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# Classification of Second Order Functional Differential Equations with Constant Coefficients to Solvable Lie Algebras

**J. Z. Lobo***
St. Xavier's College, Mapusa – Goa

**Y. S. Valaulikar**
Goa University

**Abstract.** In this paper, we shall apply symmetry analysis to second order functional differential equations with constant coefficients. The determining equations of the admitted Lie group are constructed in a manner different from that of the existing literature for delay differential equations. We define the standard Lie bracket and make a complete classification of the second order linear functional differential equations with constant coefficients, to solvable Lie algebras. We also classify some second order nonlinear functional differential equations with constant coefficients, to solvable Lie algebras.

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*Corresponding Author

## 1 Introduction

In the case of ordinary differential equations the unknown function and its derivatives are all evaluated at the same instant, $t$. This may not always be the case as we could have differential equations, in which the unknown function occurs with various different arguments. Such differential equations are popularly known as functional differential eqautions. The simplest of these are found by expressing some derivative of the dependent variable $x$ at time $t$, which is the independent variable, in terms of $x$ and its lower derivatives, if any, at $t$ and earlier instants. Such equations are called delay differential equations. Further, if the differential equations contain the unknown function and the derivative with time delays, then such differential equations are called neutral differential equations. Neutral differential equations in particular are of importance in studying models involving flip-flop circuit [27], compartmental systems [30], etc. Several research papers deal with obtaining solutions of neutral differential equations. In [28], neutral differential equations are solved using multistep block method. Other methods of solution include implicit block method [14], and analysing discontinuities of the derivatives as studied in [1]. In general, these differential equations have a wide range of applications which include heat transfer problems, signal processing, evolution of species, traffic flow, study of epidemics, population models, prey-predator models, biological systems, population dynamics, networking problems, rolling of ships, electrical engineering, control systems, etc [15]. Well known methods to solve these kinds of differential equations include the method of steps, numerical solutions, substitutions, and power series solutions (See [5]). A great detail of literature on delay differential equations can be found in [7, 8].

Symmetries are transformations which leave an object unchanged or invariant. Symmetries make a very important tool in studying various laws governing nature. In [25] it is pointed that symmetry accounts for the regularities of the laws that are independent of some inessential circumstances. For example, reproducibility of experiments at different places and time relies on invariance laws of nature under space translation, rotation and time translation. A very important implication of symmetry in Physics and Mathematics is the existence of conservation laws. Nöether [24], in 1918, observed this connection in proving a relation between continuous symmetries and conservation laws. Even scientists like Kepler and Newton studied the motion of planets as a symmetry principle. Symmetry analysis is widely used in Mathematics, Physics and Mechanics. Sophus Lie, over a century back initiated this subject. Lie group analysis is an excellent tool in studying the properties of solutions of differential equations.

In this paper we make a complete classification of second order functional (delay and neutral) differential equations with constant coefficients to solvable Lie algebras. The presence of delay terms makes especially the higher order nonlinear differential equations difficult to solve. As there is no analytic method to solve them directly, group analysis is the best way to study the properties of delay differential equations. Most of the existing research on symmetry analysis are done by changing the space variables. However, differential equations with deviating arguments do not possess any equivalent transformations related with the change of the variables – both dependent and independent. We consider the absence of such equivalent transformations to obtain a basis for the solvable Lie algebras of such differential equations. We shall use certain facts stated later to simplify our second order differential equations. We provide a basis for the Lie algebra given by first order linear and nonlinear differential equations, for which there is no existing literature. We deal with second order differential equations by simplifying them and using an approach different from the existing literature. We also make a classification for some second order nonlinear differential equations for which there is no existing literature.

We shall be studying the differential equation

$$\Phi(t, x(t), x(t-r), x'(t), x'(t-r), x''(t), x''(t-r)) = 0,$$

where $\Phi$ is defined on $I \times D^6$ where $D$ is an open set in $\mathbb{R}$, $I$ is an interval in $\mathbb{R}$ and $r > 0$ is the delay. We assume that $\dfrac{\partial \Phi}{\partial x''(t-r)} \neq 0$. We shall find a Lie group under which these differential equations are invariant. We call this the admitted Lie group by which we mean that one solution curve is carried to another solution curve of the same equation.

In [29], symmetries of delay differential equations are obtained by defining a certain operator, equivalent to the canonical Lie-Bäcklund operator. In [26], equivalent symmetries for second order linear delay differential equation are obtained. However, it may be noted that in [26] too, an operator equivalent to the canonical Lie-Bäcklund operator and other suitable operators are defined. The research carried out in [4] exhaustively describes the Lie symmetries of systems of second order linear ordinary differential equations with constant coefficients over both real and complex fields. The research also proposes an algebraic approach to obtain bounds for the dimensions of the maximal Lie invariance algebras possessed by such systems. Further, such systems are thoroughly provided their group classification in [19, 20], with extensions to linear systems of second order ordinary differential equations with more than two equations. Higher order symmetries for ordinary differential equations are studied in [10]. A group method is suggested in [16] to study functional differential equations based on a search of symmetries of underdetermined differential equations by methods of classical and modern group analysis, using the principle of factorization. The method therein, encompasses the use of a basis of invariants consisting of universal and differential invariants. By using Lie-Bäcklund operator and invariant manifold theorem, [21] classifies second order delay differential equations to solvable Lie algebras. It is seen that [21] performs a symmetry analysis without simplification of the linear delay differential equations, the simplification of which will be seen in this paper. In addition several crucial cases are not considered in [21]. The approach for classification of delay differential equations to solvable Lie algebras is extended to some nonlinear differential equations in [22, 23]. Recently in [17], an admitted Lie group for first order delay differential equations with constant coefficients is defined, and the corresponding generators of the Lie group for this equation are obtained. The approach in [17] consists of using Lie-Bäcklund operators to obtain the determining equations. Further first order neutral differential equations with most general time delay have been studied in [18]. The drawback of the analysis in [17, 18, 26, 29] is that the inverse of the obtained classification cannot be found. The research in Lie symmetry analysis has been extended to first-order linear and nonlinear fractional differential equations by [12]. Further, [13] develops Lie symmetry analysis for second-order fractional differential equations, based on conformable fractional derivatives and presents some numerical examples to illustrate the approach. In this paper we use the ideas from [2, 3, 11] for ordinary differential equations, (we particularly use only Taylor's theorem) to extend the study and results of obtaining symmetries for linear and nonlinear functional differential equations. Unlike the existing research, our approach does not lead to magnification in the delay terms when obtaining the determining equations.

The rest of this paper is organised as follows: The next section gives some preliminaries – definitions, examples and existing results. The following section extends the results for ordinary differential equations to functional differential equations, by obtaining a Lie type invariance condition using Taylor's theorem for a function of several variables. In the sections to follow, each section will consist of two subsections — one for linear and the other for nonlinear differential equations with constant coefficients. Each section will independently be concerned with (i) Second order delay differential equations (ii) Second order neutral differential equations. We conclude with representation of our results, which are the basis for the Lie algebras, in a tabular form.

## 2 Preliminaries

In this section, some preliminaries are given.

**Definition 2.1.** ([6]) *In general, we consider transformations* $\bar{t}_i = f_i(t_j, \delta), \;\; i, j = 1, 2, \cdots, n.$, *which continuously depend on the parameter* $\delta$.
*Further we assume that, for each* $i$, $f_i$ *is a smooth function of the variables* $t_j$ *and has convergent Taylor series in* $\delta$.
*These set of transformations are said to form a group if:*

1. *Two transformations carried out in succession are equivalent to another transformation of the set*

*2. There is a transformation for which the source and image points coincide.*

*3. Each transformation has an inverse.*

**Remark 2.2.** The associativity law for groups follows from the *closure* property.

**Remark 2.3.** In general, the order in which the transformations are carried out matters. If the order does not matter, then we label the group as **abelian**.

Having defined Lie groups, we now turn to define Lie algebras and the terminologies involved:

**Definition 2.4.** ([11]) *A Lie algebra $L$ is a vector space over a real or complex field $\mathbb{F}$ together with a binary operation "," satisfying the following properties:*

1. *(Bilinearity) For $S_1, S_2 \in L$ and $a, b \in \mathbb{F}$,*
   $[S_1, aS_2 + bS_3] = a[S_1, S_2] + b[S_1, S_3]$,
   *and*
   $[aS_1 + bS_2, S_3] = a[S_1, S_3] + b[S_2, S_3]$.
2. *(Anti-symmetry) For $S_1, S_2 \in L$,*
   $[S_1, S_2] = -[S_2, S_1]$.
3. *(Jacobi identity) For $S_1, S_2, S_3 \in L$,*
   $[S_1, [S_2, S_3]] + [S_2, [S_3, S_1]] + [S_3, [S_1, S_2]] = 0$.

**Definition 2.5.** (*Basis for a Lie algebra*)([11]) *A finite set of infinitesimal generators $\{S_1, S_2, \cdots, S_m\}$ is said to be a basis for the Lie algebra $L$ if:*

1. *$S_i \in L$ and $\{S_1, S_2, \cdots, S_m\}$ forms a basis for the vector space $L$,*
2. $[S_i, S_j] = \sum c_{ijk} S_k$.

The coefficients $c_{ijk}$ are called the structure constants of the Lie algebra, $i, j, k = 1, 2, \cdots, m$.

**Definition 2.6.** (*Solvable Lie algebra*)([11]) *A Lie algebra $L$ is said to be solvable if there exists a sequence*

$$L = L_r \supset L_{r-1} \supset \cdots \supset L_1,$$

*of sub-algebras of dimensions $r, r-1, \cdots, 1$, respectively such that $\forall s = 2, 3, \cdots, r$, $L_{s-1}$ is an ideal in $L_s$.*

**Definition 2.7.** (*Commutator*)([11]) *Let $S_i = \omega_{i\alpha}(x)\dfrac{\partial}{\partial x_\alpha}$, $S_j = \omega_{j\beta}(x)\dfrac{\partial}{\partial x_\beta}$, where $x = (x_1, x_2, \cdots, x_n)$. Then the commutator of $S_i$ and $S_j$ is defined as:*

$$[S_i, S_j] = S_iS_j - S_jS_i = \sum_{\alpha,\beta=1}^{n}\Big[\left(\omega_{i\alpha}(x)\frac{\partial}{\partial x_\alpha}\right)\left(\omega_{j\beta}(x)\frac{\partial}{\partial x_\beta}\right) - \left(\omega_{j\beta}(x)\frac{\partial}{\partial x_\beta}\right)\left(\omega_{i\alpha}(x)\frac{\partial}{\partial x_\alpha}\right)\Big] = \sum_{\beta=1}^{n}\Upsilon_\beta(x)\frac{\partial}{\partial x_\beta},$$

*where $\Upsilon_\beta(x) = \sum_{\alpha=1}^{n}\left[\omega_{i\alpha}(x)\dfrac{\partial \omega_{j\beta}(x)}{\partial x_\alpha} - \omega_{j\beta}(x)\dfrac{\partial \omega_{i\beta}(x)}{\partial x_\beta}\right]$.*

**Example 2.8.** Let $S_1 = t\dfrac{\partial}{\partial t}$, $\quad S_2 = tx\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right)$, then,

$$\begin{aligned}
[S_1, S_2] &= S_1S_2 - S_2S_1 \\
&= \left(t\frac{\partial}{\partial t}\right)\left(tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\right) - \left(tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\right)\left(t\frac{\partial}{\partial t}\right) \\
&= tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right) \\
&= S_2.
\end{aligned}$$

**Definition 2.9.** (*Derived algebra*)([11]) *Let $\{S_1, \cdots, S_r\}$ be a basis of a Lie algebra $L_r$. The linear span of the commutators $[S_i, S_j]$ of all possible pairs of the basis operators is an ideal denoted by $L_r'$ and is called the derived algebra. The higher-order derived algebras are defined recursively: $L_r^{(n+1)} = (L_r^{(n)})'$, $n \in \mathbb{N}$.*

**Theorem 2.10.** *In any algebra $L_r, r > 2$, there exists a two-dimensional sub-algebra. Moreover, any operator $S \in L_r$, can be included in a two-dimensional sub-algebra.*

**Theorem 2.11.** *An algebra $L_r$ is solvable if and only if the derived algebra of some order equals zero: $L_r^{(n)} = 0$ for some $n > 0$. In particular, any two-dimensional algebra is solvable.*

**Theorem 2.12.** *Any two infinitesimal generators of a $r-$ parameter Lie group satisfy $[S_i, S_j] = \sum c_{ijk} S_k$, where $i, j, k = 1, 2, \cdots, r$.*

Symmetry analysis for ordinary differential equations is described below. The work done for ordinary differential equations stands as a motivation for our research.
For a second order ordinary differential equation expressed as $\frac{d^2\bar{x}}{d\bar{t}^2} = G(\bar{t}, \bar{x}, \frac{d\bar{x}}{d\bar{t}})$, if $\bar{t} = f_1(t, x; \delta), \bar{x} = f_2(t, x; \delta)$, where $f_1$ and $f_2$ are smooth functions in $t$ and $x$ having a convergent Taylor series in $\delta$, then

$$\omega(t, x) = \frac{\partial f_1(t, x; 0)}{\partial \delta}, \quad \Upsilon(t, x) = \frac{\partial f_2(t, x; 0)}{\partial \delta}.$$

$\omega$ and $\Upsilon$ are called coefficients of the infinitesimal transformations or simply infinitesimals.
Then using invariance, we get the Lie Invariance condition for second order ordinary differential equations as

$$\begin{aligned}\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)G - 3\omega_x x' G \\ = \omega G_t + \Upsilon G_x + \Upsilon_{[t]} G_{x'},\end{aligned}$$

where $\Upsilon_{[t]} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.

## 3 Lie Type Invariance Condition for Second Order Functional Differential Equations

In this section, we extend the results for ordinary differential equations to functional differential equations. The notation $x^r$ whenever it appears will denote $x(t - r)$.
We establish the following Lie type invariance condition for second order neutral differential equations.

**Theorem 3.1.** *Consider the second order neutral differential equation*

$$\frac{d^2x}{dt^2} = F(t, x, x(t-r), x'(t), x'(t-r), x''(t-r)),$$

*where $F$ be defined on a 6-dimensional space $I \times D^5$, $D$ is an open set in $\mathbb{R}$ and $I$ is any interval in $\mathbb{R}$. Then the Lie type invariance condition is given by*

$$\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} + \Upsilon^r_{[tt]} F_{x''(t-r)} =$$
$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x'',$$

*where,*

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega),$$

$$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega), \quad \textit{where} \quad D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + \cdots,$$

$$\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r,$$

$$\Upsilon^r_{[tt]} = (\Upsilon_{tt})^r + (2(\Upsilon_{tx})^r - (\omega_{tt})^r)x'(t-r) + ((\Upsilon_{xx})^r - 2(\omega_{tx})^r)x'(t-r)^2$$
$$- (\omega_{xx})^r x'(t-r)^3 + ((\Upsilon_x)^r - 2(\omega_t)^r)x''(t-r) - 3(\omega_x)^r x'(t-r)x''(t-r)),$$

*and $\omega^r = \omega(t-r, x(t-r))$, $\Upsilon^r = \Upsilon(t-r, x(t-r))$.*

**Proof.** Let the neutral differential equation be invariant under the Lie group

$$\bar{t} = t + \delta\omega(t,x) + O(\delta^2), \quad \bar{x} = x + \delta\Upsilon(t,x) + O(\delta^2).$$

We then naturally define $\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$ and
$\overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r, x(t-r)) + O(\delta^2)$.

With the notations, $\omega^r = \omega(t-r, x(t-r))$, and $\Upsilon^r = \Upsilon(t-r, x(t-r))$, it follows that,

$$\begin{aligned}\overline{x'(t-r)} &= \tfrac{d\bar{x}}{d\bar{t}}(\overline{t-r}) \\ &= x'(t-r) + (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) \\ &- (x'(t-r))^2(\omega_x)^r)\delta + O(\delta^2).\end{aligned}$$

Considering the second-order extended infinitesimals, we can write

$$\begin{aligned}
\frac{d^2\bar{x}}{d\bar{t}^2} &= \frac{d}{d\bar{t}}\left(\frac{d\bar{x}}{d\bar{t}}\right) \\
&= \frac{\dfrac{d}{dt}\left[\dfrac{dx}{dt} + [D_t(\Upsilon) - x' D_t(\omega)]\delta + O(\delta^2)\right]}{1 + \delta D_t(\omega) + O(\delta^2)} \\
&= \left(\frac{d^2x}{dt^2} + D_t(\Upsilon_{[t]})\delta + O(\delta^2)\right)(1 - \delta D_t(\omega) + O(\delta^2)) \\
&= \frac{d^2x}{dt^2} + (D_t(\Upsilon_{[t]}) - D_t(\omega)x'')\delta + O(\delta^2).
\end{aligned}$$

So, $\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega)$.
As $\Upsilon_{[t]}$ contains $t, x$ and $x'$, we need to extend the definition of $D_t$.
Let $D_t = \dfrac{\partial}{\partial t} + x'\dfrac{\partial}{\partial x} + x''\dfrac{\partial}{\partial x'} + \cdots$.
Expanding $\Upsilon_{[tt]}$, gives,

$$\begin{aligned}
\Upsilon_{[tt]} = \Upsilon_{tt} &+ (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 \\
&- \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.
\end{aligned}$$

It follows that,

$$\begin{aligned}
\overline{x''(t-r)} &= \frac{d^2\bar{x}}{d\bar{t}^2}(\overline{t-r}) \\
&= x''(t-r) + \Big[(\Upsilon_{tt})^r + (2(\Upsilon_{tx})^r - (\omega_{tt})^r)x'(t-r) \\
&+ ((\Upsilon_{xx})^r - 2(\omega_{tx})^r)x'(t-r)^2 - (\omega_{xx})^r x'(t-r)^3 \\
&+ ((\Upsilon_x)^r - 2(\omega_t)^r)x''(t-r) - 3(\omega_x)^r \\
&\quad x'(t-r)x''(t-r)\Big]\delta + O(\delta^2).
\end{aligned}$$

Let $\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r$ and
$\Upsilon^r_{[tt]} = (\Upsilon_{tt})^r + (2(\Upsilon_{tx})^r - (\omega_{tt})^r)x'(t-r) + ((\Upsilon_{xx})^r - 2(\omega_{tx})^r)$
$x'(t-r)^2 - (\omega_{xx})^r x'(t-r)^3 + ((\Upsilon_x)^r - 2(\omega_t)^r)x''(t-r)$
$- 3(\omega_x)^r x'(t-r)x''(t-r))$.
For invariance,

$$\frac{d^2\bar{x}}{d\bar{t}^2} = F(\bar{t}, \bar{x}, \overline{x(t-r)}, \frac{d\bar{x}}{d\bar{t}}, \frac{d\bar{x}}{d\bar{t}}(\overline{t-r}), \frac{d^2\bar{x}}{d\bar{t}^2}(\overline{t-r})).$$

This gives,

$$\begin{aligned}\frac{d^2x}{dt^2} + \Upsilon_{[tt]}\delta + O(\delta^2) &= F(t + \delta\omega + O(\delta^2), x + \delta\Upsilon + O(\delta^2), x(t-r) + \delta\Upsilon^r \\ &\quad + O(\delta^2), \frac{dx}{dt} + \delta\Upsilon_{[t]} + O(\delta^2), \frac{dx}{dt}(t-r) + \Upsilon^r_{[t]}\delta \\ &\quad + O(\delta^2), \frac{d^2x}{dt^2}(t-r) + \Upsilon^r_{[tt]}\delta + O(\delta^2)) \\ &= F(t, x, x(t-r), x'(t), x'(t-r), x''(t-r)) + \\ &\quad (\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} \\ &+ \Upsilon^r_{[tt]} F_{x''(t-r)})\delta + O(\delta^2).\end{aligned}$$

Comparing the coefficient of $\delta$, we get

$$\begin{aligned}&\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} + \Upsilon^r_{[tt]} F_{x''(t-r)} = \\ &\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.\end{aligned} \tag{1}$$

The above obtained equation (1) is a Lie type invariance condition. □

We can define a prolonged operator (the general infinitesimal generator associated with the Lie algebra) as below, for the second order neutral differential equation as:

$$\zeta = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)}.$$

We then, naturally define the extended operator, for second order neutral differential equations as:

$$\begin{aligned}\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)} \\ + \Upsilon_{[tt]}\frac{\partial}{\partial x''} + \Upsilon^r_{[tt]}\frac{\partial}{\partial x''(t-r)}.\end{aligned}$$

Defining, $\Delta = x''(t) - F(t, x(t), x(t-r), x'(t), x'(t-r), x''(t-r)) = 0$, we get,

$$\zeta^{(1)}\Delta = \Upsilon_{[tt]} - \omega F_t - \Upsilon F_x - \Upsilon^r F_{x(t-r)} - \Upsilon_{[t]} F_{x'(t)} - \Upsilon^r_{[t]} F_{x'(t-r)} - \Upsilon^r_{[tt]} F_{x''(t-r)}. \tag{2}$$

Comparing equation (2) and equation (1), we get

$$\begin{aligned}\Upsilon_{[tt]} &= \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 \\ &\quad - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.\end{aligned}$$

On substituting $x'' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the second order neutral differential equation which is $\zeta^{(1)}\Delta\mid_{\Delta=0}= 0$, from which we shall obtain the determining equations.

**Remark 3.2.** If the term $x''(t-r)$ is absent, then the corresponding second order neutral differential equation reduces to a second order delay differential equation.

We conclude this section by proving two very elementary results which we shall be using in our subsequent sections:

**Proposition 3.3.** *If the linear differential equation is given by*

$$x''(t) + bx'(t) + cx'(t-r) + dx''(t-r) + ex(t) + jx(t-r) = m(t), \quad (3)$$

*then by employing a change of variables namely* $\bar{t} = t, \bar{x} = x - \tilde{x}$*, where* $\tilde{x}$ *is a solution of equation (3), we can convert the given non-homogeneous linear differential equation to a homogeneous one, namely*
$x''(t) + bx'(t) + cx'(t-r) + dx''(t-r) + ex(t) + jx(t-r) = 0.$

**Proof.** The proposition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + \tilde{x}(\bar{t})$ in (3), by noting that
$\tilde{x}''(t) + b\tilde{x}'(t) + c\tilde{x}'(t-r) + d\tilde{x}''(t-r) + e\tilde{x}(t) + j\tilde{x}(t-r) = m(t).$ □

The next proposition is particularly useful in simplifying second order differential equations.

**Proposition 3.4.** *If the linear differential equation is given by*

$$\begin{aligned}x''(t) + a_1(t)x'(t) + b_1(t)x'(t-r) + c_1(t)x(t) \\ + d_1(t)x(t-r) + k_1(t)x''(t-r) = 0,\end{aligned} \quad (4)$$

*where* $a_1(t), b_1(t), c_1(t), d_1(t)$ *and* $k_1(t)$ *are twice differentiable functions with variable coefficients, then by making a suitable transformation, equation (4) can be reduced to a one in which the first order ordinary derivative term is missing.*

**Proof.** By employing a change, $x = u(t)s(t)$, where $u(t) \neq 0$ is some twice differentiable function in $t$ and with $s(t)$ satisfying
$s(t) = exp(-\int^{t} \frac{a_1(\xi)d\xi}{2}) + s_0$, where $s_0$ is an arbitrary constant, equation (4) can be reduced to
$u''(t)+b_2(t)u'(t-r)+c_2(t)u(t)+d_2(t)u(t-r)+k_2(t)u''(t-r) = 0$, where
$b_2(t) = \frac{b_1(t)s(t-r) + 2k(t)s'(t-r)}{s(t)}$, $c_2(t) = \frac{s''(t) + a_1(t)s'(t) + c_1(t)s(t)}{s(t)}$,
$d_2(t) = \frac{b_1(t)s'(t-r) + d_1(t)s(t-r) + k_1(t)s''(t-r)}{s(t)}$ and $k_2(t) = \frac{k_1(t)}{u(t)}$.
□

**Remark 3.5.** It should be noted that this transformation does not affect the symmetries of equation (4).

# 4 Classification of Second Order Delay Differential Equations to Solvable Lie Algebras

## 4.1 The Linear Case

We shall make a classification of

$$x''(t) + \alpha x'(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) = 0.$$

By using Proposition 3.4, we make a classification of

$$x''(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) = 0. \tag{5}$$

The extension and prolongation operator for equation (5) is given by,

$$\begin{aligned}\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} \\ + \Upsilon^r_{[t]}\frac{\partial}{\partial x^{r\prime}} + \Upsilon_{[tt]}\frac{\partial}{\partial x''}.\end{aligned} \tag{6}$$

Applying the operator defined by equation (6), to the delay equation $g(t) = t - r$, we get $\omega(t, x) = \omega(t - r, x(t - r))$.

Applying the operator defined by equation (6), to equation (5), we get,

$$\Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xx}x'^3+(\Upsilon_x-2\omega_t)(-\beta x'(t-r)$$
$$-\gamma x-\rho x(t-r))-3\omega_x x'(-\beta x'(t-r)-\gamma x-\rho x(t-r))+\beta(\Upsilon_t^r+(\Upsilon_x^r-\omega_t^r)x^{r\prime}$$
$$-\omega_x^r {x^{r\prime}}^2)+\gamma\Upsilon+\rho\Upsilon^r=0. \quad (7)$$

Splitting equation (7) with respect to the constant term we get,

$$\Upsilon_{tt}+\beta\Upsilon_t^r+\gamma\Upsilon+\rho\Upsilon^r=0. \quad (8)$$

Splitting equation (7) with respect to $x$ we get,

$$\gamma(\Upsilon_x-\omega_t)=0. \quad (9)$$

Splitting equation (7) with respect to $x'$ we get,

$$2\Upsilon_{tx}=\omega_{tt}.$$

Splitting equation (7) with respect to $x'^2$ we get,

$$\Upsilon_{xx}=2\omega_{tx}. \quad (10)$$

Splitting equation (7) with respect to $x'^3$ we get,

$$\omega_{xx}=0.$$

Splitting equation (7) with respect to $x'x^{r\prime}, xx'$ or $x'x^r$, we get,

$$\omega_x=0. \quad (11)$$

Splitting equation (7) with respect to $x^r$ we get,

$$-\rho(\Upsilon_x-2\omega_t)=0. \quad (12)$$

Splitting equation (7) with respect to ${x^{r\prime}}^2$ we get,

$$-\beta\omega_x^r=0. \quad (13)$$

Splitting equation (7) with respect to $x^{r\prime}$ we get,

$$-\beta(\Upsilon_x-2\omega_t)+\beta(\Upsilon_x^r-\omega_t^r)=0. \quad (14)$$

From equation (11) and (13), we get $\omega = \omega(t)$.
From equation (10), $\Upsilon = A(t)x + \theta(t)$.
From equation (9) or (12), we get,

$$\omega_t = \frac{1}{2}A(t). \tag{15}$$

From equation (14) and using equation (15), we get
$\Upsilon^r = \frac{1}{2}A(t)x + \psi(t-r)$.

The following theorems make a complete group classification of the second order delay differential equation. The notation $u$ is used to denote $x^r$.

**Theorem 4.1.** *The delay differential equation given by equation (5) for which* $\beta \neq 0, \gamma \neq -\frac{\rho}{2}$ *admits a three dimensional group generated by*

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\frac{\partial}{\partial x}, \quad S_3 = x\frac{\partial}{\partial u},$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_4^i = -\left(\frac{1}{2(\gamma+\frac{\rho}{2})} + \frac{4(\gamma+\frac{\rho}{2})}{\beta}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{2}{\beta}A_t + \frac{4(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]$$
$$\frac{\partial}{\partial x} + \left[\psi - x\left(\frac{1}{\beta}A_t + \frac{2(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta, \gamma, \rho$ be arbitrary non-zero constants, $\gamma \neq -\frac{\rho}{2}$. Then from equation (8), we get,

$$A_{tt} + \frac{\beta}{2}A_t + \gamma A + \frac{\rho}{2}A = 0, \tag{16}$$

and $\theta_{tt} + \beta\psi_t + \gamma\theta + \rho\psi = 0$.
Solving equation (16) using equation (15), we get,

$$\omega = c_1 - \frac{A_t + \frac{\beta}{2}A}{2(\gamma+\frac{\rho}{2})}, \tag{17}$$

where $c_1$ is an arbitrary constant. From equation (17),

$$A(t) = c_2 - \frac{2}{\beta}A_t - \frac{4\omega(\gamma + \frac{\rho}{2})}{\beta},$$

where $c_2 = \dfrac{4c_1(\gamma + \frac{\rho}{2})}{\beta}$.
This yields,

$$\Upsilon = \left(c_2 - \frac{2}{\beta}A_t - \frac{4\omega(\gamma + \frac{\rho}{2})}{\beta}\right) x + \theta,$$

and,

$$\Upsilon^r = \left(c_3 - \frac{1}{\beta}A_t - \frac{2\omega(\gamma + \frac{\rho}{2})}{\beta}\right) x + \psi,$$

where $c_3 = \dfrac{c_2}{2}$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= \left(c_1 - \frac{A_t + \frac{\beta}{2}A}{2(\gamma + \frac{\rho}{2})}\right)\frac{\partial}{\partial t} + \left[\left(c_2 - \frac{2}{\beta}A_t - \frac{4\omega(\gamma + \frac{\rho}{2})}{\beta}\right) x + \theta\right] \\ &\quad \frac{\partial}{\partial x} + \left[\left(c_3 - \frac{1}{\beta}A_t - \frac{2\omega(\gamma + \frac{\rho}{2})}{\beta}\right) x + \psi\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by
$S_1 = \dfrac{\partial}{\partial t}, \quad S_2 = x\dfrac{\partial}{\partial x}, \quad S_3 = x\dfrac{\partial}{\partial u}.$

With $g = 2\omega_t$, we get

$$S_4 = -\left(\frac{1}{2(\gamma+\frac{\rho}{2})} + \frac{\beta}{4(\gamma+\frac{\rho}{2})}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{2}{\beta}A_t + \frac{4(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]$$
$$\frac{\partial}{\partial x} + \left[\psi - x\left(\frac{1}{\beta}A_t + \frac{2(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial u}$$

is the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $S_1$ | 0 | 0 | 0 |
| $S_2$ | 0 | 0 | $S_3$ |
| $S_3$ | 0 | $-S_3$ | 0 |

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

**Theorem 4.2.** *The delay differential equation given by equation (5) for which* $\beta \neq 0, \gamma = -\frac{\rho}{2}$ *admits a four dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right],$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_5^i = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + (\psi - \frac{\beta x\omega}{4})\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta, \gamma, \rho$ be arbitrary non-zero constants, $\gamma = -\frac{\rho}{2}$. Then from equation (8), we get,

$$\omega_{ttt} + \frac{\beta}{2}\omega_{tt} = 0, \tag{18}$$

and $\theta_{tt} + \beta\psi_t + \frac{\rho}{2}\theta + \rho\psi = 0$.
Solving equation (18) we get,

$$\omega = c_6 t + c_7 - \frac{A}{\beta}, \tag{19}$$

where $c_4, c_5$ are arbitrary constants and $c_6 = \dfrac{2c_4}{\beta}, c_7 = \dfrac{2c_5}{\beta}$. From equation (19),

$$A(t) = c_8 t + c_9 - \beta\omega,$$

where $c_8 = \beta c_6, \quad c_9 = \beta c_7$. This yields,

$$\Upsilon = (c_8 t + c_9 - \beta\omega)x + \theta(t),$$

and,

$$\Upsilon^r = \frac{1}{2}(c_8 t + c_9 - \beta\omega)x + \psi(t - r).$$

The infinitesimal generator is given by

$$\begin{aligned}
\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\
&= \left(c_6 t + c_7 - \frac{A}{\beta}\right)\frac{\partial}{\partial t} + [(c_8 t + c_9 - \beta\omega)x + \theta]\frac{\partial}{\partial x} \\
&+ [\frac{1}{2}(c_8 t + c_9 - \beta\omega)x + \psi]\frac{\partial}{\partial x^r}.
\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t}, \quad S_2 = \dfrac{\partial}{\partial t}$,
$S_3 = tx\left[\dfrac{\partial}{\partial x} + \dfrac{1}{2}\dfrac{\partial}{\partial u}\right], \quad S_4 = x\left[\dfrac{\partial}{\partial x} + \dfrac{1}{2}\dfrac{\partial}{\partial u}\right]$ with
$S_5 = -\dfrac{A}{\beta}\dfrac{\partial}{\partial t} + (\theta - \beta x\omega)\dfrac{\partial}{\partial x} + (\psi - \dfrac{\beta x\omega}{4})\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
|---|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | $S_3$ | 0 |
| $S_2$ | $S_2$ | 0 | $S_4$ | 0 |
| $S_3$ | $-S_3$ | $-S_4$ | 0 | 0 |
| $S_4$ | 0 | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3, S_4\}$ is a solvable Lie algebra. □

**Theorem 4.3.** *The delay differential equation given by equation (5) for which $\beta = 1, \gamma = 0 = \rho$ admits a three dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_3 = \frac{\partial}{\partial u},$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_4^i = -A\frac{\partial}{\partial t} + (\theta - x\omega)\frac{\partial}{\partial x} - \left(\theta_t + \frac{x\omega}{2}\right)\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta = 1$, $\gamma = 0 = \rho$. Then equation (8) becomes $\Upsilon_{tt} + \Upsilon_t^r = 0$, which yields,

$$A_{tt} + \frac{1}{2}A_t = 0, \tag{20}$$

and $\psi = -\theta_t + c_{10}$.
Solving equation (20) we get,

$$\omega = c_{11}t + c_{12} - A(t), \tag{21}$$

where $c_{10}, c_{11}, c_{12}$ are arbitrary constants. From equation (21),

$$A(t) = c_{11}t + c_{12} - \omega.$$

This yields,

$$\Upsilon = (c_{11}t + c_{12} - \omega)x + \theta(t),$$

and,

$$\Upsilon^r = \frac{1}{2}(c_{11}t + c_{12} - \omega)x + c_{10} - \theta_t.$$

The infinitesimal generator is given by

$$\begin{aligned}
\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\
&= (c_{11}t + c_{12} - A(t))\frac{\partial}{\partial t} + [(c_{11}t + c_{12} - \omega)x + \theta(t)]\frac{\partial}{\partial x} \\
&+ [\frac{1}{2}(c_{11}t + c_{12} - \omega)x + c_{10} - \theta_t]\frac{\partial}{\partial x^r}.
\end{aligned}$$

The Lie algebra is spanned by
$S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right],$

$$S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_3 = \frac{\partial}{\partial u}$$

and $S_4 = -A\dfrac{\partial}{\partial t} + (\theta - x\omega)\dfrac{\partial}{\partial x} - \left(\theta_t + \dfrac{x\omega}{2}\right)\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | 0 |
| $S_2$ | $S_2$ | 0 | 0 |
| $S_3$ | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

**Theorem 4.4.** *The delay differential equation given by equation (5) for which $\beta \neq 0, \gamma = 0 = \rho$ admits a five dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right],$$

$$S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u},$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_6^i = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + \left(\frac{\theta_t}{\beta} + \beta x\omega\right)\frac{\partial}{\partial u}.$$

**Proof.** Let $\gamma = 0 = \rho$, $\beta$ be an arbitrary non zero constant. Then equation (8) becomes $\Upsilon_{tt} + \beta\Upsilon_t^r = 0$, which yields,

$$A_{tt} + \frac{\beta}{2}A_t = 0, \tag{22}$$

and $\psi = c_{14} - \dfrac{\theta_t}{\beta}$, where $c_{13}$ is an arbitrary constant and $c_{14} = \dfrac{c_{13}}{\beta}$. Solving equation (22) we get,

$$\omega = c_{17}t + c_{18} - \frac{\theta}{\beta}, \tag{23}$$

where $c_{15}, c_{16}$ are arbitrary constants and $c_{17} = \frac{c_{15}}{\beta}, \quad c_{18} = \frac{c_{16}}{\beta}$. From equation (23),

$$A(t) = c_{15}t + c_{16} - \beta\omega.$$

This yields,

$$\Upsilon = (c_{15}t + c_{16} - \beta\omega)x + \theta(t),$$

and,

$$\Upsilon^r = \frac{1}{2}(c_{15}t + c_{16} - \beta\omega)x + c_{14} - \frac{\theta_t}{\beta}.$$

The infinitesimal generator is given by

$$\begin{aligned}
\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\
&= (c_{17}t + c_{18} - \frac{\theta}{\beta})\frac{\partial}{\partial t} + [(c_{15}t + c_{16} - \beta\omega)x + \theta(t)]\frac{\partial}{\partial x} \\
&+ \left[\frac{1}{2}(c_{15}t + c_{16} - \beta\omega)x + c_{14} - \frac{\theta_t}{\beta}\right]\frac{\partial}{\partial x^r}.
\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}$, $S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u}$ with $S_6 = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + \left(\frac{\theta_t}{\beta} + \beta x\omega\right)\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | $S_3$ | 0 | 0 |
| $S_2$ | $S_2$ | 0 | $S_4$ | 0 | 0 |
| $S_3$ | $-S_3$ | $-S_4$ | 0 | 0 | 0 |
| $S_4$ | 0 | 0 | 0 | 0 | 0 |
| $S_5$ | 0 | 0 | 0 | 0 | 0 |

Then $L = \{S_1, S_2, S_3, S_4, S_5\}$ is a solvable Lie algebra. $\square$

### 4.2 A Nonlinear Case

We make a classification of

$$x''(t) + x'(t) + x'(t-r)x(t) = 0. \tag{24}$$

Applying the operator defined by equation (5), to the delay equation $g(t) = t - r$, we get equation $\omega(t, x) = \omega(t - r, x(t - r))$.
Applying the operator defined by equation (5), to equation (24), we get,

$$\begin{aligned} &\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x'' + \Upsilon_t \\ &+ (\Upsilon_x - \omega_t)x' - \omega_x x'^2 + x^{r\prime}\Upsilon + x[\Upsilon_t^r + (\Upsilon_x^r - \omega_t^r)x^{r\prime} - \omega_x^r x^{r\prime 2}] = 0. \end{aligned} \tag{25}$$

Splitting equation (25) with respect to constant term, $x'$, $x'^2$, $x'^3$, $x''$, $x'x''$, $x^{r\prime}$ and $x^{r\prime 2}$ respectively, we get,

$$\Upsilon_{tt} + \Upsilon_t + x\Upsilon_t^r = 0, \tag{26}$$

and,

$$\begin{aligned} &2\Upsilon_{tx} - \omega_{tt} + \Upsilon_x - \omega_t = 0,\ \Upsilon_{xx} - 2\omega_{tx} - \omega_x = 0,\ \omega_{xx} = 0, \\ &\Upsilon_x - 2\omega_t = 0,\ \omega_x = 0,\ \Upsilon + x(\Upsilon_x^r - \omega_t^r) = 0,\ x\omega_x^r = 0. \end{aligned}$$

From these equations we get, $\omega = \omega(t)$, $\quad \Upsilon = A(t)x + \theta(t)$, $\Upsilon^r = \frac{1}{2}A(t)x + \psi(t - r)$, where $A(t) = 2\omega_t$.
Substituting the values of $\Upsilon, \Upsilon^r$ in equation (26) and solving it, we get, $A(t) = c_{19},\ \psi = c_{20},\ \theta = c_{21} - \theta_t$, and

$$\omega = c_{22}t + c_{23},$$

$$\Upsilon = c_{19}x + c_{21} - \theta_t, \quad \Upsilon^r = c_{24}x + c_{20},$$

where $c_{19}, c_{20}, c_{21}, c_{22}, c_{23}, c_{24}$ are arbitrary constants.
The infinitesimal generator is given by

$$\begin{aligned} \zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= (c_{22}t + c_{23})\frac{\partial}{\partial t} + (c_{19}x + c_{21} - \theta_t)\frac{\partial}{\partial x} + (c_{24}x + c_{20})\frac{\partial}{\partial x^r}. \end{aligned}$$

The Lie algebra is spanned by
$S_1 = x\dfrac{\partial}{\partial x}$, $S_2 = \dfrac{\partial}{\partial x}$, $S_3 = t\dfrac{\partial}{\partial t}$,
$S_4 = \dfrac{\partial}{\partial t}$, $S_5 = x\dfrac{\partial}{\partial u}$, $S_6 = \dfrac{\partial}{\partial u}$ and $S_7 = -\theta_t\dfrac{\partial}{\partial x}$ is the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ |
|---|---|---|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | 0 | 0 | $S_5$ | 0 |
| $S_2$ | $S_2$ | 0 | 0 | 0 | $S_6$ | 0 |
| $S_3$ | 0 | 0 | 0 | $-S_4$ | 0 | 0 |
| $S_4$ | 0 | 0 | $S_4$ | 0 | 0 | 0 |
| $S_5$ | $-S_5$ | $-S_6$ | 0 | 0 | 0 | 0 |
| $S_6$ | 0 | 0 | 0 | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ is a solvable Lie algebra.

**Remark 4.5.** For the non-homogeneous nonlinear second order delay differential equation $x''(t)+x'(t)+x'(t-r)x(t) = h(t)$, we get exactly the same generators as in the homogeneous case, only that $S_7 = (\theta_t - y)\dfrac{\partial}{\partial x}$ is the corresponding infinite dimensional Lie sub-algebra, where
$y = c_{22}\int t\, h' dt + c_{23}h$.

# 5 Classification of Second Order Neutral Differential Equations to Solvable Lie Algebras

## 5.1 The Linear Case

We shall make a classification of

$$x''(t) + \alpha x'(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) + \kappa x''(t-r) = 0.$$

By using the Theorem 3.1, we make a classification of

$$x''(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) + \kappa x''(t-r) = 0. \tag{27}$$

The extension and prolongation operator for equation (5) is given by,

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x^{r\prime}} + \Upsilon_{[tt]}\frac{\partial}{\partial x''} + \Upsilon^r_{[tt]}\frac{\partial}{\partial x^{r\prime\prime}}. \tag{28}$$

Applying the operator defined by equation (28), to the delay equation $g(t) = t - r$, we get equation $\omega(t, x) = \omega(t - r, x(t - r))$.
Applying the operator defined by equation (28), to equation (27), we get,

$$\begin{aligned} 0 = {} & \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)(-\beta x'(t-r) \\ & -\gamma x - \rho x(t-r) - \kappa x''(t-r)) - 3\omega_x x'(-\beta x'(t-r) - \gamma x - \rho x(t-r) - \kappa x''(t-r)) \\ & + \beta\Big[\Upsilon^r_t + (\Upsilon^r_x - \omega^r_t)x^{r\prime} - \omega^r_x {x^{r\prime}}^2\Big] + \gamma\Upsilon + \rho\Upsilon^r + \kappa\Big[\Upsilon^r_{tt} + (2\Upsilon^r_{tx} - \omega^r_{tt})x^{r\prime} \\ & + (\Upsilon^r_{xx} - 2\omega^r_{tx}){x^{r\prime}}^2 - \omega^r_{xx}{x^{r\prime}}^3 + (\Upsilon^r_x - 2\omega^r_t)x^{r\prime\prime} - 3\omega^r_x x^{r\prime}x^{r\prime\prime}\Big]. \end{aligned} \tag{29}$$

Splitting equation (29) with respect to the constant term we get,

$$\Upsilon_{tt} + \beta\Upsilon^r_t + \gamma\Upsilon + \rho\Upsilon^r + \kappa\Upsilon^r_{tt} = 0. \tag{30}$$

Splitting equation (29) with respect to $x$ we get,

$$\gamma(\Upsilon_x - 2\omega_t) = 0. \tag{31}$$

Splitting equation (29) with respect to $x'$ we get,

$$2\Upsilon_{tx} - \omega_{tt} = 0.$$

Splitting equation (29) with respect to $x'^2$ we get,

$$\Upsilon_{xx} - 2\omega_{tx} = 0. \tag{32}$$

Splitting equation (29) with respect to $x'^3$ we get,

$$\omega_{xx} = 0.$$

Splitting equation (29) with respect to $x'x^{r\prime}, xx', x'x^r$, or $x'x^{r\prime\prime}$ we get,

$$\omega_x = 0. \tag{33}$$

Splitting equation (29) with respect to $x^r$ we get,

$$-\rho(\Upsilon_x - 2\omega_t) = 0. \tag{34}$$

Splitting equation (29) with respect to ${x^{r}}'^{2}$ we get,

$$-\beta\omega^r_x + \kappa(\Upsilon^r_{xx} - 2\omega^r_{tx}) = 0.$$

Splitting equation (29) with respect to $x^{r'}$ we get,

$$-\beta(\Upsilon_x - 2\omega_t) + \beta(\Upsilon^r_x - \omega^r_t) + \kappa(2\Upsilon^r_{tx} - \omega^r_{tt}) = 0.$$

Splitting equation (29) with respect to ${x^{r}}'^{3}$ we get,

$$-\kappa\omega^r_{xx} = 0.$$

Splitting equation (29) with respect to $x^{r\prime}$ or $x^{r\prime\prime}$ we get,

$$-\kappa\omega^r_x = 0. \tag{35}$$

Splitting equation (29) with respect to $x^{r\prime\prime}$ we get,

$$-\kappa(\Upsilon_x - 2\omega_t) + \kappa(\Upsilon^r_x - 2\omega^r_t) = 0. \tag{36}$$

From equation (33), we get $\omega = \omega(t)$.
From equation (32), $\Upsilon = A(t)x + \theta(t)$.
From equation (31) or (34), we get,

$$\omega_t = \frac{1}{2}A(t). \tag{37}$$

From equation (36) and using equations (35) and (37), we get
$\Upsilon^r = A(t)x + \psi(t - r)$.

The following theorems make a complete group classification of the second order neutral differential equation. The notation $u$ is used to denote $x^r$:

**Theorem 5.1.** *The neutral differential equation given by equation (27) for which $\beta \neq 0, \kappa \neq 0, \gamma \neq -\rho$ admits a two dimensional group generated by*

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$

*with the infinite dimensional Lie sub-algebra given by*

$$\begin{aligned} S_3^i = &-\left(\frac{1+\kappa}{2(\gamma+\rho)}g_t + \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t} \\ &+\left[\theta - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial x} \\ &+\left[\psi - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial u}. \end{aligned}$$

**Proof.** Let $\beta, \gamma, \rho, \kappa$ be arbitrary non-zero constants, $\gamma \neq -\rho$. Then from equation (30), we get,

$$(1+\kappa)A_{tt} + \beta A_t + (\gamma+\rho)A = 0, \tag{38}$$

and $\theta_{tt} + \beta\psi_t + \gamma\theta + \rho\psi + \kappa\psi_{tt} = 0$.
Solving equation (38) by using equation (37), we get,

$$\omega = c_2 - \frac{1+\kappa}{2(\gamma+\rho)}A_t - \frac{\beta}{2(\gamma+\rho)}A, \tag{39}$$

where $c_1$ is an arbitrary constant and $c_2 = \dfrac{c_1}{\gamma+\rho}$. From equation (39),

$$A(t) = c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega,$$

where $c_3 = \dfrac{2c_2(\gamma+\rho)}{\beta}$.
This yields,

$$\Upsilon = \left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \theta,$$

and,

$$\Upsilon^r = \left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \psi.$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_2 - \frac{1+\kappa}{2(\gamma+\rho)}A_t - \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t}\\ &+ \left[\left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \theta\right]\frac{\partial}{\partial x}\\ &+ \left[\left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \psi\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right).$$

With $g = 2\omega_t$, we get

$$\begin{aligned}S_3 = &-\left(\frac{1+\kappa}{2(\gamma+\rho)}A_t + \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t}\\ &+ \left[\theta - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial x}\\ &+ \left[\psi - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial u}\end{aligned}$$

is the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ |
|---|---|---|
| $S_1$ | 0 | 0 |
| $S_2$ | 0 | 0 |

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra. □

**Corollary 5.2.** *For the neutral differential equation given by equation (27), we obtain the same result as in Theorem 5.1, if either $\gamma$ or $\rho$ is* 0.

**Theorem 5.3.** *The neutral differential equation given by equation (27) for which $\beta \neq 0, \gamma = -\rho, \kappa \neq -1$, admits a four dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_5^i = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} + \left[\psi - \frac{2\beta x\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta, \gamma, \rho, \kappa$ be arbitrary non-zero constants, $\gamma = -\rho, \kappa \neq -1$. Then from equation (30), we get,

$$(1+\kappa)\omega_{ttt} + \beta\omega_{tt} = 0, \tag{40}$$

and $\theta_{tt} + \beta\psi_t + \gamma(\theta - \psi) + \kappa\psi_{tt} = 0$.
Solving equation (40) we get,

$$\omega = c_6 t + c_7 - \frac{1+\kappa}{2\beta}A, \tag{41}$$

where $c_4, c_5$ are arbitrary constants and $c_6 = \frac{c_4}{\beta}, c_7 = \frac{c_5}{\beta}$. From equation (41),

$$A(t) = c_8 t + c_9 - \frac{2\beta}{A}\omega,$$

where $c_8 = \frac{2c_1}{1+\kappa}, \quad c_9 = \frac{2c_2}{1+\kappa}$. This yields,

$$\Upsilon = \left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right)x + \theta(t),$$

and,

$$\Upsilon^r = \left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right)x + \psi(t-r).$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_6 t + c_7 - \frac{1+\kappa}{2\beta}A\right)\frac{\partial}{\partial t} + \left[\left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right)x + \theta\right]\frac{\partial}{\partial x}\\ &+ \left[\left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right)x + \psi\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t}$, $S_2 = \frac{\partial}{\partial t}$, $S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$, $S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$ with $S_5 = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} + \left[\psi - \frac{2\beta x\omega}{1+\kappa}\right]\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by,

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
|---|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | $S_3$ | 0 |
| $S_2$ | $S_2$ | 0 | $S_4$ | 0 |
| $S_3$ | $-S_3$ | $-S_4$ | 0 | 0 |
| $S_4$ | 0 | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3, S_4\}$ is a solvable Lie algebra. □

**Theorem 5.4.** *The neutral differential equation given by equation (27) for which $\beta \neq 0, \gamma = -\rho, \kappa = -1$, admits a three dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_4^i = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta, \gamma, \rho, \kappa$ be arbitrary non-zero constants, $\gamma = -\rho, \kappa = -1$. Then equation (30) becomes $\Upsilon_{tt} + \beta\Upsilon_t^r + \gamma(\Upsilon - \Upsilon^r) - \Upsilon_{tt}^r = 0$, which yields,

$$A_{tt} = 0, \tag{42}$$

and $\theta_{tt} + \beta\psi_t + \gamma(\theta - \psi) - \psi_{tt} = 0$.
Solving equation (42) we get,

$$\omega = c_{10}t + c_{11}, \tag{43}$$

where $c_{10}, c_{11}$ are arbitrary constants. From equation (43),

$$A(t) = c_{12},$$

where $c_{12} = 2c_{10}$.
This yields,

$$\Upsilon = c_{12}x + \theta(t),$$

and,

$$\Upsilon^r = c_{12}x + \psi.$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{10}t + c_{11})\frac{\partial}{\partial t} + (c_{12}x + \theta)\frac{\partial}{\partial x} + (c_{12}x + \psi)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by
$S_1 = t\frac{\partial}{\partial t}$, $S_2 = \frac{\partial}{\partial t}$, $S_3 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$ and
$S_4 = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ is the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | 0 |
| $S_2$ | $S_2$ | 0 | 0 |
| $S_3$ | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

**Theorem 5.5.** *The neutral differential equation given by equation (27) for which $\beta \neq 0, \gamma = 0 = \rho, \kappa \neq 0$, admits a five dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$$

$$S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u},$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_6^i = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} - \left[\frac{\kappa}{\beta}\psi_t + \frac{1}{\beta} + \frac{2\beta\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta, \kappa$ be arbitrary non-zero constants, $\gamma = 0 = \rho$.
Then equation (30) becomes $\Upsilon_{tt} + \beta\Upsilon_t^r + \kappa\Upsilon_{tt}^r = 0$, which yields,

$$(1+\kappa)A_{tt} + \beta A_t = 0, \tag{44}$$

and $\psi = c_{13} - \frac{\kappa}{\beta}\psi_t - \frac{1}{\beta}\theta_t$, where $c_{13}$ is an arbitrary constant.
Solving equation (44) we get,

$$\omega = c_{16}t + c_{17} - \frac{1+\kappa}{2\beta}A, \tag{45}$$

where $c_{14}, c_{15}$ are arbitrary constants and $c_{16} = \frac{c_{14}}{\beta}, \quad c_{17} = \frac{c_{15}}{\beta}$. From equation (45),

$$A(t) = c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega,$$

where $c_{18} = \frac{2c_{16}\beta}{1+\kappa}$ and $c_{19} = \frac{2c_{17}\beta}{1+\kappa}$. This yields,

$$\Upsilon = \left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + \theta,$$

and,

$$\Upsilon^r = \left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + c_{13} - \frac{\kappa}{\beta}\psi_t - \frac{1}{\beta}\theta_t.$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_{16}t + c_{17} - \frac{1+\kappa}{2\beta}A\right)\frac{\partial}{\partial t} + \left[\left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + \theta\right]\frac{\partial}{\partial x}\\ &+ \left[\left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + c_{13} - \frac{\kappa}{\beta}\psi_t - \frac{1}{\beta}\theta_t\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}$,
$S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u}$ with

$$S_6 = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} - \left[\frac{\kappa}{\beta}\psi_t + \frac{1}{\beta} + \frac{2\beta\omega}{1+\kappa}\right]\frac{\partial}{\partial u}$$

as the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | $S_3$ | 0 | 0 |
| $S_2$ | $S_2$ | 0 | $S_4$ | 0 | 0 |
| $S_3$ | $-S_3$ | $-S_4$ | 0 | 0 | 0 |
| $S_4$ | 0 | 0 | 0 | 0 | 0 |
| $S_5$ | 0 | 0 | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3, S_4, S_5\}$ is a solvable Lie algebra. □

**Theorem 5.6.** *The neutral differential equation given by equation (27) for which $\beta = 1, \kappa = 1$, admits a three dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right] \quad S_3 = \frac{\partial}{\partial u},$$

*with the infinite dimensional Lie sub-algebra given by*

$$S_4^i = -A\frac{\partial}{\partial t} + [\theta - \omega x]\frac{\partial}{\partial x} - [\omega x + (\theta_t + \psi_t)]\frac{\partial}{\partial u}.$$

**Proof.** Let $\beta = 1 = \kappa$, $\gamma = -\rho, \kappa \neq -1$. Then equation (30) becomes $\Upsilon_{tt} + \Upsilon_t^r + \Upsilon_{tt}^r = 0$, which yields,

$$2A_{tt} + A_t = 0, \tag{46}$$

and $\psi = c_{20} - (\theta_t + \psi_t)$, where $c_{20}$ is an arbitrary constant.
Solving equation (46) we get,

$$\omega = c_{21}t + c_{22} - A, \tag{47}$$

where $c_{21}, c_{22}$ are arbitrary constants. From equation (47),

$$A(t) = c_{21}t + c_{22} - \omega.$$

This yields,

$$\Upsilon = (c_{21}t + c_{22} - \omega)x + \theta,$$

and,

$$\Upsilon^r = (c_{21}t + c_{22} - \omega)x + c_{20} - (\theta_t + \psi_t).$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{21}t + c_{22} - A)\frac{\partial}{\partial t} + [(c_{21}t + c_{22} - \omega)x + \theta]\frac{\partial}{\partial x}\\ &+ [(c_{21}t + c_{22} - \omega)x + c_{20} - (\theta_t + \psi_t)]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$,
$S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$, $\quad S_3 = \frac{\partial}{\partial u}$ with
$S_4 = -A\frac{\partial}{\partial t} + [\theta - \omega x]\frac{\partial}{\partial x} - [\omega x + (\theta_t + \psi_t)]\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | 0 |
| $S_2$ | $S_2$ | 0 | 0 |
| $S_3$ | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

## 5.2 A Nonlinear Case

We make a classification of

$$x''(t) + x''(t-r) + x'(t-r) + x'(t)x(t) = 0. \tag{48}$$

Applying the operator defined by equation (27), to the delay equation $g(t) = t - r$, we get equation $\omega(t, x) = \omega(t - r, x(t - r))$.
Applying the operator defined by equation (27), to equation (48), we get,

$$\begin{aligned} &\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' \\ &-3\omega_x x'x'' + \Upsilon^r_{tt} + (2\Upsilon^r_{tx} - \omega^r_{tt})x^{r\prime} + (\Upsilon^r_{xx} - 2\omega^r_{tx})x^{r\prime 2} - \omega^r_{xx}x^{r\prime 3} + (\Upsilon^r_x - 2\omega^r_t)x^{r\prime\prime} \\ &- 3\omega^r_x x^{r\prime}x^{r\prime\prime} + \Upsilon^r_t + (\Upsilon^r_x - \omega^r_t)x^{r\prime} - \omega^r_x x^{r\prime 2} + x'\Upsilon + x[\Upsilon_t + (\Upsilon_x - \omega_t)x' \\ &- \omega_x x'^2] = 0. \end{aligned} \tag{49}$$

Splitting equation (49) with respect to constant term, $x'$, $x'^2$, $x'^3$, $x''$, $x'x''$, $x^{r\prime}$, $x^{r\prime 2}$, $x^{r\prime 3}$, $x^{r\prime\prime}$ and $x^{r\prime}x^{r\prime\prime}$ respectively, we get,

$$\Upsilon_{tt} + \Upsilon^r_{tt} + \Upsilon^r_t + x\Upsilon_t = 0, \tag{50}$$

and,

$$\begin{aligned} &2\Upsilon_{tx} - \omega_{tt} + \Upsilon = 0,\ \Upsilon_{xx} - 2\omega_{tx} = 0, \\ &\omega_{xx} = 0,\ \Upsilon_x - 2\omega_t = 0,\ \omega_x = 0, \\ &2\Upsilon^r_{tx} - \omega^r_{tt} + \Upsilon^r_x - \omega^r_t = 0,\ \Upsilon^r_{xx} - 2\omega^r_{tx} - \omega^r_x = 0, \\ &\omega^r_{xx} = 0,\ \Upsilon^r_x - 2\omega^r_t = 0,\ \omega^r_x = 0. \end{aligned}$$

From these equations we get, $\omega = \omega(t)$, $\Upsilon = A(t)x + \theta(t)$, $\Upsilon^r = \dfrac{1}{2}A(t)x + \psi(t - r)$, where $A(t) = 2\omega_t$.
Substituting the values of $\Upsilon, \Upsilon^r$ in equation (50) and solving it, we get, $A(t) = c_{23}$, $\theta = c_{24}$, $\psi = c_{25} - \psi_t$, and,

$$\omega = c_{26}t + c_{27},$$

$$\Upsilon = c_{23}x + c_{24}, \quad \Upsilon^r = c_{23}x + c_{25} - \psi_t,$$

where $c_{23}, c_{24}, c_{25}, c_{27}$ are arbitrary constants and $c_{26} = \frac{c_{23}}{2}$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{26}t + c_{27})\frac{\partial}{\partial t} + (c_{23}x + c_{24})\frac{\partial}{\partial x} + (c_{23}x + c_{25} - \psi_t)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by
$S_1 = t\frac{\partial}{\partial t}$, $S_2 = \frac{\partial}{\partial t}$, $S_3 = x\Big[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\Big]$,
$S_4 = \frac{\partial}{\partial x}$, $S_5 = \frac{\partial}{\partial u}$ and $S_6 = -\psi_t\frac{\partial}{\partial x}$ is the infinite dimensional Lie sub-algebra.

The commutator table is given by

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $S_1$ | 0 | $-S_2$ | 0 | 0 | 0 |
| $S_2$ | $S_2$ | 0 | 0 | 0 | 0 |
| $S_3$ | 0 | 0 | 0 | $-S_4 - S_5$ | 0 |
| $S_4$ | 0 | 0 | $S_4 + S_5$ | 0 | 0 |
| $S_5$ | 0 | 0 | 0 | 0 | 0 |

.

Then $L = \{S_1, S_2, S_3, S_4, S_5\}$ is a solvable Lie algebra.

**Remark 5.7.** For the non-homogeneous nonlinear second order neutral differential equation $x''(t) + x''(t-r) + x'(t-r) + x'(t)x(t) = h(t)$, we get exactly the same generators as in the homogeneous case, only that $S_6 = (y - \psi_t)\frac{\partial}{\partial x^r}$ is the corresponding infinite dimensional Lie sub-algebra, where $y = c_{26}\int t\, h' dt + c_{27}h$.

## 6 Conclusion

With the notation $L_n^m$, where $m$ denotes the dimension of the solvable Lie algebra and $S^i$ to mean the infinite dimensional Lie sub-algebra, the entire classification of second order functional differential equations with constant coefficients to solvable Lie algebras is summarized below:

**Table 1:** Group Classification of Second Order Differential Equations

| Type of Functional Differential Equation | Basis for the Lie Algebra | Solvable Lie algebra |
|---|---|---|
| $x''(t)+\beta x'(t-r)+\gamma x(t)+\rho x(t-r)=0,$ $\gamma\neq\frac{\rho}{2}.$ | $S_1=\frac{\partial}{\partial t},\quad S_2=x\frac{\partial}{\partial x},\quad S_3=x\frac{\partial}{\partial u},$ $S_4^i=-\left(\frac{1}{2(\gamma+\frac{\rho}{2})}+\frac{4(\gamma+\frac{\rho}{2})}{\beta}A\right)\frac{\partial}{\partial t}+\left[\theta-x\left(\frac{2}{\beta}A_t+\frac{4(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial x}+\left[\psi-x\left(\frac{1}{\beta}A_t+\frac{2(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$ | $L_1^3$ |
| $x''(t)+\beta x'(t-r)+\frac{\rho}{2}x(t)+\rho x(t-r)=0.$ | $S_1=t\frac{\partial}{\partial t},\quad S_2=\frac{\partial}{\partial t},$ $S_3=tx\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_4=x\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_5^i=-\frac{A}{\beta}\frac{\partial}{\partial t}+(\theta-\beta x\omega)\frac{\partial}{\partial x}+(\psi-\frac{\beta x\omega}{4})\frac{\partial}{\partial u}.$ | $L_2^4$ |
| $x''(t)+x'(t-r)=0.$ | $S_1=t\frac{\partial}{\partial t}+tx\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_2=\frac{\partial}{\partial t}+x\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],\quad S_3=\frac{\partial}{\partial u},$ $S_4^i=-A\frac{\partial}{\partial t}+(\theta-x\omega)\frac{\partial}{\partial x}-\left(\theta_t+\frac{x\omega}{2}\right)\frac{\partial}{\partial u}.$ | $L_3^3$ |

**Table 2:** Group Classification of Second Order Differential Equations

| Type of Functional Differential Equation | Basis for the Lie Algebra | Solvable Lie algebra |
|---|---|---|
| $x''(t) + \beta x'(t-r) = 0.$ | $S_1 = t\frac{\partial}{\partial t}, S_2 = \frac{\partial}{\partial t}, S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u},$ $S_6^i = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + \left(\frac{\theta_t}{\beta} + \beta x\omega\right)\frac{\partial}{\partial u}.$ | $L_4^5$ |
| $x''(t) + x'(t) + x'(t-r)x(t) = v(t).$ | $S_1 = x\frac{\partial}{\partial x}, \quad S_2 = \frac{\partial}{\partial x},$ $S_3 = t\frac{\partial}{\partial t}, \quad S_4 = \frac{\partial}{\partial t},$ $S_5 = x\frac{\partial}{\partial u}, \quad S_6 = \frac{\partial}{\partial u},$ $S_7^i = (\theta_t - y)\frac{\partial}{\partial x}.$ | $L_5^6$ |
| $x''(t)+\beta x'(t-r)+\gamma x(t)+\rho x(t-r)+\kappa x''(t-r) = 0,$ $\gamma \neq -\rho.$ | $S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$ $S_3^i = -\left(\frac{1+\kappa}{2(\gamma+\rho)}g_t + \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial x} + \left[\psi - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$ | $L_6^2$ |
| $x''(t)+\beta x'(t-r)+\gamma(x(t)-x(t-r))+\kappa x''(t-r) = 0,$ $\kappa \neq -1.$ | $S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t},$ $S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$ $S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$ $S_5^i = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} + \left[\psi - \frac{2\beta x\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$ | $L_7^4$ |

**Table 3:** Group Classification of Second Order Differential Equations

| Type of Functional Differential Equation | Basis for the Lie Algebra | Solvable Lie algebra |
|---|---|---|
| $x''(t)+\beta x'(t-r)+\gamma(x(t)-x(t-r))-x''(t-r)=0.$ | $S_1=t\frac{\partial}{\partial t}, S_2=\frac{\partial}{\partial t}, S_3=x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_4^i=\theta\frac{\partial}{\partial x}+\psi\frac{\partial}{\partial u}.$ | $L_8^3$ |
| $x''(t)+\beta x'(t-r)+\kappa x''(t-r)=0.$ | $S_1=t\frac{\partial}{\partial t}, S_2=\frac{\partial}{\partial t}, S_3=tx\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_4=x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right], \quad S_5=\frac{\partial}{\partial u},$ $S_6^i=-\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t}+\left[\theta-\frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} - \left[\frac{\kappa}{\beta}\psi_t+\frac{1}{\beta}+\frac{2\beta\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$ | $L_9^5$ |
| $x''(t)+x'(t-r)+x''(t-r)=0.$ | $S_1=t\frac{\partial}{\partial t}+tx\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_2=\frac{\partial}{\partial t}+x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right], \quad S_3=\frac{\partial}{\partial u},$ $S_4^i=-A\frac{\partial}{\partial t}+[\theta-\omega x]\frac{\partial}{\partial x}-[\omega x+(\theta_t+\psi_t)]\frac{\partial}{\partial u}.$ | $L_{10}^3$ |
| $x''(t)+x'(t)+x'(t-r)x(t)=v(t).$ | $S_1=t\frac{\partial}{\partial t}, \quad S_2=\frac{\partial}{\partial t}, \quad S_3=\frac{\partial}{\partial x},$ $S_4=x\Big[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\Big], \quad S_5=\frac{\partial}{\partial u},$ $S_6^i=(z-\psi_t)\frac{\partial}{\partial u}.$ | $L_{11}^5$ |

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**Jervin Zen Lobo**
Department of Mathematics
Assistant Professor of Mathematics
St. Xavier's College, Mapusa - Goa
PIN: 403507
India
E-mail: zenlobo1990@gmail.com

**Y. S. Valaulikar**
Department of Mathematics
Associate Professor of Mathematics
Goa University, Taleigao Plateau
PIN: 403206
India
E-mail: ysv@unigoa.ac.in, ysvgoa@gmail.com