Journal of Mathematical Extension
Vol. 16, No. 8, (2022) (7)1-13
URL: https://doi.org/10.30495/JME.2022.1801
ISSN: 1735-8299
Original Research Paper

Adjoint of Sandwich Weighted Composition Operator on Weighted Hardy Spaces

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Abstract. Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. A sandwich weighted composition operator $S_{\psi,\varphi}$ takes an analytic map f on the open unit disc $\mathbb{D}$ to the map $(\psi.f' o\varphi)'$, where φ is an analytic map of $\mathbb{D}$ into itself and ψ is an analytic map on $\mathbb{D}$. In this paper, we compute the adjoint of a sandwich weighted composition operator $S_{\psi,\varphi}$ on weighted Hardy spaces.

AMS Subject Classification: 30H10; 47A05; 47B33.
Keywords and Phrases: Weighted composition operator, evaluation kernel, weighted Hardy spaces.

1 Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that $\beta_0 = 1$ and $\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$. Then for $1 \leq p < \infty$, the

Received: September 2020; Accepted: March 2021
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weighted Hardy space $H^p(\beta)$ is the Banach space of all analytic functions f on the open unit disk $\mathbb{D}$ defined by

$$H^p(\beta) = \Big\{ f : z \to \sum_{n=0}^{\infty} a_n z^n \quad s.t \quad \|f\|^p_{H^p(\beta)} = \sum_{n=0}^{\infty} |a_n|^p \beta_n^p < \infty \Big\},$$

where $\|.\|_{H^p(\beta)}$ is a norm on $H^p(\beta)$. If $\beta \equiv 1$, then $H^p(\beta)$ becomes the classical Hardy space H^p. For $p = 2$, $H^2(\beta)$ is a Hilbert space with respect to the inner product

$$< f, g >= \sum_{n=0}^{\infty} a_n.\bar{b_n}\beta_n^2,$$

where $f(z) = \sum_{n=0}^{\infty} a_n.z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n.z^n$ are elements of $H^2(\beta)$. For a detailed discussion on $H^p(\beta)$ one can see [12].
Let $w \in \mathbb{D}$. Then evaluation kernel $K_w \in H^2(\beta)$ is defined by

$$K_w(z) = \sum_{n=0}^{\infty} \frac{\bar{w}^n}{\beta_n^2} z^n, \forall \ \ z \in \mathbb{D}.$$

Clearly $\|K_w\|^2 = \sum_{n=0}^{\infty} |w|^{2n}/\beta_n^2$, where $\|K_w\|$ is an increasing function of $|w|$. For $f(z) = \sum_{n=0}^{\infty} a_n.z^n$ and $w \in \mathbb{D}$, we have

$$\langle f, K_w \rangle = \sum_{n=0}^{\infty} \frac{a_n w^n}{\beta_n^2}.\beta_n^2 = \sum_{n=0}^{\infty} a_n w^n = f(w).$$

Let φ be an analytic map from the open unit disc $\mathbb{D}$ into itself. The operator that takes the analytic map f to $fo\varphi$ is a composition operator and is usually denoted by C_φ. A natural generalization of the composition operator is an operator that takes f to $\psi.fo\varphi$, where ψ is a fixed analytic map. This operator which is called a weighted composition operator and denoted by $W_{\psi,\varphi}$, is defined as

$$W_{\psi,\varphi} f = M_\psi C_\varphi f = \psi.fo\varphi.$$

Let D be the differentiation operator defined by $Df = f'$. Then the generalized weighted composition operator on the weighted Hardy space $H^p(\beta)$ is given as

$$M_\psi C_\varphi D f = \psi.f' o\varphi,$$

where f' denotes the derivative of the function f.
Note that the operator $M_\psi C_\varphi D$ induces many known operators. If $\psi(z) = 1$, then $M_\psi C_\varphi D = C_\varphi D$, and if $\psi(z) = \varphi'(z)$, then we get the operator DC_φ, known as the product of composition and differentiation operators. These two operators have been studied in [4], [6], [8], and [13]. If we put $\varphi(z) = z$, then $M_\psi C_\varphi D = M_\psi D$, that is, the product of multiplication and differentiation operators. Similarly the sandwich weighted composition operator $DM_\psi C_\varphi D$ on $H^p(\beta)$ is defined as

$$DM_\psi C_\varphi D f = (\psi . f' o\varphi)', \forall \ \ f \in H^p(\beta).$$

Yousefi[14], introduced the study of composition operators on weighted Hardy spaces. In [4], Hibschweiler and Portony defined the product $C_\varphi D$ and DC_φ and studied the boundedness and compactness of these operators between Bergman and Hardy spaces by using the Carleson-type measure, whereas in [8], the author studied the boundedness and compactness of $C_\varphi D$ and DC_φ between Hardy type spaces. This paper is organised as follows.
In the second section, we compute the adjoint of the sandwich weighted composition operator $DM_\psi C_\varphi D$ using power series method, whereas in third section, we compute the adjoint of the sandwich weighted composition operator $DM_\psi C_\varphi D$ for the evaluation kernels on weighted Hardy spaces. For the sake of simplicity we denote the sandwich weighted composition operator $DM_\psi C_\varphi D$ by $S_{\psi,\varphi}$, the sandwich composition operator $DC_\varphi D$ by C^φ and the sandwich multiplication operator $DM_\psi D$ by M^ψ.

2 Computation of Adjoint of the Operator $S_{\psi,\varphi}$ Using Power Series

In this section, we compute the adjoint of a sandwich weighted composition operator on weighted Hardy space $H^2(\beta)$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic functions so that the sandwich weighted composition operator $S_{\psi,\varphi} = DM_\psi C_\varphi D \in B(H^2(\beta))$, the Banach algebra of bounded linear operators on $H^2(\beta)$. For $f \in H^2(\beta)$, we shall write

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \tag{1}$$

and

$$\|f\|^2_{H^2(\beta)} = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \beta^2_{n.}$$

To avoid ambiguity, we may often write $(f)^h$ for $\widehat{f}$. Let $T : H^2(\beta) \to H^2(\beta)$ be defined by

$$\widehat{(Tf)}(n) = \begin{cases} 0, & for \;\; n = 0 \\ \frac{n}{\beta_n^2} \sum_{k=0}^{\infty} \widehat{f}(k) \overline{\widehat{h}_n(k)} \beta_k^2, & for \;\; n \geq 1, \end{cases}$$

where $h_n = \psi.D\varphi^{n-1} + \psi'.\varphi^{n-1}$. In the following Theorem we show that the adjoint $S^{\star}_{\psi,\varphi}$ of sandwich weighted composition operator $S_{\psi,\varphi}$ is equal to T.

Theorem 2.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic maps so that $S_{\psi,\varphi} \in B(H^2(\beta))$. Then $S^{\star}_{\psi,\varphi} = T$.*

Proof. Let $f, g \in H^2(\beta)$. Then

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \;\; and \;\; g(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n.$$

Therefore

$$f'(z) = \sum_{n=1}^{\infty} n \widehat{f}(n) z^{n-1} \;\; and \;\; f''(z) = \sum_{n=2}^{\infty} n(n-1) \widehat{f}(n) z^{n-2}.$$

Further

$$\begin{aligned} (S_{\psi,\varphi} f)(z) &= (DM_\psi C_\varphi D) f(z) = D(\psi.f' o\varphi)(z) \\ &= (\psi.f'' o\varphi.\varphi' + \psi'.f' o\varphi)(z) \end{aligned}$$

$$\begin{aligned} &= \sum_{n=2}^{\infty} n(n-1)\widehat{f}(n)\varphi^{n-2}(z).\varphi'(z).\psi(z) \\ &+ \sum_{n=1}^{\infty} n\widehat{f}(n)\varphi^{n-1}(z).\psi'(z). \end{aligned}$$

Hence

$$\begin{aligned} S_{\psi,\varphi}f &= \sum_{n=2}^{\infty} n(n-1)\widehat{f}(n)\varphi^{n-2}.\varphi'.\psi \\ &+ \sum_{n=1}^{\infty} n\widehat{f}(n)\varphi^{n-1}.\psi' \quad \forall\ f \in H^2(\beta). \end{aligned} \tag{2}$$

Now

$$\begin{aligned} \langle f, Tg\rangle &= \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{Tg}(n)}\beta_n^2 \\ &= \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\Big(\frac{n}{\beta_n^2}\sum_{k=0}^{\infty}\widehat{g}(k)\overline{\widehat{h}_n(k)}\beta_k^2\Big)}\beta_n^2 \\ &= \sum_{k=0}^{\infty}\Big(\sum_{n=1}^{\infty} n\widehat{f}(n)\widehat{h}_n(k).\overline{\widehat{g}(k)}\Big)\beta_k^2 \\ &= \sum_{k=0}^{\infty}\Big(\sum_{n=1}^{\infty} n\widehat{f}(n)\big(\psi.D\varphi^{n-1}+\psi'.\varphi^{n-1}\big)^h(k)\Big)\overline{\widehat{g}(k)}\beta_k^2 \\ &= \sum_{k=0}^{\infty}\Big(\sum_{n=1}^{\infty} n\widehat{f}(n)(\psi.D\varphi^{n-1})^h(k) \\ &+ \sum_{n=1}^{\infty} n\widehat{f}(n).(\psi'.\varphi^{n-1})^h(k)\Big)\overline{\widehat{g}(k)}\beta_k^2 \\ &= \sum_{k=0}^{\infty}\Big(\sum_{n=2}^{\infty} n(n-1)\widehat{f}(n)\psi.\varphi^{n-2}\varphi' \\ &+ \sum_{n=1}^{\infty} n\widehat{f}(n)\psi'.\varphi^{n-1}\Big)^h(k)\overline{\widehat{g}(k)}\beta_k^2. \end{aligned}$$

Using equation (2), we get

$$\begin{aligned}\langle f, T_g\rangle &= \sum_{k=0}^{\infty} (\widehat{S_{\psi,\varphi} f})(k)\overline{\widehat{g}(k)}\beta_k^2 \\ &= \langle S_{\psi,\varphi} f, g\rangle \quad for\ all \quad f, g \in H^2(\beta).\end{aligned}$$

This implies that $S^{\star}_{\psi,\varphi} = T$. $\square$

Corollary 2.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map such that the sandwich composition operator $C^{\varphi} = DC_{\varphi}D \in H^2(\beta)$. Then the adjoint of C^{φ} is given by*

$$(C^{\varphi})^{\star} f = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$C_n = \begin{cases} 0, \quad if \ \ n = 0 \\ \frac{n}{\beta_n^2} \sum_{k=0}^{\infty} \widehat{f}(k).D\overline{\widehat{\varphi}^{n-1}(k)}\beta_k^2, \quad if \ \ n \geq 1. \end{cases}$$

Proof. Putting $\psi \equiv 1$ in Theorem 2.1, the proof follows. $\square$

Corollary 2.3. *Let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic map such that the sandwich multiplication operator $M^{\psi} = DM_{\psi}D \in H^2(\beta)$. Then the adjoint of M^{ψ} is given by*

$$(M^{\psi})^{\star} f = \sum_{n=0}^{\infty} d_n z^n,$$

where

$$d_n = \begin{cases} 0, \quad if \ \ n = 0 \\ \frac{n}{\beta_n^2}\big[\sum_{k=0}^{\infty}(n-1)\widehat{f}(k)(\widehat{e_{n-2}.\psi})(k) + \sum_{k=0}^{\infty} \widehat{f}(k)(\widehat{e_{n-1}.\psi'})(k)\big]\beta_k^2, \\ if \ \ n \geq 1 \end{cases}$$

where $e_n : \mathbb{D} \to \mathbb{C}$ is defined as $e_n(z) = z^n \ \ \forall\ z \in \mathbb{D}$.

Proof. Putting $\varphi(z) = z$, in Theorem 2.1, the proof follows. $\square$

3 Computation of Adjoint of the Operator $S_{\psi,\varphi}$ for Evaluation Kernels

In this section, we compute the adjoint of the sandwich weighted composition operator $S_{\psi,\varphi}$ of the evaluation kernels on weighted Hardy space $H^2(\beta)$. For this, we need following Lemma.

Lemma 3.1. *Let $f \in H^2(\beta)$ and $K_w(z)$ be the evaluation kernel. Then*

$$\langle f, K_w^{[1]}(z)\rangle = f'(w) \quad \textit{and} \quad \langle f, K_w^{[2]}(z)\rangle = f''(w),$$

where $K_w^{[1]}$ and $K_w^{[2]}$ are the first and second order derivatives of K_w with respect to w respectively.

Proof. Let $f \in H^2(\beta)$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k.z^k,$$

$$f'(w) = \sum_{k=1}^{\infty} ka_k.w^{k-1} \tag{3}$$

and

$$f''(w) = \sum_{k=2}^{\infty} k(k-1)a_k.w^{k-2}. \tag{4}$$

Now

$$K_w(z) = \sum_{k=0}^{\infty} \frac{z^k.\overline{w}^k}{\beta_k^2}.$$

Therefore,

$$K_w^{[1]}(z) = \sum_{k=1}^{\infty} \frac{kz^k.\overline{w}^{k-1}}{\beta_k^2} \tag{5}$$

and

$$K_w^{[2]}(z) = \sum_{k=2}^{\infty} \frac{k(k-1)z^k.\overline{w}^{k-2}}{\beta_k^2}. \tag{6}$$

Using equations (3) and (5), we get $\langle f, K_w^{[1]}(z)\rangle = f'(w)$ and by using (4) and (6), we get $\langle f, K_w^{[2]}(z)\rangle = f''(w)$. □

Theorem 3.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic such that the sandwich weighted composition operator $S_{\psi,\varphi} : H^2(\beta) \to H^2(\beta)$ is bounded. Then the adjoint $S^*_{\psi,\varphi}$ of $S_{\psi,\varphi}$ is given by*

$$S^*_{\psi,\varphi}K_w = \overline{\psi'(w)}K^{[1]}_{\varphi(w)} + \overline{\psi(w).\varphi'(w)}.K^{[2]}_{\varphi(w)}, \forall \ \ w \in \mathbb{D}.$$

Proof. Let $f \in H^2(\beta)$. Then for $w \in \mathbb{D}$,

$$\begin{aligned} \langle f, S^*_{\psi,\varphi}K_w\rangle &= \langle S_{\psi,\varphi}f, K_w\rangle \\ &= \langle (\psi.f'o\varphi)', K_w\rangle = (\psi.f'o\varphi)'(w) \\ &= (\psi'.f'o\varphi + \psi.f''o\varphi.\varphi')(w) \\ &= \psi'(w).f'(\varphi(w)) + \psi(w).f''(\varphi(w))\varphi'(w). \end{aligned}$$

Using Lemma 3.1, we get

$$\begin{aligned} \langle f, S^*_{\psi,\varphi}K_w\rangle &= \psi'(w)\langle f, K^{[1]}_{\varphi(w)}\rangle + \psi(w).\varphi'(w)\langle f, K^{[2]}_{\varphi(w)}\rangle \\ &= \langle f, \overline{\psi'(w)}K^{[1]}_{\varphi(w)} + \overline{\psi(w).\varphi'(w)}.K^{[2]}_{\varphi(w)}\rangle. \end{aligned}$$

This implies that

$$S^*_{\psi,\varphi}K_w = \overline{\psi'(w)}K^{[1]}_{\varphi(w)} + \overline{\psi(w).\varphi'(w)}.K^{[2]}_{\varphi(w)} \ \ \forall \ \ w \in \mathbb{D}.$$

□

Corollary 3.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic map. Then the adjoint of the sandwich composition operator $C^\varphi = DC_\varphi D : H^2(\beta) \to H^2(\beta)$ is given by*

$$(C^\varphi)^\star K_w = \overline{\varphi'(w)}.K^{[2]}_{\varphi(w)} \ \ \forall \ \ w \in \mathbb{D},$$

where $(C^\varphi)^\star$ is the adjoint of C^φ.

Proof. The result followings by putting $\psi(w) \equiv 1$ in Theorem 3.2. $\square$

Corollary 3.4. *Let $\psi : \mathbb{D} \to \mathbb{C}$ be analytic map. Then the adjoint $(M^{\psi})^{\star}$ of the sandwich multiplication operator $M^{\psi} = DM_{\psi}D : H^2(\beta) \to H^2(\beta)$ is given by*

$$(M^{\psi})^{\star} K_w = (\overline{\psi}.K_w^{[1]})' \quad \forall \quad w \in \mathbb{D}.$$

Proof. By putting $\varphi(w) = w$ in Theorem 3.2, we get

$$(M^{\psi})^{\star} K_w = \overline{\psi'(w)}.K_w^{[1]} + \overline{\psi(w)}.K_w^{[2]}.$$

Since derivative of conjugate is equal to conjugate of derivative, we see that

$$\begin{aligned}(M^{\psi})^{\star} K_w &= \overline{\psi'(w).\overline{K_w^{[1]}} + \psi(w).\overline{K_w^{[2]}}} \\ &= \overline{(\psi.\overline{K_w^{[1]}})'} \\ &= (\overline{\psi}.K_w^{[1]})'.\end{aligned}$$

$\square$

Theorem 3.5. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic such that the sandwich composition operator $C^{\varphi} \in B(H^2(\beta))$. Then $|\varphi'(w)| \leq \|C^{\varphi}\|.\frac{\|K_w\|_{H^2(\beta)}}{\|K^{[2]}_{\varphi(w)}\|_{H^2(\beta)}}$ $for\ all \quad w \in \mathbb{D}$.*

Proof. Let $f_w = \frac{K_w}{\|K_w\|_{H^2(\beta)}}$. Then $\|f_w\|_{H^2(\beta)} = 1$. Since C^{φ} is bounded, we have

$$\| \ (C^{\varphi})^{\star} f_w \ \|_{H^2(\beta)} \leq \|(C^{\varphi})^{\star}\|.\|f_w\|_{H^2(\beta)} = \|C^{\varphi}\|.$$

That is

$$\|(C^{\varphi})^{\star} \frac{K_w}{\|K_w\|_{H^2(\beta)}}\|_{H^2(\beta)} \leq \|C^{\varphi}\|$$

or

$$\|(C^{\varphi})^{\star} K_w\|_{H^2(\beta)} \leq \|C^{\varphi}\| \|K_w\|_{H^2(\beta)}.$$

By using Corollary 3.3, we have

$$\|\overline{\varphi'(w)}.K^{[2]}_{\varphi(w)}\|_{H^2(\beta)} \leq \|C^{\varphi}\|\|K_w\|_{H^2(\beta)}.$$

This implies that

$$|\varphi'(w)| \leq \|C^{\varphi}\|.\frac{\|K_w\|_{H^2(\beta)}}{\|K^{[2]}_{\varphi(w)}\|_{H^2(\beta)}}.$$

This complete the proof. □

Theorem 3.6. *Let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic map such that the sandwich multiplication operator $M^{\psi} \in B(H^2(\beta))$. Then for all $w \in \mathbb{D}$,*

$$\|(\overline{\psi}K^{[1]}_w)'\|_{H^2(\beta)} \leq \|M^{\psi}\|.\|K_w\|_{H^2(\beta)}.$$

Proof. Let $f_w = \frac{K_w}{\|K_w\|_{H^2(\beta)}}$. Then $\|f_w\|_{H^2(\beta)} = 1$. Since M^{ψ} is bounded,

$$\|(M^{\psi})^* f_w\|_{H^2(\beta)} \leq \|(M^{\psi})^*\|.\|f_w\|_{H^2(\beta)}.$$

That is

$$\begin{aligned} \|(M^{\psi})^*.\frac{K_w}{\|K_w\|_{H^2(\beta)}}\|_{H^2(\beta)} &\leq \|M^{\psi}\| \\ \|(M^{\psi})^* K_w\|_{H^2(\beta)} &\leq \|M^{\psi}\|.\|K_w\|_{H^2(\beta)}. \end{aligned}$$

Hence

$$\|(\overline{\psi}K^{[1]}_w)'\|_{H^2(\beta)} \leq \|M^{\psi}\|.\|K_w\|_{H^2(\beta)}.$$

□

Theorem 3.7. *Let φ be an analytic self-map of the open unit disc $\mathbb{D}$, such that the sandwich composition operator $C^{\varphi} \in B(H^2(\beta))$. If $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$, then $\|C^{\varphi}\|$ is bounded below by $\frac{\alpha\sqrt{2}}{\beta_2\sqrt{\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}}}$, where $\alpha = \|\varphi'(w)\|_{H^2(\beta)}$ and $w \in \mathbb{D}$ s.t $\varphi'(w) \neq 0$.*

Proof. By the Theorem 3.5, we have

$$|\varphi'(w)| \leq \|C^{\varphi}\|.\frac{\|K_w\|_{H^2(\beta)}}{\|K^{[2]}_{\varphi(w)}\|_{H^2(\beta)}} \quad \forall \quad w \in \mathbb{D}. \tag{7}$$

Now

$$\|K_w\|^2_{H^2(\beta)} = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2}.$$

This implies that

$$\|K_w\|_{H^2(\beta)} < \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}. \tag{8}$$

Now from equation (6) of Lemma 3.1, we have

$$\|K^{[2]}_w\|^2_{H^2(\beta)} = \sum_{n=2}^{\infty} \frac{n^2(n-1)^2\|w\|^{2(n-2)}}{\beta_n^2}$$

$$\geq \frac{2}{\beta_2^2}. \tag{9}$$

Using equations (8) and (9) in (7), we get

$$|\varphi'(w)| \leq \|C^{\varphi}\|.\frac{\beta_2}{\sqrt{2}}\sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}$$

$$\frac{\alpha\sqrt{2}}{\beta_2\sqrt{\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}}} \leq \|C^{\varphi}\|.$$

This complete the proof. □

Acknowledgements
The authors are thankful to the editor and referees for quitting out several mathematical errors which improved the manuscript a lot.

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