_3\} = \{p_1, p_3\}\{p_3p_1p_3^{-1}, p_2\}$;

(vi) ${}^{p_1}\{p_2, p_3\} = \{p_1p_2p_1^{-1}, p_1p_3p_1^{-1}\}$;

(vii) $\{\partial(\alpha_1)p_1, \partial(\alpha_2)p_2\}\alpha_2^{p_2}\alpha_1 = \alpha_1^{p_1}\alpha_2\{p_1, p_2\}$;
for all $\alpha_1, \alpha_2 \in P_1$ and $p, p_1, p_2, p_3 \in P_0$.

Proof. It is straightforward. □

Definition 5.9. *A Γ_equivariant braided crossed polymodule is a braided crossed polymodule $\partial : P_1 \longrightarrow P_0$ equipped with an polyaction by a polygroup Γ and the braiding are assumed to be Γ_equivariant in the sense that ${}^{\sigma}\{p_1, p_2\} = \{{}^{\sigma}p_1, {}^{\sigma}p_2\}$.*

Example 5.10. According to Lemma 3.2, every group is a polygroup. Let N be a normal subgroup of a group G so that the quotient group $\frac{G}{N}$ is abelian, On the other hand , let N be a normal subgroup in G which contains the derived group of G. Then $(N, G, i, \mu, [,])$ is a braided crossed module, where $i : N \longrightarrow G$ is an inclusion, $\mu : G \longrightarrow \mathrm{Aut}(N)$ is defined by conjugation and $\eta : G \times G \longrightarrow N$, $\eta(a, b) = [a, b] = aba^{-1}b^{-1}$.

Definition 5.11. *A morphism between braided crossed polymodules is a morphism between crossed polymodules which is compatible with the braiding map $\{-, -\}$.*

Example 5.12. We know that if (P_1, P_0, ∂_1) and (P'_1, P'_0, ∂_2) are braided crossed polymodules, then a morphism

$$(f_1, f_0) : (P_1, P_0, \partial_1) \longrightarrow (P'_1, P'_0, \partial_2)$$

Of braided crossed polymodules is given by a morphism of crossed polymodules such that

$$\{-,-\}(f_0 \times f_0) = f_1\{-,-\}.$$

Let $f, g : (P_1, P_0, \partial_1) \longrightarrow (P_1', P_0', \partial_2)$ be two morphisms of braided crossed polymodules and D denotes the set $D = \{p_1 \in P_1 | \, f(p_1) = g(p_1)\}$, then $\{D, d\}$ has the structure of a braided crossed polymodule, and the inclusion $h : \{D, d\} \longrightarrow (P_1, P_0, \partial_1)$ is a morphism of braided crossed polymodules.

Theorem 5.13. *If*

$$\begin{CD} P_1 @>{\bar{p}_1}>> \Gamma_1 \\ @V{\partial}VV @VV{\partial'}V \\ P_0 @>>{\bar{p}_0}> \Gamma_0 \end{CD}$$

Diagram (2)

is a crossed polysquare, then Diagram (3)

$$\begin{CD} P_1 @= {id_{p_1}} @= P_1 \\ @V{\bar{\partial}}VV @VV{\partial}V \\ P_0 \times_{\Gamma_0} \Gamma_1 @>{pP_0}>> P_0 \end{CD}$$

Diagram (3)

gives rise to a crossed polysquare (that is a crossed polymodule of crossed polymodules) with actions, strong homorphism pP_0 and function $\hat{h} : P_1 \times (P_0 \times_{\Gamma_0} \Gamma_1) \longrightarrow \mathcal{P}^*(P_1)$ defined as following:

(a) the polyaction of P_0 on P_1 is the polyaction of the crossed polymodule $\partial : P_1 \longrightarrow P_0$;

(b) *the polyaction of* P_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$ *is defined by* ${}^{p}(p_2, \beta_2) = \{(x, y) \,|x \in pp_2p^{-1},\, y \in {}^{p}\beta_2\}$;

(c) $pP_0 : P_0 \times_{\Gamma_0} \Gamma_1 \longrightarrow P_0$ *is the canonical projection on* P_0.

(d) $\hat{h}(\alpha, (p_2, \beta_2)) := \alpha^{p_2}\alpha^{-1}$, *where the function* h *is given by the crossed polysquare in definition 3.7, and* $\hat{h}(\alpha, (p_2, p_2)) = h(\bar{p}_1(\alpha), p_2)$.

Proof. The polyaction of P_0 on $P_0 \times_{\Gamma_0} \Gamma_1$ is well defined. So we want to check the five properties making diagram a crossed polysquare.

1. The map $\mathrm{id}_{P_1} : P_1 \longrightarrow P_1$ is the action of P_0. The map $\bar{\partial}$ preserves the action of P_0:

$$\begin{aligned}
\bar{\partial}({}^{p}\alpha) &= (\partial({}^{p}\alpha), \bar{p}_1({}^{p}\alpha)) = \{(x, y) \,|\, x \in p\partial(\alpha)p^{-1},\, y \in \bar{p}_1({}^{\bar{p}_0(p)}\alpha)\} \\
&= \{(x, y) \,|\, x \in p\partial(\alpha)p^{-1},\, y \in {}^{\bar{p}_0(p)}\bar{p}_1(\alpha)\} \\
&= \{(x, y) \,|\, x \in p\partial(\alpha)p^{-1},\, y \in {}^{p}\bar{p}_1(\alpha)\} \\
&={}^{p} (\partial(\alpha), \bar{p}_1(\alpha)) ={}^{p} \bar{\partial}(\alpha).
\end{aligned}$$

But ∂ is a crossed polymodule, because diagram (2) is a crossed polysquare and so pP_0 is a crossed polymodule, because

$$\begin{aligned}
pP_0({}^{p}(p_2, \beta_2)) &= pP_0\{(x, y) \,|\, x \in pp_2p^{-1},\, y \in^{p} \beta_2\} \\
&= \{x|\, x \in pp_2p^{-1}\} \\
&= ppP_0(p_2, \beta_2)p^{-1};
\end{aligned}$$

$$\begin{aligned}
{}^{pP_0(p_2,\beta_2)}(p_2', \beta_2') &={}^{p_2} (p_2, \beta_2) = \{(x, y) \,|\, x \in p_2p_2'p_2^{-1},\, y \in {}^{p_2}\beta_2'\} \\
&= \{(x, y) \,|\, x \in p_2p_2'p_2^{-1},\, y \in {}^{\bar{p}_0(p_2)}\beta_2'\} \\
&= \{(x, y) \,|\, x \in p_2p_2'p_2^{-1},\, y \in {}^{\partial'(\beta_2)}\beta_2'\} \\
&= \{(x, y) \,|\, x \in p_2p_2'p_2^{-1},\, y \in \beta_2\beta_2'\beta_2^{-1}\} \\
&= (p_2, \beta_2)(p_2', \beta_2')(p_2, \beta_2)^{-1}.
\end{aligned}$$

But $\partial : P_1 \longrightarrow P_0$ is a crossed polymodule, so $pP_0\bar{\partial} = \partial\mathrm{id}_{P_1}$ is a crossed polymodule.

2. We have $\mathrm{id}_{P_1}\left(\hat{h}(\alpha, (p_2, \beta_2))\right) = \alpha^{\,p_2}\alpha^{-1} = \alpha^{(p_2,\beta)}\alpha^{-1}$.

Moreover, we have $\bar{\partial}\hat{h}(\alpha,(g_2,\beta_2)) = {}^{\alpha}(g_2,\beta_2)(g_2,\beta_2)^{-1}$, because

$$\begin{array}{l}
\bar{\partial}\hat{h}(\alpha,(p_2,\beta_2))\\
= \{(x,y)\,|\,x\in\partial(\alpha\,{}^{p_2}\alpha^{-1}),\ y\in\bar{p}_1(\alpha\,{}^{p_2}\alpha^{-1})\}\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\bar{p}_1(\alpha)\,{}^{p_2}\bar{p}_1(\alpha)^{-1}\}\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\bar{p}_1(\alpha)\,{}^{\bar{p}_0(p_2)}\bar{p}_1(\alpha)^{-1}\}\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\bar{p}_1(\alpha)\,{}^{\partial'(\beta_2)}\bar{p}_1(\alpha)^{-1}\}\\
\{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\bar{p}_1(\alpha)\beta_2\bar{p}_1(\alpha)^{-1}\beta_2^{-1}\};
\end{array}$$

and

$$\begin{array}{l}
{}^{\alpha}(p_2,\beta_2)(p_2,\beta_2)^{-1}\\
={}^{\partial(\alpha)}\,(p_2,\beta_2)(p_2,\beta_2)^{-1}\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1},\ y\in\,{}^{\partial(\alpha)}\beta_2\}(p_2^{-1},\beta_2^{-1})\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\,{}^{\bar{p}_0(\partial(\alpha))}\beta_2\beta_2^{-1}\}\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\,{}^{\partial'(\bar{p}_1(\alpha))}\beta_2\beta_2^{-1}\}\\
= \{(x,y)\,|\,x\in\partial(\alpha)p_2\partial(\alpha)^{-1}p_2^{-1},\ y\in\bar{p}_1(\alpha)\beta_2\bar{p}_1(\alpha)^{-1}\beta_2^{-1}\}.
\end{array}$$

3. We have

$$\hat{h}(\mathrm{id}_{P_1}(\alpha),(p_2,\beta_2)) = \hat{h}(\alpha,(p_2,\beta_2)) = \alpha\,{}^{p_2}\alpha^{-1} = \alpha\,{}^{(p_2,\beta_2)}\alpha^{-1}$$

and

$$\begin{array}{rl}
\hat{h}(\alpha,\bar{\partial}(\alpha')) & = \hat{h}(\alpha,(\partial(\alpha'),\bar{p}_1(\alpha'))) = \alpha\,{}^{\partial(\alpha')}\alpha^{-1}\\
& = \alpha\alpha'\alpha^{-1}\alpha'^{-1} ={}^{\partial(\alpha)}\,\alpha'\alpha'^{-1} ={}^{\alpha}\,\alpha'\alpha'^{-1}
\end{array}$$

4. We have

$$\begin{array}{rl}
\hat{h}(\alpha\alpha',(p_2,\beta_2)) & = \alpha\alpha'\,{}^{p_2}(\alpha\alpha')^{-1}\\
& = \alpha\alpha'\,{}^{p_2}\alpha'^{-1}\,{}^{p_2}\alpha^{-1}\\
& = \alpha\alpha'\,{}^{p_2}\alpha'^{-1}\alpha^{-1}\alpha^{p_2}\alpha^{-1}\\
& = \alpha\hat{h}(\alpha',(p_2,\beta_2))\alpha^{-1}\hat{h}(\alpha,(p_2,\beta_2))\\
& ={}^{\alpha}\,\hat{h}(\alpha',(p_2,\beta_2))\hat{h}(\alpha,(p_2,\beta_2));
\end{array}$$

and

$$\begin{array}{rl}
\hat{h}(\alpha,(p_2,\beta_2)(p_2',\beta_2')) & = \alpha^{p_2p_2'}\alpha^{-1} = \alpha\,{}^{p_2}\alpha^{-1}\,{}^{p_2}\alpha\,{}^{p_2p_2'}\alpha^{-1}\\
& = \alpha\,{}^{p_2}\alpha^{-1}\,{}^{p_2}(\alpha\,{}^{p_2'}\alpha^{-1})\\
& = \hat{h}(\alpha,(p_2,\beta_2))\,{}^{p_2}\hat{h}(\alpha,(p_2',\beta_2'))\\
& = \hat{h}(\alpha,(p_2,\beta_2))\,{}^{(p_2,\beta_2)}\hat{h}(\alpha,(p_2',\beta_2')).
\end{array}$$

5.
$$\begin{aligned}\hat{h}({}^{p}\alpha,{}^{p}(p_2,\beta_2)) &= {}^{p}\alpha^{\,pp_2p^{-1}}({}^{p}\alpha^{-1}) = {}^{p}\alpha^{\,pp_2}\alpha^{-1}\\ &= {}^{p}(\alpha^{p_2}\alpha^{-1}) = {}^{p}\hat{h}(\alpha,(p_2,\beta_2)).\end{aligned}$$

□

Example 5.14. If P is a polygroup and P_1, P_2 , normal subpolygroups, and $L = P_1 \cap P_2$;

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & P_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_2 & \xrightarrow[\lambda]{} & P \end{array}$$

with all maps the evident inclusions, all polycation by conjugation, and $h : P_1 \times P_2 \longrightarrow P^*(L)$ given by $h(p_1, p_2) = [p_1, p_2]$, then the diagram is a crossed polysquare and

$$\begin{array}{ccc} L & \xrightarrow{id_{L_1}} & L \\ {\scriptstyle\bar{\partial}}\downarrow & & \downarrow{\scriptstyle\partial} \\ P_2 \times_P P_1 & \xrightarrow[p_{P_2}]{} & P_2 \end{array}$$

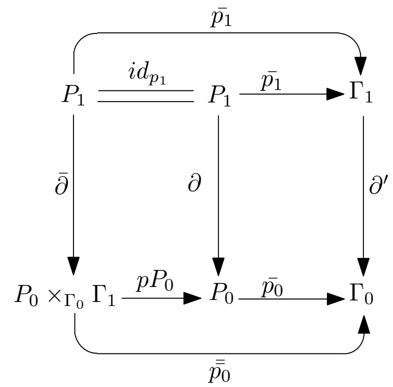
gives rise to a crossed polysquare of crossed polymodules.

If $\langle \bar{p}_1, \bar{p}_0 \rangle$ is just a morphism of crossed polymodules, then Diagram (3) is still a crossed polysquare. This is a generalization of the well-known fact in the category of polygroups that if $\partial : P_1 \longrightarrow P_0$ is a morphism of polygroups then $\operatorname{Ker}\partial \longrightarrow P_1$ is a crossed polymodule of polygroups.

Theorem 5.15. *If Diagram (2) is a crossed polysquare, then the outer diagram gives rise to a crossed polysquare with polyactions and function* $\bar{h} : \Gamma_1 \times (P_0 \times_{\Gamma_0} \Gamma_1) \longrightarrow \mathcal{P}^*(P_1)$ *defined as following:*

(i) The polyaction of Γ_0 *on* P_1 *is induced by the polyaction of* $\partial' : \Gamma_1 \longrightarrow \Gamma_0$ *on* $\partial : P_1 \longrightarrow P_0$*;*

(ii) The polyaction of Γ_0 *on* Γ_1 *is the polyaction of the crossed polymodule* $\partial' : \Gamma_1 \longrightarrow \Gamma_0$*;*



(iii) The polyaction of Γ_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$ *is defined by* ${}^{\sigma}(p_2, \beta_2) = ({}^{\sigma}p_2, {}^{\sigma}\beta_2)$*;*

(iv) $\bar{h}(\beta, (p_2, \beta_2)) := h(\beta, p_2)$ *where the function* h *is given by the crossed polysquare structure of Diagram (2).*

Proof. The polyaction of Γ_0 on $P_0 \times_{\Gamma_0} \Gamma_1$ is well defined. $\bar{\bar{p}}_0$ is a polygroup strong homomorphism because $\bar{p}_0$ is and Diagram (2) commutes. Now we want to check the five properties making this diagram a crossed polysquare.

(i) The map $\bar{p}_1$ preserves the polyaction of Γ_0 because Diagram (2) is a crossed polysquare.

The map $\bar{\partial}$ preserves the polyactions of Γ_0:

$$\bar{\partial}({}^{\sigma}\alpha) = (\partial({}^{\sigma}\alpha), \bar{p}_1({}^{\sigma}\alpha)) = ({}^{\sigma}\partial(\alpha), {}^{\sigma}\bar{p}_1(\alpha)) = {}^{\sigma}(\partial(\alpha), \bar{p}_1(\alpha)) = {}^{\sigma}\bar{\partial}(\alpha).$$

∂' is a crossed polymodule because Diagram (2) is a crossed polysquare and we want to prove that $\bar{\bar{p}}_0$ is a crossed polymodule. The pre-crossed polymodule property holds because $\bar{p}_0$ satisfies the pre-crossed polymodule property. It also holds the Peitfer condition:

$$\begin{aligned} {}^{\bar{\bar{p}}_0(p_2,\beta_2)}(p_2', \beta_2') &= {}^{\bar{p}_0(p_2)}(p_2', \beta_2') = \left({}^{\bar{p}_0(p_2)}p_2', {}^{\bar{p}_0(p_2)}\beta_2'\right) \\ &= \{(x, y) \mid x \in {}^{\bar{p}_0(p_2)}p_2',\, y \in {}^{\bar{p}_0(p_2)}\beta_2'\} \\ &= \{(x, y) \mid x \in p_2 p_2' p_2^{-1},\, y \in \beta_2 \beta_2' \beta_2^{-1}\}; \end{aligned}$$

also

$$\begin{aligned} (p_2, \beta_2)(p_2', \beta_2')(p_2, \beta_2)^{-1} &= (p_2, \beta_2)(p_2', \beta_2')(p_2^{-1}, \beta_2^{-1}) \\ &= \{(x, y) \mid x \in p_2 p_2' p_2^{-1},\, y \in \beta_2 \beta_2' \beta_2^{-1}\} \end{aligned}$$

$\bar{\bar{p}}_0\bar{\partial} = \partial'\bar{p}_1$ is a crossed polymodule because Diagram (2) is a crossed polysquare.

(ii)

$$\begin{aligned}\bar{p}_1\left(\hat{h}(\beta,(p_2,\beta_2))\right) &= \bar{p}_1(h(\beta,p_2)) = \beta^{\,p_2}\beta^{-1}\\ &= \beta^{\,\bar{p}_0(p_2)}\beta^{-1} = \beta^{\,\bar{\bar{p}}_0(p_2,\beta_2)}\beta^{-1} = \beta^{\,(p_2,\beta_2)}\beta^{-1}.\end{aligned}$$

Now we want to show that $\bar{\partial}\hat{h}(\beta,(p_2,\beta_2)) = {}^{\beta}(p_2,\beta_2)(p_2,\beta_2)^{-1}$. We develop the two members separately:

$$\begin{aligned}\bar{\partial}\hat{h}(\beta,(p_2,\beta)) &= \left(\partial\hat{h}(\beta,(p_2,\beta_2)),\bar{p}_1\hat{h}(\beta,(p_2,\beta_2))\right)\\ &= (\partial h(\beta,p_2),\bar{p}_1 h(\beta,p_2))\\ &= \left\{(x,y) \mid \quad x\in {}^{\beta}p_2p_2^{-1},\, y\in\beta^{\,p_2}\beta^{-1}\right\};\end{aligned}$$

$$\begin{aligned}{}^{\beta}(p_2,\beta_2)(p_2,\beta_2)^{-1} &= {}^{\partial'(\beta)}(p_2,\beta_2)(p_2,\beta_2)^{-1}\\ &= \left({}^{\partial'(\beta)}p_2,\ {}^{\partial'(\beta)}\beta\right)(p_2^{-1},\beta_2^{-1})\\ &= \left\{(x,y) \mid \ x\in {}^{\partial'(\beta)}p_2p_2^{-1},\, y\in {}^{\partial'(\beta)}\beta_2\beta_2^{-1}\right\}\\ &= \left\{(x,y) \mid \ x\in {}^{\beta}p_2p_2^{-1},\, y\in \beta^{\partial'(\beta_2)}\beta^{-1}\right\}\\ &= \left\{(x,y) \mid \ x\in {}^{\beta}p_2p_2^{-1},\, y\in \beta^{\bar{p}_0(p_2)}\beta^{-1}\right\}\\ &= \left\{(x,y) \mid \ x\in {}^{\beta}p_2p_2^{-1},\, y\in \beta^{p_2}\beta^{-1}\right\}.\end{aligned}$$

(iii)

$$\begin{aligned}\hat{h}(\bar{p}_1(\alpha),(p_2,\beta_2)) &= h(\bar{p}_1(\alpha),p_2) = \alpha^{p_2}\alpha^{-1}\\ &= \alpha^{\,(p_2,\beta_2)}\alpha^{-1};\\ \hat{h}(\beta,\bar{\partial}(\alpha)) &= \hat{h}(\beta,(\partial(\alpha),\bar{p}_1(\alpha))) = h(\beta,\partial(\alpha))\\ &= {}^{\beta}\alpha\alpha^{-1}.\end{aligned}$$

(iv)

$$\begin{aligned}\hat{h}(\beta\beta',(p_2,\beta_2)) &= h(\beta\beta',p_2) = {}^{\beta}h(\beta',p_2)h(\beta,p_2)\\ &= {}^{\beta}\hat{h}(\beta',(p_2,\beta_2))\hat{h}(\beta,(p_2,\beta_2));\\ \hat{h}(\beta,(p_2,\beta_2)(p_2',\beta_2')) &= h(\beta,p_2p_2') = h(\beta,p_2)^{\,p_2}h(\beta,p_2')\\ &= \hat{h}(\beta,(p_2,\beta_2))^{\,(p_2,\beta_2)}\hat{h}(\beta,(p_2',\beta_2')).\end{aligned}$$

(v)

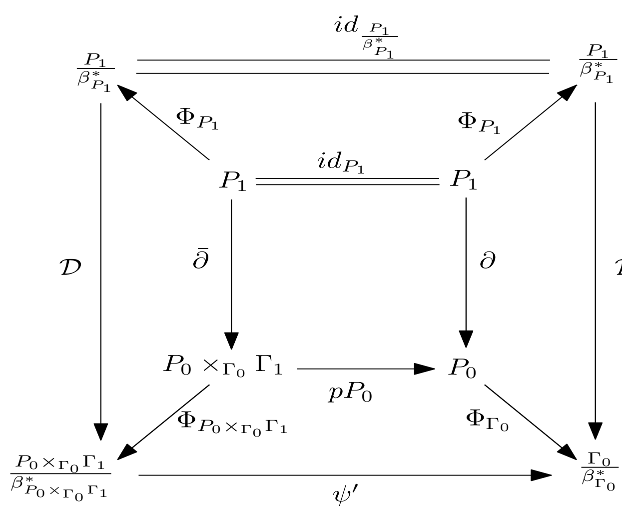
$$\begin{aligned}\hat{h}({}^{\sigma}\beta, {}^{\sigma}(p_2, \beta_2)) &= \hat{h}({}^{\sigma}\beta, ({}^{\sigma}p_2, {}^{\sigma}\beta_2)) \\ &= h({}^{\sigma}\beta, {}^{\sigma}p_2) = {}^{\sigma}h(\beta, p_2) = {}^{\sigma}\hat{h}(\beta, (p_2, \beta_2)).\end{aligned}$$

□

Theorem 5.16. *If*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

is a crossed polysquare, then the outer diagram is a crossed square with



actions and function, $\bar{h} : \frac{P_1}{\beta^*_{P_1}} \times \frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$ *defined as following:*

(a) the action of $\frac{P_0}{\beta^*_{P_0}}$ *on* $\frac{P_1}{\beta^*_{P_1}}$ *is induced by the polyaction of* P_0 *on* P_1*;*

(b) the action of $\frac{P_0}{\beta^*_{P_0}}$ *on* $\frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}}$ *is induced by the polyaction of* P_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$*;*

(c) the map $\bar{h} : \frac{P_1}{\beta^*_{P_1}} \times \frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$ *is*

$$\bar{h}\left(\beta^*_{P_1}(p_1), \beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p_0, \gamma_1)\right) = \beta^*_{P_1}(h(p_1, (p_0, \gamma_1))).$$

Proof. The action $\frac{P_0}{\beta^*_{P_0}}$ on $\frac{P_1}{\beta^*_{P_1}}$ and $\frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}}$ is well defined. We now want to check the five properties making this diagram a crossed square.

(i) The map $\mathcal{D}$ preserves the action of $\frac{P_0}{\beta^*_{P_0}}$; i.e., we have

$$\mathcal{D}({}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)) ={}^{\beta^*_{P_0}(p_0)} \mathcal{D}(\beta^*_{P_1}(p_1)).$$

Because

$$\begin{aligned} \mathcal{D}({}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)) &= \mathcal{D}(\beta^*_{P_1}(x)), \qquad \text{for all } x \in \bigcup_{\substack{y \in \beta^*_{P_1}(p_1) \\ z \in \beta^*_{P_0}(p_0)}} z_y \\ &= \beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(\partial(x)) \end{aligned}$$

and ${}^{\beta^*_{P_0}(p_0)}\mathcal{D}(\beta^*_{P_1}(p_1)) ={}^{\beta^*_{P_0}(p_0)} (\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(\partial(p_1)) = \beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(x)$ for all $x \in \bigcup_{\substack{y \in \beta^*_{P_1}(\partial(p_1)) \\ z \in \beta^*_{P_0}(p_0)}} z_y$. Also, the map ψ preserves the action of $\frac{P_0}{\beta^*_{P_0}}$. $\mathcal{D}'$ is a crossed module, because Diagram (2) is a crossed polysquare, and we want prove that ψ' is a crossed module. In fact, suppose that $p_0 \in P_0$ and $(p_0, \gamma_1) \in P_0 \times_{\Gamma_0} \Gamma_1$ are arbitrary. We have

$$\begin{aligned} &\psi'({}^{\beta^*_{P_0}(p'_0)}\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p_0, \gamma_1) \\ &= \psi'(\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(z)) \text{ for all } z \in {}^{p'_0}(p_0, \gamma_1) \\ &= \psi'(\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p'_0 p_0 p_0'^{-1}, {}^{p'_0}\gamma_1)) \\ &= \beta^*_{P_0}(p'_0)\psi'(\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p_0, \gamma_1))(\beta^*_{P_0}(p'_0))^{-1} \end{aligned}$$

and

$$\begin{aligned} &{}^{\psi'(\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p_0, \gamma_1))}\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p'_0, \gamma'_1) \\ &= {}^{\beta^*_{P_0}(pP_0(p_0))}\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p'_0, \gamma'_1) \\ &= \beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(z), \text{ for all } z \in {}^{pP_0(p_0)}(p'_0, \Gamma'_1) \\ &= \beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p_0, \gamma_1)\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p'_0, \gamma'_1)\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}(p_0, \gamma_1)^{-1}. \end{aligned}$$

$\psi'\mathcal{D} = \mathcal{D}' \operatorname{id}_{\frac{P_1}{\beta^*_{P_1}}}$ is a crossed module because $\partial : P_1 \longrightarrow P_0$ is a crossed polymodule.

(ii)

$$\begin{aligned}
\operatorname{id}_{\frac{P_1}{\beta^*_{P_1}}}(\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))) &= \beta^*_{P_1}(p_1)^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)\\
&= \beta^*_{P_1}(p_1)^{(,\beta^*_{\Gamma_1}(\gamma_1))}\beta^*_{P_1}(p_1)^{-1}
\end{aligned}$$

Now we want to prove that

$$\begin{aligned}
&\mathcal{D}(\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\\
&= {}^{\beta^*_{P_1}(p_1)}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}
\end{aligned}$$

and we develop the two members separately:

$$\begin{aligned}
&\mathcal{D}(\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1)^{\ \beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1),\bar{p}_1(\beta^*_{P_1}(p_1)^{\ \beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}))\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1)\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_1}(p_1))^{-1}\beta^*_{P_0}(p_0)^{-1},\\
&\qquad\qquad \bar{p}_1(\beta^*_{P_1}(p_1)^{\ \bar{p}_0(\beta^*_{P_0}(p_0))}\bar{p}_1(\beta^*_{P_1}(p_1))^{-1})\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1)\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_1}(p_1))^{-1}\beta^*_{P_0}(p_0)^{-1},\\
&\qquad\qquad \bar{p}_1(\beta^*_{P_1}(p_1))^{\ \partial'(\beta^*_{\Gamma_1}(\gamma_1))}\bar{p}_1(\beta^*_{P_1}(p_1))^{-1})\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1)\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_0}(p_0))^{-1}\beta^*_{P_0}(p_0)^{-1},\\
&\qquad\qquad \bar{p}_1(\beta^*_{P_1}(p_1))\beta^*_{\Gamma_1}(\gamma_1)\bar{p}_1(\beta^*_{P_0}(p_0))^{-1}\beta^*_{\Gamma_1}(\gamma_1)^{-1});
\end{aligned}$$

also,

$$\begin{aligned}
&{}^{\beta^*_{P_1}(p_1)}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}\\
&= {}^{\mathcal{D}'(\beta^*_{P_1}(p_1))}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1))\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_1}(p_1)^{-1},\\
&\qquad\qquad {}^{\mathcal{D}'(\beta^*_{P_1}(p_1))}\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)^{-1})\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1))\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_1}(p_1))^{-1}\beta^*_{P_0}(p_0)^{-1},\\
&\qquad\qquad {}^{\bar{p}_0(\mathcal{D}'(\beta^*_{P_1}(p_1)))}\beta^*_{\Gamma_1}(\Gamma_1)\beta^*_{\Gamma_1}(\Gamma_1)^{-1})\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1))\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_1}(p_1))^{-1}\beta^*_{P_0}(p_0)^{-1},\\
&\qquad\qquad {}^{\partial'(\bar{p}_1(\beta^*_{P_1}(p_1)))}\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1})\\
&= (\mathcal{D}'(\beta^*_{P_1}(p_1))\beta^*_{P_0}(p_0)\mathcal{D}'(\beta^*_{P_1}(p_1))^{-1}\beta^*_{P_0}(p_0)^{-1},\\
&\qquad\qquad \bar{p}_1(\beta^*_{P_1}(p_1))\beta^*_{\Gamma_1}(\gamma_1)\bar{p}_1(\beta^*_{P_0}(p_0))^{-1}\beta^*_{\Gamma_1}(\gamma_1)^{-1}).
\end{aligned}$$

(iii)

$$\begin{aligned}
&\hat{h}(\mathrm{id}_{\frac{p_1}{\beta^*_{P_1}}}(\beta^*_{P_1}(p_1)),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\\
&=\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\\
&=\beta^*_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}\\
&=\beta^*_{P_1}(p_1)^{\,(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\beta^*_{P_1}(p_1)^{-1};
\end{aligned}$$

also

$$\begin{aligned}
\hat{h}(\beta^*_{P_1}(p_1),\bar{\partial}(\beta^{*'}_{P_1}(p_1))) &= \hat{h}(\beta^*_{P_1}(p_1),(\mathcal{D}'(\beta^{*'}_{P_1}(p_1)),\bar{p}_1(\beta^{*'}_{P_1}(p_1)))\\
&= \beta^*_{P_1}(p_1)^{\,\mathcal{D}'(\beta^{*'}_{P_1}(p_1))}\beta^*_{P_1}(p_1)^{-1}\\
&= \beta^*_{P_1}(p_1)^{\,\mathcal{D}'(\beta^{*'}_{P_1}(p_1))}\beta^*_{P_1}(p_1)^{-1}\\
&= \beta^*_{P_1}(p_1)\beta^{*'}_{P_1}(p_1)\beta^*_{P_1}(p_1)^{-1}\beta^{*'}_{P_1}(p_1)^{-1}\\
&=^{\mathcal{D}'(\beta^*_{P_1}(p_1))}\ \beta^{*'}_{P_1}(p_1)\beta^*_{P_1}(p_1)^{-1}\\
&=^{\beta^*_{P_1}(p_1)}\ \beta^{*'}_{P_1}(p_1)\beta^{*'}_{P_1}(p_1)^{-1}.
\end{aligned}$$

(iv)

$$\begin{aligned}
&\hat{h}(\beta^*_{P_1}(p_1)\beta^{*'}_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\\
&=\beta^*_{P_1}(p_1)\beta^{*'}_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}(\beta^*_{P_1}(p_1)\beta^{*'}_{P_1}(p_1))^{-1}\\
&=\beta^*_{P_1}(p_1)\beta^{*'}_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}\beta^{*'}_{P_1}(p_1)^{-1\,\beta^*_{P_0}(p_0)}\beta^{*'}_{P_1}(p_1)^{-1}\\
&=\beta^*_{P_1}(p_1)\beta^{*'}_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}\beta^{*'}_{P_1}(p_1)^{-1}\beta^*_{P_1}(p_1)^{-1}\beta^*_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}\\
&=\beta^*_{P_1}(p_1)\hat{h}(\beta^{*'}_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\beta^*_{P_1}(p_1)^{-1}\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\\
&=^{\beta^*_{P_1}(p_1)}\ \hat{h}(\beta^{*'}_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)));
\end{aligned}$$

also

$$\begin{aligned}
&\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1)))\\
&=\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)\beta^{*'}_{\Gamma_1}(\gamma_1)))\\
&=\beta^*_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)}\beta^{*'}_{P_1}(p_1)\\
&=\beta^*_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1\,\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}\\
&=\beta^*_{P_1}(p_1)^{\,\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1\,\beta^*_{P_0}(p_0)}(\beta^*_{P_1}(p_1)^{\,\beta^{*'}_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1})\\
&=\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))^{\,\beta^*_{P_0}(p_0)}\hat{h}(\beta^*_{P_1}(p_1),(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1)))\\
&=\hat{h}(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))^{\,(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\hat{h}(\beta^*_{P_1}(p_1),(\beta^{*'}_{P_0}(p_0).\beta^{*'}_{\Gamma_1}(\gamma_1))).
\end{aligned}$$

(v)

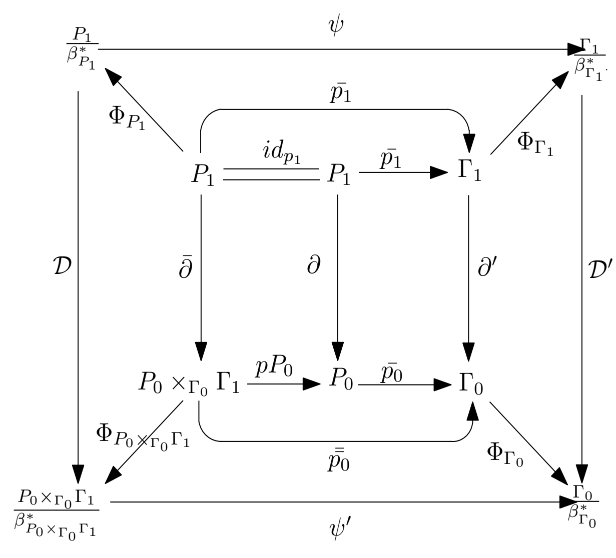
$$
\begin{aligned}
&\hat{h}({}^{\beta^{*'}_{P_0}(p_0)}\beta^*_{P_1}(p_1),\ {}^{\beta^{*'}_{P_0}(p_0)}(\beta^*_{P_0}(p_0), \beta^*_{\Gamma_1}(\gamma_1)))\\
&= \hat{h}({}^{\beta^{*'}_{P_0}(p_0)}\beta^*_{P_1}(p_1), (\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)^{-1},\ {}^{\beta^{*'}_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma_1)))\\
&= {}^{\beta^{*'}_{P_0}(p_0)}\beta^*_{P_1}(p_1)\ {}^{\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)^{-1}}({}^{\beta^{*'}_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1})\\
&= {}^{\beta^{*'}_{P_0}(p_0)}\beta^*_{P_0}(p_0)\ {}^{\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}\\
&= {}^{\beta^{*'}_{P_0}(p_0)}(\beta^*_{P_1}(p_1)\ {}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1})\\
&= {}^{\beta^{*'}_{P_0}(p_0)}\hat{h}(\beta^*_{P_1}(p_1), (\beta^*_{P_0}(p_0), \beta^*_{\Gamma_1}(\gamma_1))).
\end{aligned}
$$

□

Theorem 5.17. *If*

$$
\begin{array}{ccc}
P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\
{\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\
P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0
\end{array}
$$

is a crossed polysquare, then the outer diagram is a crossed square with



actions and function,

$$\hat{h} : \frac{\Gamma_1}{\beta^*_{\Gamma_1}} \times \frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$$

defined as following:

(a) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_1}{\beta^*_{P_1}}$ *is induced by the polyaction of* Γ_0 *on* P_1*;*

(b) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{\Gamma_1}{\beta^*_{\Gamma_1}}$ *is induced by the polyaction of* Γ_0 *on* Γ_1*;*

(c) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}}$ *is induced by the polyaction of* Γ_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$*;*

(d) *the map* $\hat{h} : \frac{\Gamma_1}{\beta^*_{\Gamma_1}} \times \frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0} \Gamma_1}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$ *is*

$$\hat{h}\left(\beta^*_{\Gamma_1}(\gamma_1), \beta^*_{P_0 \times_{\Gamma_0} \gamma_1}(p_0, \gamma_1)\right) = \beta^*_{P_1}(h(\gamma_1, (p_0, \gamma_1))).$$

Proof. The action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ on $\frac{P_0 \times_{\Gamma_0} (\gamma_1)}{\beta^*_{P_0} \times_{\Gamma_0} (\Gamma_1)}$ and $\frac{\Gamma_1}{\beta^*_{\Gamma_1}}$ and $\frac{P_1}{\beta^*_{P_1}}$ is well defined. ψ' is a group homomorphism because $\bar{p}_0$ is and Diagram (1) commutes. Now we want to check the five properties making this diagram a crossed square.

(i) The map ψ preserves the action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ because Diagram (1) is a crossed polysquare. The map $\mathcal{D}$ preserves the actions of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$:

$$\begin{aligned}
\mathcal{D}({}^{\sigma}\beta^*_{P_1}(p_1)) &= (\partial({}^{\sigma}\beta^*_{P_1}(p_1)), \psi({}^{\sigma}(\beta^*_{P_1}(p_1))) \\
&= ({}^{\sigma}\partial(\beta^*_{P_1}(p_1)), {}^{\sigma}\psi(\beta^*_{P_1}(p_1))) \\
&= {}^{\sigma}(\partial(\beta^*_{P_1}(p_1)), \psi(\beta^*_{P_1}(p_1))) \\
&= {}^{\sigma}\mathcal{D}(\beta^*_{P_1}(p_1)).
\end{aligned}$$

$\mathcal{D}'$ is a crossed module. We want to prove that ψ' is a crossed module. The pre-crossed module property holds because $\bar{p}_0$ satisfies the pre-crossed polymodule property. It also holds the Peiffer condition:

$$\begin{aligned}
&{}^{\psi'(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1))\\
&={}^{\bar{p}_0(\beta^*_{P_0}(p_0))}\,(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1))\\
&=\left({}^{\bar{p}_0(\beta^*_{P_0}(p_0))}\beta^{*'}_{P_0}(p_0),\ {}^{\bar{p}_0(\beta^*_{P_0}(p_0))}\beta^{*'}_{\Gamma_1}(\gamma_1)\right)\\
&=\left(\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\ {}^{\mathcal{D}'(\beta^*_{\Gamma_1}(\gamma_1))}\beta^{*'}_{\Gamma_1}(\gamma_1)\right)\\
&=\left(\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)\beta^{*'}_{\Gamma_1}(\gamma_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\right);
\end{aligned}$$

also

$$\begin{aligned}
&(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}\\
&=(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)^{-1})\\
&=(\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0),^{-1},\beta^*_{\Gamma_1}(\gamma_1)\beta^{*'}_{\Gamma_1}(\gamma_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}).
\end{aligned}$$

$\psi'\mathcal{D}=\mathcal{D}'\psi$ is a crossed module.

(ii)

$$\begin{aligned}
\psi\left(\hat{h}(\beta^{*'}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\right) &=\psi\left(h(\beta^{*'}_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0))\right)\\
&=\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma_1)^{-1}\\
&=\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\bar{p}_0(\beta^*_{P_0}(p_0))}\beta^*_{\Gamma_1}(\gamma_1)^{-1}\\
&=\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\psi'(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\beta^*_{\Gamma_1}(\gamma_1)^{-1}\\
&=\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\beta^*_{\Gamma_1}(\gamma_1)^{-1}
\end{aligned}$$

Now we want to show that

$$\begin{aligned}
&\mathcal{D}\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right)\\
&=\ {}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}.
\end{aligned}$$

We develop the we members separately:

$$\begin{aligned}
&\mathcal{D}\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right)\\
&=\left(\partial\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right),\psi\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right)\right)\\
&=\left(\partial h(\beta^{*'}_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)),\psi h(\beta^{*'}_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0))\right)\\
&=\left({}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma_1)^{-1}\right).
\end{aligned}$$

But

$$
\begin{aligned}
&{}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\left(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)\right)^{-1}\\
&= {}^{\mathcal{D}'(\beta^{*'}_{\Gamma_1}(\gamma_1))}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}\\
&= \left({}^{\mathcal{D}'(\beta^{*'}_{\Gamma_1}(\gamma_1))}\beta^{*'}_{P_0}(p_0),\ {}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\beta^*_{\Gamma_1}(\gamma_1)\right)\left(\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)^{-1}\right)\\
&= \left({}^{\mathcal{D}'(\beta^{*'}_{\Gamma_1}(\gamma_1))}\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\ {}^{\mathcal{D}'(\beta^{*'}_{\Gamma_1}(\gamma_1))}\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\right)\\
&= \left({}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\mathcal{D}'(\beta^*_{\Gamma_1}(\gamma_1))}\beta^{*'}_{\Gamma_1}(\gamma_1)^{-1}\right)\\
&= \left({}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\bar{p}_0(\beta^*_{P_0}(p_0))}\beta^{*'}_{\Gamma_1}(\gamma_1)^{-1}\right)\\
&= \left({}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^{*'}_{\Gamma_1}(\gamma_1)\,{}^{\beta^*_{P_0}(p_0)}\beta^{*'}_{\Gamma_1}(\gamma_1)^{-1}\right).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\hat{h}\left(\psi(\beta^*_{P_1}(p_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)))\right) &= h\left(\psi(\beta^*_{P_1}(p_1)),\beta^*_{P_0}(p_0)\right)\\
&= \beta^*_{P_1}(p_1)\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}\\
&= \beta^*_{P_1}(p_1)\,{}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\beta^*_{P_1}(p_1)^{-1};
\end{aligned}
$$

$$
\begin{aligned}
\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),\mathcal{D}(\beta^*_{P_1}(p_1))\right) &= \hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),\partial(\beta^*_{P_1}(p_1),\psi(\beta^*_{P_1}(p_1)))\right)\\
&= h\left(\beta^{*'}_{\Gamma_1}(\gamma_1),\partial(\beta^*_{P_1}(p_1))\right)\\
&= {}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\beta^*_{P_1}(p_1)\beta^*_{P_1}(p_1)^{-1}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
&\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1)\beta^{*''}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right)\\
&= h\left(\beta^{*'}_{\Gamma_1}(\gamma_1)\beta^{*''}_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\right)\\
&= {}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}h\left(\beta^{*''}_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\right)h\left(\beta^{*'}_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\right)\\
&= {}^{\beta^{*'}_{\Gamma_1}(\gamma_1)}\hat{h}\left(\beta^{*''}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right)\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))\right);
\end{aligned}
$$

$$\begin{aligned}
&\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1), (\beta^{*}_{P_0}(p_0), \beta^{*}_{\Gamma_1}(\gamma_1))(\beta^{*'}_{P_0}(p_0), \beta^{*'''}_{\Gamma_1}(\gamma_1))\right)\\
&= \hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1), (\beta^{*}_{P_0}(p_0)\beta^{*'}_{P_0}(p_0), \beta^{*}_{\Gamma_1}(\gamma_1)\beta^{*'''}_{\Gamma_1}(\gamma_1))\right)\\
&= h\left(\beta^{*'}_{\Gamma_1}(\gamma_1), \beta^{*}_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\right)\\
&= h\left(\beta^{*'}_{\Gamma_1}(\gamma_1), \beta^{*}_{P_0}(p_0)\right)\ {}^{\beta^{*}_{P_0}(p_0)}h\left(\beta^{*'}_{\Gamma_1}(\gamma_1), \beta^{*'''}_{\Gamma_1}(\gamma_1)\right)\\
&= \hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1), (\beta^{*}_{P_0}(p_0), \beta^{*}_{\Gamma_1}(\gamma_1))\right)\ {}^{(\beta^{*}_{P_0}(p_0), \beta^{*}_{\Gamma_1}(\gamma_1))}\\
&\qquad\qquad \hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1), (\beta^{*'}_{P_0}(p_0), \beta^{*'''}_{\Gamma_1}(\gamma_1))\right).
\end{aligned}$$

(v)

$$\begin{aligned}
&\hat{h}\left({}^{\sigma}\beta^{*'}_{\Gamma_1}(\gamma_1), {}^{\sigma}(\beta^{*}_{P_0}(p_0), \beta^{*}_{\Gamma_1}(\gamma_1))\right)\\
&= \hat{h}\left({}^{\sigma}\beta^{*'}_{\Gamma_1}(\gamma_1), ({}^{\sigma}\beta^{*}_{P_0}(p_0), {}^{\sigma}\beta^{*}_{\Gamma_1}(\gamma_1))\right)\\
&= h\left({}^{\sigma}\beta^{*'}_{\Gamma_1}(\gamma_1)\, {}^{\sigma}\beta^{*}_{P_0}(p_0)\right)\\
&= {}^{\sigma}h\left(\beta^{*'}_{\Gamma_1}(\gamma_1), \beta^{*}_{P_0}(p_0)\right)\\
&= {}^{\sigma}\hat{h}\left(\beta^{*'}_{\Gamma_1}(\gamma_1), (\beta^{*}_{P_0}(p_0), \beta^{*}_{\Gamma_1}(\gamma_1))\right),
\end{aligned}$$

and the proof is completed. □

6 Conclusion

Polygroups are certain subclass of hypergroups. Already, Davvaz and Alp applied the notion of crossed modules to polygroups and introduced crossed polymodules. On the other hand, the fundamental relations make a connection between polygroups and groups. We investigated crossed polysquares and using the notion of fundamental relations we obtained a crossed square from a crossed polysquare. Finally, crossed polysquare version of homotopy kernels is studied. The foundations that we made through this paper can be used to get an insight into other types of algebraic hyperstructures.

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