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Power Modified Lindley Distribution: Theory and Applications

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Abstract. The power version of the modified Lindley distribution is introduced in this paper, offering a new two-parameter lifetime distribution. As a main interest, it provides a motivated alternative to the Weibull and power Lindley distributions. We discuss its main characteristics and properties, including shapes of the probability density and hazard rate functions, incomplete moments, crude moments, variance, skewness, kurtosis and order statistics. Then, a statistical study of the model is developed. The parameters are estimated by the maximum likelihood method. A simulation study examines the numerical comportment of the bias and mean square error of the maximum likelihood estimates of the parameters. Application of the new model to three data sets is presented, showing that the model has a better fit behavior in comparison to some other well-known lifetime models, including the Weibull and power Lindley models.

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1 Introduction

The Lindley distribution pioneered by [24] has received a lot of attention during the last decades. Its primary characterization is the cumulative density function (cdf), which is defined as

$$G(x;\theta) = 1 - \left[1 + \frac{\theta x}{1+\theta}\right] e^{-\theta x}, \quad x > 0, \tag{1}$$

where $\theta > 0$, and $G(x;\theta) = 0$ for $x \leq 0$. The Lindley distribution is considered as a mixture of exponential distribution (with parameter θ) and gamma distribution (with shape parameter 2 and rate parameter θ). [17] have conducted a detailed study of various properties and applications of the Lindley distribution in reliability analysis. It was discovered that it may provide a better fit than the exponential distribution, among the most important facts.

Because of having only one parameter, there are some situations where the Lindley distribution does not provide enough flexibility for analyzing different types of lifetime data. Many researchers have proposed modified or generalized forms of the one-parameter Lindley distribution to address this issue. Some of these modifications or generalizations are the discrete Poisson-Lindley distribution by [28], zero-truncated Poisson-Lindley distribution by [18], size-biased Poisson-Lindley distribution by [16], negative binomial Lindley distribution by [36], two-parameter weighted Lindley distribution by [19], two-parameter Lindley distribution by [31], quasi Lindley distribution by [32], inverse Lindley distribution by [34], gamma Lindley distribution by [37], transmuted Lindley distribution by [26], extended Lindley distribution by [8], Akash distribution by [29], quasi Akash distribution by [30], weighted Akash distribution by [33], three-parameter generalized Lindley distribution by [15], new weighted Lindley distribution by [7], wrapped modified Lindley distribution by [14], Weibull Marshall-Olkin Lindley distribution by [2], inverted modified Lindley distribution by [12], and sum and difference of two Lindley distributions by [13]. In addition, [35] provided a review study on the Lindley distribution and its generalizations.

In particular, a generalization of the Lindley distribution, called power Lindley (PL) distribution, introduced by [20] aims to apply the power function x^α in the cdf given as (1) in order to increase its overall flexibility. Thus, the considered cdf is defined by

$$G(x;\alpha,\theta) = G(x^\alpha;\theta) = 1 - \left[1 + \frac{\theta x^\alpha}{1+\theta}\right] e^{-\theta x^\alpha}, \quad x > 0, \tag{2}$$

with $\alpha > 0$, and $G(x;\alpha,\theta) = 0$ for $x \leq 0$. This cdf is such that, if X denotes a random variable following the Lindley distribution, then $X^{1/\alpha}$ follows the PL distribution. The probability density function (pdf) of the PL distribution is a two-component mixture of a Weibull distribution (with shape parameter α and rate parameter θ), and a generalized gamma distribution (with shape parameter 2α and rate parameter θ). Then, it is shown in [20] that the parameter α can have an important role in the pliant properties of some crucial functions, such as the corresponding pdf and hazard rate function (hrf). The PL distribution has been studied and generalized by many authors in recent years. The developments include the generalized power Lindley distribution by [25], exponentiated power Lindley distribution by [6], extended power Lindley distribution by [3], alpha power transformed power Lindley distribution by [21] and exponentiated generalized power Lindley distribution by [27].

On the other hand, a weighted modification of the former Lindley distribution, called the modified Lindley (ML) distribution, has been proposed by [11]. It is defined by the following cdf:

$$F(x;\theta) = 1 - \left[1 + \frac{\theta x}{1+\theta} e^{-\theta x}\right] e^{-\theta x}, \qquad x > 0,$$

with $\theta > 0$, and $F(x;\theta) = 0$ for $x \leq 0$. In some sense, in comparison to the former Lindley distribution, the polynomial function x in the bracket is weighted by the one-parameter exponential function $e^{-\theta x}$ in such a way that (i) the definition of the cdf remains manageable and (ii) the following stochastic ordering holds: $G(x;\theta) \leq F(x;\theta) \leq H(x;\theta)$, where $G(x;\theta)$ and $H(x;\theta)$ are the cdfs of the Lindley and exponential distributions with parameter θ, respectively. Thus, it provides a motivated alternative to the Lindley and exponential distributions, keeping only one parameter and an overall simplicity. In [11], the fitting behavior of the ML model is illustrated by the consideration of three popular real data sets, outperforming the Lindley and exponential models in this regard.

In this study, as the PL distribution is for the Lindley distribution, we propose a new generalization of the ML distribution by the use of the one-parameter power function x^α; we consider the cdf given as $F(x;\alpha,\theta) = F(x^\alpha;\theta)$. The corresponding distribution is called the power modified Lindley (PML) distribution. That is, if X denotes a random variable following the ML distribution, then $X^{1/\alpha}$ follows the PML distribution. We thus defined a new two-parameter lifetime distribution satisfying the following desirable stochastic ordering property: $G(x;\alpha,\theta) \leq F(x;\alpha,\theta) \leq H(x;\alpha,\theta)$, where $G(x;\alpha,\theta)$ and $H(x;\alpha,\theta)$ are the cdfs of the PL and Weibull distributions with parameters α and θ, respectively. In this way, we develop an intermediate model between the Weibull and PL models, both well known for their relevance in data fitting. As a first objective, we describe the main properties of the PML distribution, with an emphasis on the moments. Then, the inferential properties of the related model are examined by the use of the maximum likelihood method. Application is provided to three real data sets, showing that it can be more suitable to fit data in comparison to the former Weibull, PL, exponentiated power Lindley and three-parameter generalized Lindley models.

We organize the rest of the paper as follows. Sect. 2 completes the presentation of the PML distribution by expressing some functions of interest. Sect. 3 is devoted to some of its important properties. The inferential aspect of the PML distribution is discussed in Sect. 4, with a simulation study and application of the associated model to three real data sets. Some conclusions are drawn in Sect. 5.

2 Power Modified Lindley Distribution

The fundamentals of the PML distribution are now presented, beginning with the main related functions of interest.

2.1 Functions of interest

First, we recall that the PML distribution is specified by the following cdf:

$$F(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta x^{\alpha}}{1+\theta} e^{-\theta x^{\alpha}}\right] e^{-\theta x^{\alpha}}, \quad x > 0, \tag{3}$$

with $\alpha > 0$ and $\theta > 0$, and $F(x;\alpha,\theta) = 0$ for $x \leq 0$. It is constructed by the composition of the cdf of the former ML distribution and the power function x^{α}. The expression of the survival function immediately follows:

$$S(x;\alpha,\theta) = 1 - F(x;\alpha,\theta) = \left[1 + \frac{\theta x^{\alpha}}{1+\theta} e^{-\theta x^{\alpha}}\right] e^{-\theta x^{\alpha}}, \qquad x > 0, \tag{4}$$

and $S(x;\alpha,\theta) = 1$ for $x \leq 0$. Also, upon differentiation of $F(x;\alpha,\theta)$ with respect to the variable x, the corresponding pdf is given by

$$f(x;\alpha,\theta) = \frac{\theta\alpha}{1+\theta} x^{\alpha-1} e^{-2\theta x^{\alpha}} \left[(1+\theta)e^{\theta x^{\alpha}} + 2\theta x^{\alpha} - 1\right], \quad x > 0, \tag{5}$$

and $f(x;\alpha,\theta) = 0$ for $x \leq 0$. The corresponding hrf is given as

$$h(x;\alpha,\theta) = \frac{f(x;\alpha,\theta)}{S(x;\alpha,\theta)} = \alpha\theta x^{\alpha-1} \left[\frac{\theta x^{\alpha} - 1}{(1+\theta)e^{\theta x^{\alpha}} + \theta x^{\alpha}} + 1\right], \quad x > 0, \tag{6}$$

and $h(x;\alpha,\theta) = 0$ for $x \leq 0$. The functions $F(x;\alpha,\theta)$ and $S(x;\alpha,\theta)$ fully characterize the PML distribution. The functions $f(x;\alpha,\theta)$ and $h(x;\alpha,\theta)$ play complementary roles; they are useful for identifying some crucial statistical features of the lifetime PML model. Further characteristics on these functions are given below.

2.2 Analysis of the pdf

This part is devoted to the pdf of the PML distribution, $f(x;\alpha,\theta)$, as described in (5). A remark on the structure of $f(x;\alpha,\theta)$ is given below. It can be expressed as a linear combination of listed pdfs of the literature.

Indeed, for $x > 0$, we can write

$$f(x;\alpha,\theta) = \theta\alpha x^{\alpha-1}e^{-\theta x^{\alpha}} + \frac{1}{2(1+\theta)}\left[(2\theta)^2\alpha x^{2\alpha-1}e^{-2\theta x^{\alpha}} - 2\theta\alpha x^{\alpha-1}e^{-2\theta x^{\alpha}}\right], \quad (7)$$

and, more explicitly,

$$f(x;\alpha,\theta) = f_1(x;\alpha,\theta) + \frac{1}{2(1+\theta)}\left[f_2(x;\alpha,\theta) - f_3(x;\alpha,\theta)\right],$$

where $f_1(x;\alpha,\theta) = \theta\alpha x^{\alpha-1}e^{-\theta x^{\alpha}}$, $x > 0$, is the pdf of the Weibull distribution with parameters α and θ, $f_2(x;\alpha,\theta) = (2\theta)^2\alpha x^{\alpha-1}x^{\alpha}e^{-2\theta x^{\alpha}}$, $x > 0$ is the pdf of the generalized gamma distribution with parameters 2, 2θ and α, and $f_3(x;\alpha,\theta) = 2\theta\alpha x^{\alpha-1}e^{-2\theta x^{\alpha}}$, $x > 0$, is the pdf of the Weibull distribution with parameters 2θ and α, with standard zero values for these pdfs for $x < 0$. Thus, we can use this linear representation to provide some properties of the PML distribution, an approach that we will consider in Section 3.

An asymptotic study gives

$$\lim_{x\to 0} f(x;\alpha,\theta) = \begin{cases} +\infty & \text{if } \alpha < 1 \\ \dfrac{\theta^2}{1+\theta} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}, \quad \lim_{x\to+\infty} f(x;\alpha,\theta) = 0.$$

We see that α plays a determinant role in these limits, mainly when x tends to 0. Also, the critical point(s) for $f(x;\alpha,\theta)$ is(are) given as the solution(s) of the following equation: $\{\log[f(x;\alpha,\theta)\}' = 0$, which can be reduced to

$$(\alpha-1)\frac{1}{x} + \alpha\theta x^{\alpha-1}\left[\frac{(1+\theta)e^{\theta x^{\alpha}}+2}{(1+\theta)e^{\theta x^{\alpha}}+2\theta x^{\alpha}-1} - 2\right] = 0.$$

The maximum point(s) represent(s) the mode(s) of the PML distribution. These critical points or mode(s) can be approximated numerically by the use of any mathematical software.

We conclude this part by showing some plots for $f(x;\alpha,\theta)$ for selected values of the parameters in Figure 1.

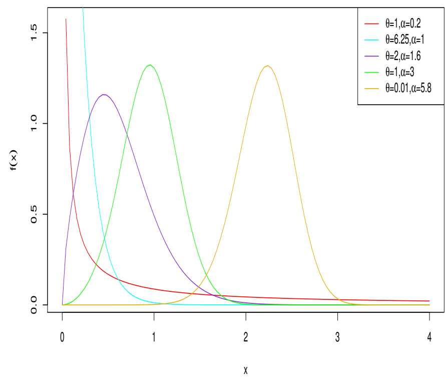

Figure 1: Curves of the pdf of the PML distribution for various values of α and θ

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Figure 1 shows a variety of non-monotonic shapes, including reverse J-shaped, symmetric, skewed to the left or the right, and unimodal shapes.

2.3 Analysis of the hrf

Here, we focus on the hrf of the PML distribution, $h(x;\alpha,\theta)$, as described in (6). First, the following limits are obtained:

$$\lim_{x\to 0} h(x;\alpha,\theta) = \begin{cases} +\infty & \text{if } \alpha < 1 \\ \dfrac{\theta^2}{1+\theta} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases},$$

$$\lim_{x\to +\infty} h(x;\alpha,\theta) = \begin{cases} 0 & \text{if } \alpha < 1 \\ \theta & \text{if } \alpha = 1 \\ +\infty & \text{if } \alpha > 1 \end{cases}.$$

The influence of α on these limits is therefore unequivocal.

Also, the critical point(s) for $h(x;\alpha,\theta)$ is(are) given as the solution(s) of the following equation: $\{\log[h(x;\alpha,\theta)\}'=0$, which can be reduced to

$$(\alpha-1)\frac{1}{x}+\alpha\theta x^{\alpha-1}\left[\frac{(1+\theta)e^{\theta x^{\alpha}}+2}{(1+\theta)e^{\theta x^{\alpha}}+2\theta x^{\alpha}-1}+\frac{\theta x^{\alpha}-1}{(1+\theta)e^{\theta x^{\alpha}}+\theta x^{\alpha}}\right]$$
$$-\alpha\theta x^{\alpha-1}=0.$$

Since this equation is complicated to solve analytically, mathematical software is needed to approximate these critical points.

Figure 2 depicts some plots for $h(x;\alpha,\theta)$ for selected values of the parameters.

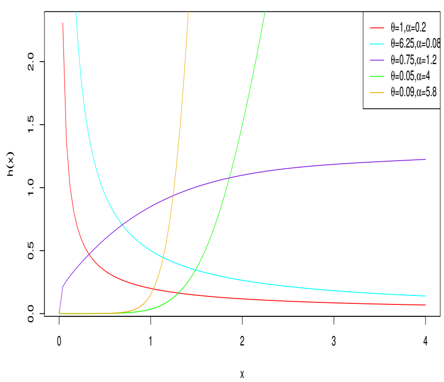

Figure 2: Curves of the hrf of the PML distribution for various values of α and θ

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Figure 2 shows a variety of monotonic shapes, such as decreasing, increasing, and reverse bathtub shapes. Thus, the hrf of the PML distribution is much more flexible than the hrf of the former ML distribution, only showing unimodal curves (see [11]).

3 Properties

This section is devoted to some important properties of the PML distribution.

3.1 Incomplete moments with application

Here, let us consider a random variable X following the PML distribution, i.e., with the cdf given by (3) or equivalently, with the pdf given as (5). First, we investigate the incomplete moment of X, which are the main ingredients to define important measures and functions that will be discussed later.

Proposition 3.1. *For any positive integer r and positive t, the r^{th} incomplete moment of X taking at t is obtained as*

$$\mu_r'(t) = \theta^{-r/\alpha}\left\{\gamma\left(\frac{r}{\alpha}+1, \theta t^{\alpha}\right) + \frac{r2^{-r/\alpha-1}}{\alpha(1+\theta)}\gamma\left(\frac{r}{\alpha}+1, 2\theta t^{\alpha}\right)\right\}$$
$$- \frac{\theta}{1+\theta}t^{r+\alpha}e^{-2\theta t^{\alpha}},$$

where $\gamma(a,x)$ denotes the lower incomplete gamma function (i.e., $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$, $a,x>0$).

Proof. First of all, let us recall that $\mu_r'(t) = \mathbb{E}(X^r I\{\{X \leq t\}\}) = \int_0^t x^r f(x;\alpha,\theta)dx$, where $I(A)$ denotes the indicator function over a certain event denoted by A and $f(x;\alpha,\theta)$ is the pdf given as (5). Now, owing to (7) and the changes of variable $y = \theta x^{\alpha}$, i.e., $x = (y/\theta)^{1/\alpha}$, or $y = 2\theta x^{\alpha}$, i.e., $x = [y/(2\theta)]^{1/\alpha}$, depending on the definition of the integral, we get

$$
\begin{aligned}
\mu_r'(t) &= \int_0^t x^r \theta\alpha x^{\alpha-1} e^{-\theta x^\alpha} dx \\
&+ \frac{1}{2(1+\theta)}\left[\int_0^t x^r (2\theta)^2 \alpha x^{2\alpha-1} e^{-2\theta x^\alpha} dx - \int_0^t x^r 2\theta\alpha x^{\alpha-1} e^{-2\theta x^\alpha} dx\right] \\
&= \theta^{-r/\alpha}\int_0^{\theta t^\alpha} y^{r/\alpha} e^{-y} dy + \frac{1}{2(1+\theta)}\left[(2\theta)^{-r/\alpha}\int_0^{2\theta t^\alpha} y^{r/\alpha+1} e^{-y} dy\right] \\
&- \frac{(2\theta)^{-r/\alpha}}{2(1+\theta)}\int_0^{2\theta t^\alpha} y^{r/\alpha} e^{-y} dy \\
&= \theta^{-r/\alpha}\left\{\gamma\left(\frac{r}{\alpha}+1, \theta t^\alpha\right) + \frac{2^{-r/\alpha-1}}{1+\theta}\left[\gamma\left(\frac{r}{\alpha}+2, 2\theta t^\alpha\right)\right]\right\} \\
&- \frac{\theta^{-r/\alpha} 2^{-r/\alpha-1}}{1+\theta}\left[\gamma\left(\frac{r}{\alpha}+1, 2\theta t^\alpha\right)\right]. \qquad (8)
\end{aligned}
$$

Now, as a known result, the lower incomplete gamma function satisfies the relation: $\gamma(a+1,x) = a\gamma(a,x) - x^a e^{-x}$. Therefore, the term in brackets in (8) can be expressed as

$$
\begin{aligned}
&\gamma\left(\frac{r}{\alpha}+2, 2\theta t^\alpha\right) - \gamma\left(\frac{r}{\alpha}+1, 2\theta t^\alpha\right) \\
&= \left(\frac{r}{\alpha}+1\right)\gamma\left(\frac{r}{\alpha}+1, 2\theta t^\alpha\right) - (2\theta t^\alpha)^{r/\alpha+1} e^{-2\theta t^\alpha} - \gamma\left(\frac{r}{\alpha}+1, 2\theta t^\alpha\right) \\
&= \frac{r}{\alpha}\gamma\left(\frac{r}{\alpha}+1, 2\theta t^\alpha\right) - (2\theta t^\alpha)^{r/\alpha+1} e^{-2\theta t^\alpha}. \qquad (9)
\end{aligned}
$$

We end the proof of Proposition 3.1 by putting (9) into (8). □

Several results follow from Proposition 3.1, including the expression of the first incomplete moment given as

$$
\begin{aligned}
\mu_1'(t) = \theta^{-1/\alpha}&\left\{\gamma\left(\frac{1}{\alpha}+1, \theta t^\alpha\right) + \frac{2^{-1/\alpha-1}}{\alpha(1+\theta)}\gamma\left(\frac{1}{\alpha}+1, 2\theta t^\alpha\right)\right\} \\
&- \frac{\theta}{1+\theta} t^{1+\alpha} e^{-2\theta t^\alpha}.
\end{aligned}
$$

This function with respect to t is involved in the definitions of various important measures and functions in probability and statistics, including various types of mean deviation, mean residual life functions and income curves, among others.

Also, by applying $t \to +\infty$ in Proposition 3.1, we can derive the crude moments of X. Indeed, the r^{th} crude moment of X is obtained as

$$\mu'_r = \mathbb{E}(X^r) = \lim_{t \to +\infty} \mu'_r(t) = \theta^{-r/\alpha} \left[1 + \frac{r 2^{-r/\alpha - 1}}{\alpha(1+\theta)} \right] \Gamma\left(\frac{r}{\alpha} + 1 \right),$$

where $\Gamma(a)$ denotes the (standard) gamma function (i.e., $\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt$, $a > 0$). By taking $\alpha = 1$, using $\Gamma(r+1) = r!$, we rediscover the r^{th} crude moments related to the former ML distribution (see [11]). One can also remark that μ'_r is a decreasing function with respect to θ, which tends to 0 when θ tends to $+\infty$. When μ'_r is viewed as a function of α, its behavior becomes more complicated, depending on $\theta \in (0,1)$ or $\theta > 1$. In all cases, if θ is fixed, μ'_r tends to 1 when α tends to $+\infty$.

The first four crude moments of X can be easily deduced as follows:

$$\mu = \mu'_1 = \theta^{-1/\alpha} \left[1 + \frac{2^{-1/\alpha - 1}}{\alpha(1+\theta)} \right] \Gamma\left(\frac{1}{\alpha} + 1 \right),$$

$$\mu'_2 = \theta^{-2/\alpha} \left[1 + \frac{2^{-2/\alpha}}{\alpha(1+\theta)} \right] \Gamma\left(\frac{2}{\alpha} + 1 \right),$$

$$\mu'_3 = \theta^{-3/\alpha} \left[1 + \frac{3 2^{-3/\alpha - 1}}{\alpha(1+\theta)} \right] \Gamma\left(\frac{3}{\alpha} + 1 \right)$$

and

$$\mu'_4 = \theta^{-4/\alpha} \left[1 + \frac{2^{-4/\alpha + 1}}{\alpha(1+\theta)} \right] \Gamma\left(\frac{4}{\alpha} + 1 \right).$$

Based on μ and μ'_2, the variance of X is given by

$$\sigma^2 = \theta^{-2/\alpha} \left\{ \left[1 + \frac{2^{-2/\alpha}}{\alpha(1+\theta)} \right] \Gamma\left(\frac{2}{\alpha} + 1 \right) - \left[1 + \frac{2^{-1/\alpha - 1}}{\alpha(1+\theta)} \right]^2 \Gamma\left(\frac{1}{\alpha} + 1 \right)^2 \right\}.$$

Owing to the standard binomial theorem, the r^{th} central moment of X is given by the following finite linear representation:

$$\begin{aligned}\mu_r &= \mathbb{E}[(X-\mu)^r] \\ &= \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu^{r-k} \theta^{-k/\alpha} \left[1 + \frac{k 2^{-k/\alpha-1}}{\alpha(1+\theta)}\right] \Gamma\left(\frac{k}{\alpha}+1\right).\end{aligned}$$

The general coefficient of X is deduced as $C_r = \mu_r/\sigma^r$, covering the skewness and kurtosis coefficients of X given by $\sqrt{\beta_1} = C_3$ and $\beta_2 = C_4$.

Table 1 indicates numerical values for the first four crude moments of X, $\sqrt{\beta_1}$ and β_2, for selected values for α and θ. In particular, in Table 1, one can see that the PML distribution can be left and right skewed, and symmetric. This last aspect is illustrated in the table with the special values $\alpha = 3.49005$ and $\theta = 10$ for which the skewness is near equal to zero, i.e., $\sqrt{\beta_1} \approx 9.1800 \times 10^{-7}$. Also, the PML distribution has a versatile kurtosis; it can be platykurtic (corresponding to $\beta_2 < 3$), mesokurtic (corresponding to $\beta_2 = 3$, near attained in the table with the values $\alpha = 3.65$ and $\theta = 0.01$) and leptokurtic (corresponding to $\beta_2 > 3$).

3.2 Order statistics

From the modeling of various real-life phenomena involving the mixing of minimum and maximum random variables, the concept of order statistics is born. We may refer the reader to [9] and [5]. Here, we discuss some properties of the order statistics of the PML distribution. Firstly, the pdf of the m^{th} order statistic the PML distribution, denoted by $X_{(m)}$, is defined as

$$f_{X_{(m)}}(x;\alpha,\theta) = n\binom{n-1}{m-1} f(x;\alpha,\theta) F(x;\alpha,\theta)^{m-1} S(x;\alpha,\theta)^{n-m}, \quad x \in \mathbb{R}.$$

Table 1: Numerical values for the first four crude moments of X, $\sqrt{\beta_1}$ and β_2 for various choices of parameters

Parameters	μ	μ_2'	μ_3'	μ_4'	$\sqrt{\beta_1}$	β_2
$\alpha = 0.5$ $\theta = 3$	0.2361	0.3055	0.9992	6.1694	6.48045	85.2287
$\alpha = 1.6$ $\theta = 2$	0.6206	0.5181	0.5289	0.6308	0.8743	3.9639
$\alpha = 5.8$ $\theta = 0.01$	2.2036	4.9504	11.3199	26.3129	-0.1936	3.1293
$\alpha = 30$ $\theta = 10$	0.9106	0.8306	0.7589	0.6944	-0.9853	4.7209
$\alpha = 3.5$ $\theta = 10$	0.47099	0.2431	0.1345	0.0787	-0.0025	2.7445
$\alpha = 10$ $\theta = 5$	0.8162	0.6751	0.5650	0.4778	-0.6910	3.7484
$\alpha = 3.49005$ $\theta = 10$	0.4700	0.2422	0.1338	0.0783	9.18×10^{-7}	2.7440
$\alpha = 5$ $\theta = 1$	0.9581	0.9545	0.9819	1.0384	-0.3614	3.2298
$\alpha = 3.65$ $\theta = 0.01$	3.5417	13.1392	50.7832	203.6499	0.0663	3.0008

That is, owing to (3), (4) and (5), for $x > 0$, we have

$$f_{X_{(m)}}(x;\alpha,\theta) = n\binom{n-1}{m-1}\frac{\theta\alpha}{1+\theta}x^{\alpha-1}\left[(1+\theta)e^{\theta x^\alpha}+2\theta x^\alpha-1\right]\times \left\{1-\left[1+\frac{\theta x^\alpha}{1+\theta}e^{-\theta x^\alpha}\right]e^{-\theta x^\alpha}\right\}^{m-1}\left[1+\frac{\theta x^\alpha}{1+\theta}e^{-\theta x^\alpha}\right]^{n-m}\times e^{-\theta(n-m+2)x^\alpha}, \quad x>0,$$

and $f_{X_{(m)}}(x;\alpha,\theta) = 0$ for $x < 0$.

In particular, for $x > 0$, the pdf of $X_{(1)} = \inf(X_1, \ldots, X_n)$ is given as

$$f_{X_{(1)}}(x;\alpha,\theta) = n\frac{\theta\alpha}{1+\theta}x^{\alpha-1}\left[(1+\theta)e^{\theta x^\alpha}+2\theta x^\alpha-1\right]\times \left[1+\frac{\theta x^\alpha}{1+\theta}e^{-\theta x^\alpha}\right]^{n-1}e^{-\theta(n+1)x^\alpha},$$

and the pdf of $X_{(n)} = \sup(X_1, \ldots, X_n)$ can be set as

$$f_{X_{(n)}}(x;\alpha,\theta) = n\frac{\theta\alpha}{1+\theta}x^{\alpha-1}e^{-2\theta x^\alpha}\left[(1+\theta)e^{\theta x^\alpha}+2\theta x^\alpha-1\right]\times \left\{1-\left[1+\frac{\theta x^\alpha}{1+\theta}e^{-\theta x^\alpha}\right]e^{-\theta x^\alpha}\right\}^{n-1}.$$

Furthermore, $X_{(1)}$ enjoys a singular asymptotic distribution result as described below. First, note that, for $x > 0$,

$$\lim_{\epsilon\to 0}\frac{F(\epsilon x;\alpha,\theta)}{F(\epsilon;\alpha,\theta)} = \lim_{\epsilon\to 0}\frac{xf(\epsilon x;\alpha,\theta)}{f(\epsilon;\alpha,\theta)} = x^\alpha\lim_{\epsilon\to 0}\frac{e^{-2\theta\epsilon^\alpha x^\alpha}\left[(1+\theta)e^{\theta\epsilon^\alpha x^\alpha}+2\theta\epsilon^\alpha x^\alpha-1\right]}{e^{-2\theta\epsilon^\alpha}\left[(1+\theta)e^{\theta\epsilon^\alpha}+2\theta\epsilon^\alpha-1\right]} = x^\alpha.$$

It follows from [5, Theorem 8.3.6(ii)] that the minimal domain of attraction of the PML distribution is the standard Weibull distribution with parameters 1 and α, i.e., with cdf $K(x;\alpha) = 1 - e^{-x^\alpha}$ for $x > 0$, and $K(x;\alpha) = 0$ for $x < 0$.

4 Parametric Estimation and Application

This section is devoted to the practical features of the PML model. First, we investigate the estimation of the parameters θ and α, along with a simulation study, then applications are given for three different data sets.

4.1 Parametric estimation

Here, the parameters α and θ are assumed to be unkown. For estimating them, we propose the method of maximum likelihood. Thus, let $x_1, \ldots, x_n$ be a n independent observations from the PML distribution with unknown parameters α and θ, corresponding to data. Then, the likelihood function is given by

$$\begin{aligned} L(\alpha,\theta) &= \prod_{i=1}^{n} f(x_i;\alpha,\theta) \\ &= \frac{\theta^n\alpha^n}{(1+\theta)^n} e^{-2\theta\sum_{i=1}^{n} x_i^{\alpha}} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \prod_{i=1}^{n}\left[(1+\theta)e^{\theta x_i^{\alpha}} + 2\theta x_i^{\alpha} - 1\right]. \end{aligned}$$

The log-likelihood function follows immediately as

$$\begin{aligned} \ell(\alpha,\theta) &= \log\left[L(\alpha,\theta)\right] = n\log(\theta) + n\log(\alpha) - n\log(1+\theta) \\ &\quad - 2\theta\sum_{i=1}^{n} x_i^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[(1+\theta)e^{\theta x_i^{\alpha}} + 2\theta x_i^{\alpha} - 1\right]. \end{aligned}$$

The maximum likelihood estimates (MLEs) for α and θ, say $\hat{\alpha}$ and $\hat{\theta}$ are defined as $(\hat{\alpha},\hat{\theta}) = \arg\max_{(\alpha,\theta)\in(0,+\infty)^2} L(\alpha,\theta)$ or, equivalently, $(\hat{\alpha},\hat{\theta}) = \arg\max_{(\alpha,\theta)\in(0,+\infty)^2} \ell(\alpha,\theta)$. One can obtained these estimates by solving $\partial\ell(\alpha,\theta)/\partial\alpha = 0$ and $\partial\ell(\alpha,\theta)/\partial\theta = 0$ (simultaneously) according to α and θ, equations which can be expressed analytically as

$$\frac{n}{\alpha} - 2\theta\sum_{i=1}^{n} x_i^{\alpha}\log(x_i) + \sum_{i=1}^{n}\log(x_i) + \theta\sum_{i=1}^{n}\frac{x_i^{\alpha}\log(x_i)\left[(1+\theta)e^{\theta x_i^{\alpha}} + 2\right]}{(1+\theta)e^{\theta x_i^{\alpha}} + 2\theta x_i^{\alpha} - 1} = 0$$

and

$$\frac{n}{\theta} - \frac{n}{1+\theta} - 2\sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\frac{e^{\theta x_i^{\alpha}}\left[(1+\theta)x_i^{\alpha} + 1\right] + 2x_i^{\alpha}}{(1+\theta)e^{\theta x_i^{\alpha}} + 2\theta x_i^{\alpha} - 1} = 0.$$

These equations are complicated to solve analytically. One can use mathematical software to get numerical solutions. Under regularity conditions, the bi-dimensional normal distribution $\mathcal{N}_2((\alpha,\theta), J^{-1}(\hat{\alpha},\hat{\theta}))$ can approximate the underlying distribution of $(\hat{\alpha},\hat{\theta})$, where $J(\alpha,\theta)$ denote the following 2×2 matrix:

$$J(\alpha,\theta) = -\begin{pmatrix} \dfrac{\partial^2 \ell(\alpha,\theta)}{\partial\alpha^2} & \dfrac{\partial^2 \ell(\alpha,\theta)}{\partial\alpha\partial\theta} \\ \dfrac{\partial^2 \ell(\alpha,\theta)}{\partial\theta\partial\alpha} & \dfrac{\partial^2 \ell(\alpha,\theta)}{\partial\theta^2} \end{pmatrix},$$

whose components can be expressed analytically with mathematical developments. This asymptotic result can be used to generate asymptotic confidence intervals, statistical tests, and so on.

4.2 Simulation study

Here, we perform a simulation study evaluating the performance of the MLEs presented above for the PML model for selected values of the parameters α and θ. The simulation experiment was repeated 1000 times each with sample sizes of 20, 50, 100, 200, and the parameter combinations are as follows:
I) $\alpha = 1.95$ and $\theta = 0.85$, II) $\alpha = 0.2$ and $\theta = 0.1$, III) $\alpha = 1.5$ and $\theta = 0.5$, IV) $\alpha = 2$ and $\theta = 1$.
The algorithm for the simulation study is formalized as follows:

Step 1: Set the number of replications denoted by N.

Step 2: Set the sample size denoted by n and the values of the parameters α and θ.

Step 3: Set the initial value for the random start, denoted by x_0.

Step 4: For $j = 1,\ldots,n$, generate u_j from a random variable U_j following the unit uniform distribution.

Step 5: Update x_0 by x^* by using the Newton's formula as follows:

$$x^* = x_0 - \left\{\frac{F(x_0;\alpha,\theta) - u_j}{f(x_0;\alpha,\theta)}\right\},$$

where $F(x_0; \alpha, \theta)$ and $f(x_0; \alpha, \theta)$ are the cdf and pdf of the PML distribution at $x = x_0$ as given by (3) and (5), respectively.

Step 6: If $|x_0 - x^*| \leq \epsilon$ for small $\epsilon > 0$, ϵ being considered as a tolerance limit, then $x = x^*$ is considered as a generated value from the PML distribution with parameter α and θ, else set $x_0 = x^*$ and go to Step 5.

Step 7: Repeat Steps 4 to 6 for $j = 1, \ldots, n$ to obtain n values $x_1, \ldots, x_n$.

Step 8: Compute the MLEs of α and θ from $x_1, \ldots, x_n$.

Step 9: Repeat Steps 2 to 8, N times.

Step 10: Compute the average estimate (AE), Bias and mean square error (MSE) for each parameter, defined as

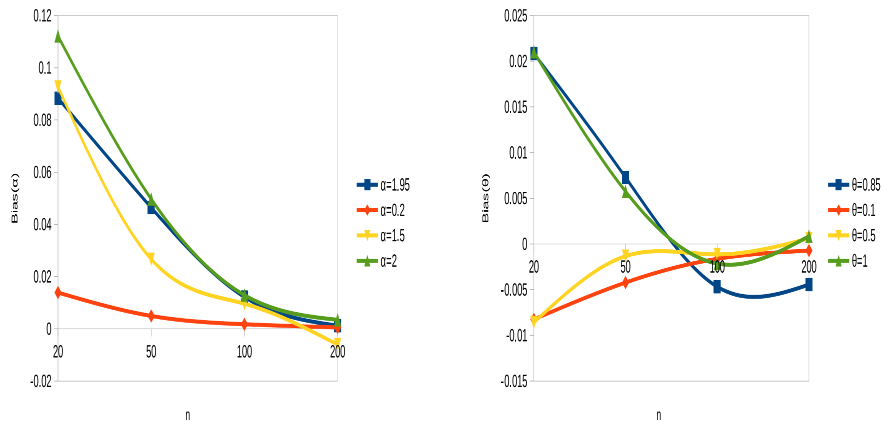
$$\mathrm{AE}(\alpha) = \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i, \quad \mathrm{Bias}(\alpha) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha),$$
$$\mathrm{MSE}(\alpha) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2,$$
$$\mathrm{AE}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i,$$
$$\mathrm{Bias}(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta), \quad \mathrm{MSE}(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2,$$

where $\hat{\alpha}_i$ and $\hat{\theta}_i$ are the MLEs of α and θ, respectively, obtained at the i^{th} replication.

Table 2 presents the AEs, Bias and MSEs values of parameters for different sample sizes. Figures 3 and 4 give the graphical representations of the Bias and MSE related to the two parameters.

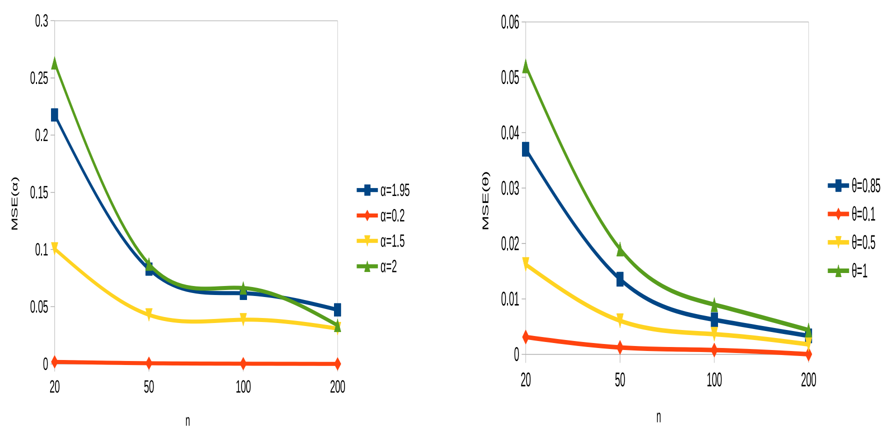
Table 2: Numerical values for the AEs, Bias and MSE of the parameters based on 1000 simulations in the setting of the PML model

	n	Parameters	AEs	Bias	MSE
I	20	α	2.038336	0.08833574	0.2176643
		θ	0.8708443	0.02084432	0.03701037
	50	α	1.996429	0.04642946	0.08285901
		θ	0.8572819	0.00728185	0.01355176
	100	α	1.96219	0.01219701	0.06173568
		θ	0.8453169	-0.004683061	0.00622736
	200	α	1.951196	0.001195866	0.04729822
		θ	0.8455547	-0.004445285	0.003326896
II	20	α	0.2138739	0.01387395	0.001786709
		θ	0.09174396	-0.008256044	0.003104728
	50	α	0.2049212	0.00492118	0.0005953382
		θ	0.09577495	-0.00422505	0.001230094
	100	α	0.2017429	0.001742857	0.0001910657
		θ	0.09837379	-0.001626206	0.0007620415
	200	α	0.200525	0.0005250294	8.89×10^{-6}
		θ	0.09927519	-0.0007248077	1.50×10^{-5}
III	20	α	1.592697	0.09269666	0.100596
		θ	0.4914032	-0.008596789	0.01619796
	50	α	1.526672	0.0266715	0.04286805
		θ	0.4986687	-0.00133133	0.00605377
	100	α	1.50961	0.009610091	0.03863887
		θ	0.49886	-0.001139978	0.003621219
	200	α	1.493986	-0.006014323	0.03088576
		θ	0.5006133	0.0006133199	0.001788008
IV	20	α	2.112133	0.1121327	0.2631284
		θ	1.021017	0.02101712	0.05205832
	50	α	2.049721	0.04972109	0.08723618
		θ	1.005752	0.005751939	0.01899681
	100	α	2.012916	0.01291555	0.06636949
		θ	0.9978486	-0.002151383	0.008961524
	200	α	2.003441	0.003441277	0.0336673
		θ	1.000845	0.0008452682	0.004373854



(a) (b)

Figure 3: Curves of the Bias for the estimates of (a) α and (b) θ for various sample sizes



(a) (b)

Figure 4: Curves of the MSE for the estimates of (a) α and (b) θ for various sample sizes

From Table 2, and Figures 3 and 4, it can be noted that, as sample size increases, the Bias decays towards zero and MSE decreases. That is, this illustrates the fact that the parent estimators are asymptotically unbiased and consistent. Hence, the maximum likelihood method works quite well to estimate the parameters of the PML model.

4.3 Application

Now, we use the previous parametric estimation for data fitting purposes. We fit the PML distribution to three data sets and compare the results with those of the fitted Weibull (W), PL, exponentiated power Lindley (EPL) and three-parameter generalized Lindley (TGL) distributions. The corresponding pdfs of these competitors are recalled below.

- For the W distribution:

$$f(x;\alpha,\theta)=\frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}e^{-(x/\theta)^{\alpha}},\quad x>0,$$

  with $\alpha>0$ and $\theta>0$, and $f(x;\alpha,\theta)=0$ for $x<0$.

- For the PL distribution:

$$f(x;\alpha,\theta)=\frac{\alpha\theta^2}{\theta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}},\quad x>0,$$

  with $\alpha>0$ and $\theta>0$, and $f(x;\alpha,\theta)=0$ for $x<0$.

- For the EPL distribution:

$$f(x;\alpha,\beta,\theta)=\frac{\alpha\theta^2\beta x^{\beta-1}}{\theta+1}(1+x^{\beta})e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\theta+1}\right)e^{-\theta x^{\beta}}\right]^{\alpha-1},$$

  $x>0$ with $\alpha>0$, $\beta>0$ and $\theta>0$, , and $f(x;\alpha,\beta,\theta)=0$ for $x<0$.

- For the TGL distribution

$$f(x;\alpha,\beta,\theta)=\frac{\alpha\theta^2(\beta+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{1+\theta\beta},\quad x>0,$$

  with $\alpha>0$, $\beta>0$ and $\theta>0$, and $f(x;\alpha,\beta,\theta)=0$ for $x<0$.

We estimate the unknown parameters of each model by the maximum likelihood method of estimation. In order to compare the five models, we consider criteria like the statistic log-likelihood ($\hat{\ell}$), Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion (AICc), Hannan-Quinn information criterion (HQIC) and the values of the Kolmogorov-Smirnov (K-S) statistic, and the corresponding p-values (p-V) for the three different data sets.

The model with the lowest AIC, BIC, AICc, HQIC and K-S and the largest p-V is considered the best.

Bladder cancer patients data

According to [23], the data represent the remission times (in months) of a random sample of 128 bladder cancer patients. The data are as follows.
{0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.2, 2.23, 0.26, 0.31, 0.73, 0.52, 4.98, 6.97, 9.02, 13.29, 0.4, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07, 0.22, 13.8, 25.74, 0.5, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36, 0.82, 0.51, 2.54, 3.7, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 1.76,8.53, 6.93, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65, 0.39, 10.34, 14.83, 34.26, 0.9, 2.69, 4.18, 5.34, 7.59, 10.66, 4.5, 20.28, 12.63, 0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 8.37, 3.36, 5.49, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76, 0.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.4, 5.85, 2.02, 12.07}

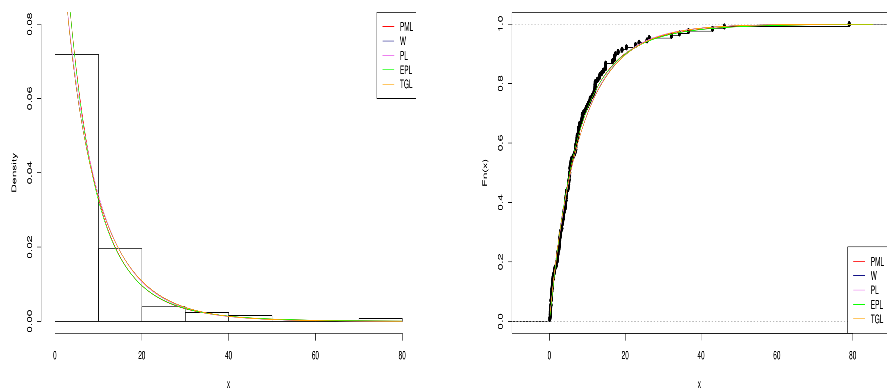
Tables 3 and 4 give the relevant numerical values for all the fitted models based on the bladder cancer patients data set. Figure 5 gives the graphs of the estimated pdfs and cdfs of the fitted models for the bladder cancer patients data set.

Table 3: Estimated values, $\hat{\ell}$, AIC and BIC for the bladder cancer patients data set

Distributions	Estimates	$-\hat{\ell}$	AIC	BIC
PML	$\boldsymbol{\hat{\alpha} = 0.7290}$ $\boldsymbol{\hat{\theta} = 0.2775}$	**401.2802**	**806.5603**	**812.2644**
W	$\hat{\alpha} = 0.9229$ $\hat{\theta} = 8.2290$	402.1907	808.3814	814.0854
PL	$\hat{\alpha} = 0.7442$ $\hat{\theta} = 0.3855$	402.2373	808.4745	814.1786
EPL	$\hat{\alpha} = 1.8412$ $\hat{\beta} = 0.5785$ $\hat{\theta} = 0.7252$	401.0833	808.1666	816.7227
TGL	$\hat{\alpha} = 0.9196$ $\hat{\beta} = 102.6147$ $\hat{\theta} = 0.1528$	402.2331	810.4661	819.0222

Table 4: AICc, HQIC and K-S with p-V for the bladder cancer patients data set

Distributions	AICc	HQIC	K-S	p-V
PML	**806.6563**	**808.878**	**0.0428**	**0.9732**
W	808.4774	810.699	0.0518	0.8817
PL	808.5705	810.7922	0.0458	0.9512
EPL	808.3601	811.643	0.0480	0.93
TGL	810.6596	813.9426	0.0524	0.8736



(a) (b)

Figure 5: Curves of the estimated (a) pdfs and (b) cdfs of the fitted models for the bladder cancer patients data set

Fatigue fracture data set

This data represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. The data was extracted from [1] and was previously used by [4] and [10]. The data are as follows.
{0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960}

Tables 5 and 6 give the relevant numerical summaries for all the fitted models based on fatigue fracture data sets. Figure 6 gives the graphs of the estimated pdfs and cdfs of the fitted models for the fatigue

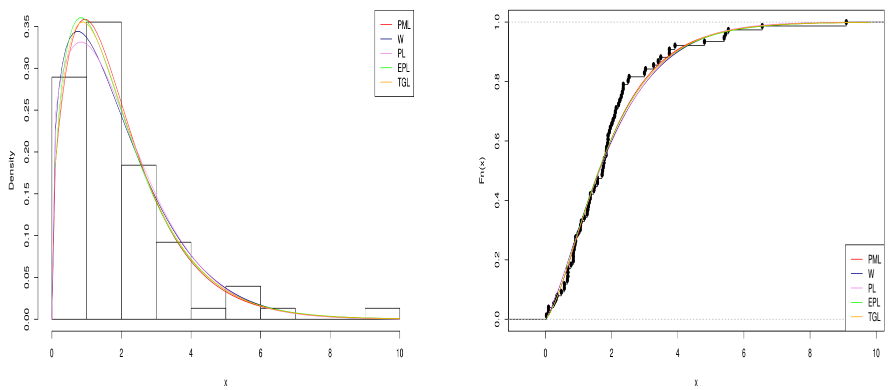
Table 5: Estimated values, $\hat{\ell}$, AIC and BIC for the fatigue fracture data set

Distributions	Estimates	$-\hat{\ell}$	AIC	BIC
PML	$\boldsymbol{\hat{\alpha} = 1.1182}$ $\boldsymbol{\hat{\theta} = 0.5324}$	**121.2194**	**246.4389**	**251.1004**
W	$\hat{\alpha} = 1.3257$ $\hat{\theta} = 2.1328$	122.5264	249.0529	253.7144
PL	$\hat{\alpha} = 0.7047$ $\hat{\theta} = 1.1423$	122.4018	248.8037	253.4652
EPL	$\hat{\alpha} = 1.5375$ $\hat{\beta} = 0.9495$ $\hat{\theta} = 1.0215$	121.8682	249.7364	256.7286
TGL	$\hat{\alpha} = 0.9931$ $\hat{\beta} = 0.1478$ $\hat{\theta} = 0.9635$	121.6506	249.3011	256.2933

Table 6: AICc, HQIC and K-S with p-V for the fatigue fracture data set

Distributions	AICc	HQIC	K-S	p-V
PML	**246.6033**	**248.3017**	**0.0964**	**0.4516**
W	249.2173	250.9157	0.1099	0.2954
PL	248.9681	250.6665	0.1123	0.2722
EPL	250.0697	252.5308	0.0992	0.4156
TGL	249.6344	252.0956	0.1020	0.3822

fracture data set.



(a) (b)

Figure 6: Curves of the estimated (a) pdfs and (b) cdfs of the fitted models for the fatigue fracture data set

March precipitation data set

This real data set represents 30 successive values of march precipitation (in inches) in Minneapolis/St Paul given by [22] and the data are as given below.
{0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05}.

Tables 7 and 8 provide the relevant numerical summaries for all the fitted models based on the march precipitation data set.

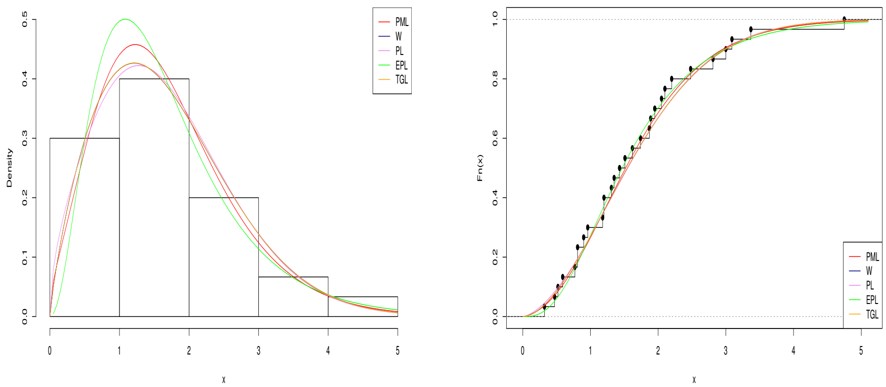
Figure 7 gives the graphs of the estimated pdfs and cdfs of the fitted models for the march precipitation data set.

Table 7: Estimated values, $\hat{\ell}$, AIC and BIC for the march precipitation data set

Distributions	Estimates	$-\hat{\ell}$	AIC	BIC
PML	$\boldsymbol{\hat{\alpha} = 1.4739}$ $\boldsymbol{\hat{\theta} = 0.4923}$	**38.5278**	**81.0556**	**83.8579**
W	$\hat{\alpha} = 1.8087$ $\hat{\theta} = 1.8924$	38.64328	81.28657	84.08896
PL	$\hat{\alpha} = 1.5263$ $\hat{\theta} = 0.6460$	38.8729	81.74579	84.54819
EPL	$\hat{\alpha} = 3.4070$ $\hat{\beta} = 0.9235$ $\hat{\theta} = 1.6304$	38.10625	82.2125	86.41609
TGL	$\hat{\alpha} = 1.8103$ $\hat{\beta} = 358.0827$ $\hat{\theta} = 0.3179$	38.6435	83.287	87.49059

Table 8: AICc, HQIC and K-S with p-V for the march precipitation data set

Distributions	AICc	HQIC	K-S	p-V
PML	**81.5000**	**81.9521**	**0.0526**	**1**
W	81.73101	82.18307	0.0689	0.9988
PL	82.19023	82.64231	0.0682	0.999
EPL	83.13558	83.55727	0.0624	0.9998
TGL	84.2100	84.6318	0.0688	0.9989



(a) (b)

Figure 7: Curves of the estimated (a) pdfs and (b) cdfs of the fitted models for the march precipitation data set

Thus, in Tables 3, 4, 5, 6, 7 and 8, the parameter estimates for the PML, W, PL, EPL and TGL models are calculated by using the maximum likelihood method. Also, $-\hat{\ell}$, AIC, BIC, AICc, HQIC and K-S with p-V are presented for the three different data sets. For all of them, based on the lowest values of the AIC, BIC, AICc, HQIC and K-S with p-V, the PML model turns out to be a better model than the W, PL, EPL and TGL models. Figures 5, 6 and 7 show the closeness of the fitted pdfs with the empirical histogram and fitted cdfs with empirical cdfs for different data sets. Based on the observations of these plots, the proposed model provides a closer fit to these data sets.

5 Conclusions

In this paper, a new two-parameter distribution, namely, PML distribution, is proposed based on power transformation over the modified Lindley distribution. We have exhibited its moments, incomplete moments, skewness, kurtosis, and order statistics. In the setting of the PML model, the unknown parameters were estimated by the maximum likelihood method of estimation. A simulation study was carried out to evaluate the bias and mean square error of the maximum likelihood estimates of the parameters. We have shown by means of three applications to real data that the proposed model can yield better fits than the famous W, PL, EPL and TGL models.

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References

[1] I. B. Abdul-Moniem and M. Seham, Transmuted Gompertz distribution, *Computational and Applied Mathematics*, 1 (2015) , 88-96.

[2] A.Z. Afify, M. Nassar, G.M. Cordeiro and D. Kumar, The Weibull Marshall-Olkin Lindley distribution: properties and estimation, *Journal of Taibah University for Science*, 14 (2020), 192-204.

[3] S. H. Alkarni, Extended power Lindley distribution: a new statistical model or non -monotone survival data, *European Journal of Statistics and Probability*, 3 (2015), 19-34.

[4] D. F. Andrews and A. M. Herzberg, *Data: a collection of problems from many fields for the student and research worker* , Springer Series in Statistics, New York (1985).

[5] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, *A First Course in Order Statistics*, John Wiley and Sons, New York (1992).

[6] S. K. Ashour and M.A. Eltehiwy, Exponentiated power Lindley distribution, *Journal of Advanced Research*, 6 (2015), 895-905.

[7] A. Asgharzadeh, H. S. Bakouch, , S. Nadarajah and F. Sharafi, A new weighted Lindley distribution with application, *Brazilian Journal of Probability and Statistics*, 30 (2016), 1-27.

[8] H. S. Bakouch, B. M. Al-Zahrani, A. A. Al-Shomrani, V. A. A. Marchi and F. Louzada, An extended Lindley distribution, *Journal of the Korean Statistical Society*, 41 (2012), 75-85.

[9] N. Balakrishnan and A. C Cohen, *Order Statistics and Inference*, Academic Press , New York (1991).

[10] R. E. Barlow, R.H. Toland and T. Freeman, A Bayesian analysis of stress rupture life of Kevlar 49/epoxy sphere-cal pressure vessels, *In 'Proc. Conference on Applications of Statistics, Marcel Dekker, New York* (1984).

[11] C. Chesneau, L. Tomy and J. Gillariose, A new modified Lindley distribution with properties and applications, *Journal of Statistics and Management Systems*,Journal of Statistics and Management Systems, (2021), 1-21.

[12] C. Chesneau, L. Tomy and J. Gillariose, The inverted modified Lindley distribution, *Journal of Statistical Theory and Practice*, 14 (2020), 1-17.

[13] C. Chesneau, L. Tomy and J. Gillariose, On a sum and difference of two Lindley distributions: theory and applications, *REVSTAT–Statistical Journal*, 18(2020), 673-695.

[14] C. Chesneau, L. Tomy and M. Jose, Wrapped modified Lindley distribution, *Journal of Statistics and Management Systems*, (2021), 1-16.

[15] N. Ekhosuehi and F. Opone, A three-parameter generalized Lindley distribution: properties and application, *Statistica*, 78 (2018), 233-249.

[16] M. E. Ghitany and D. K. Al-Mutairi, Size-biased Poisson-Lindley distribution and its applications, *Metron- International Journal of Statistics*, 66 (2008), 299-311.

[17] M. E. Ghitany, B. Atieh and S. Nadarajah, Lindley distribution and its applications, *Mathematics and Computers in Simulation*, 78 (2008a), 493-506.

[18] M. E. Ghitany, D. K. Al-Mutairi and S. Nadarajah, Zero-truncated Poisson-Lindley distribution and its applications, *Mathematics and Computers in Simulation*, 79 (2008b), 279-287.

[19] M. E. Ghitany, F. Alqallaf, D. K. Al-Mutairi and H. A. Husain,A two-parameter weighted Lindley distribution and its applications to survival data, *Mathematics and Computers in Simulation*, 81(2011), 1190-1201.

[20] M. E. Ghitany, D. K. Al-Mutairi, N. Balakrishnan and L. J. Al-Enezi, Power Lindley distribution and associated inference, *Computational Statistics and Data Analysis*, 64 (2013), 20-33.

[21] A. S. Hassan, M. Elgarhy, R. E. Mohamd and S. Alrajhi, On the alpha power transformed power Lindley distribution, *Journal of Probability and Statistics*, 2019 (2019).

[22] D. Hinkley, On quick choice of power transformations, *Applied Statistics*, 26 (1977), 67-69.

[23] E. T. Lee and J. W. Wang , *Statistical Methods for Survival Data Analysis*, Wiley, New York (2003).

[24] D. V. Lindley, Fiducial distributions and Bayes' theorem, *Journal of the Royal Statistical Society, Series A*, 20 (1958), 102-107.

[25] G. W. Liyanage and M. Parai, The generalized power Lindley distribution with its applications, *Asian Journal of Mathematics and Applications*, 2014 (2014), 1-23.

[26] F. Merovci, Transmuted Lindly distribution, *International Journal of Open Problems in Computer Science and Mathematics*, 238 (2013), 1-20.

[27] S. M. T. K. MirMostafaee, M. Alizadeh, E. Altun and S.Nadarajah, The exponentiated generalized power Lindley distribution: Properties and applications, *Applied Mathematics-A Journal of Chinese Universities*, 34 (2019), 127-148.

[28] M. Sankaran, The discrete Poisson-Lindley distribution, *Biometrics*, 26 (1970), 145-149.

[29] R. Shanker, Akash distribution and Its Applications, *International Journal of Probability and Statistics*, 4 (2015), 65-75.

[30] R. Shanker, A quasi Akash distribution. itAssam Statistical Review, 30 (2016), 135-160.

[31] R. Shanker, A. Mishra, A two parameter Lindley distribution, *Statistics in Transition New Series*, 14 (2013a), 45-56.

[32] R. Shanker, A. Mishra. A quasi Lindley distribution, *African Journal of Mathematics and Computer Science Research*, 6 (2013b), 64-71.

[33] R. Shanker and K. K Shukla, Weighted Akash distribution and its application to model lifetime data, *International Journal of Statistics*, 39 (2016), 1138-1147.

[34] V. K. Sharma, S. K. Singh, U. Singh, and F. Merovci, The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data, *Journal of Industrial and Production Engineering*, 32 (2015), 162-173.

[35] L. Tomy, A retrospective study on Lindley distribution, *Biometrics and Biostatistics International Journal*, 7 (2018), 163-169.

[36] H. Zamani and N. Ismail, Negative binomial-Lindley distribution and its application, *Journal of Mathematics and Statistics*, 6 (2010), 4-9.

[37] H. Zeghdoudi and S. Nedjar, Gamma-Lindley distribution and its application, *Journal of Applied Probability and Statistics*, 11 (2016), 129-138.

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