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Iterative Approach for a Class of Fuzzy Volterra Integral Equations Using Block Pulse Functions

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Abstract. Fuzzy Integral equations is a mathematical tool for modeling the uncertain control system and economic. In this paper, we present numerical solution of nonlinear fuzzy Volterra integral equations(NFVIEs) using successive approximations scheme and block-pulse functions. Additionally, the convergence analysis of the presented approach is investigated involving Lipschitz and several conditions and error bound between the approximate and the exact solution is provided. Finally, to approve the outcomes concerned with the theory a numerical experiment is considered.

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1 Introduction

Fuzzy integral equations is a significant tool for modeling and analysis of uncertain real-world problems in various fields of science and engineering. The topic of fuzzy integral equations was first studied by Kaleva (see [21]), Seikkala (see [31]), they converted the initial value problem for first order fuzzy differential equations to the fuzzy Volterra integral equation. In this regards, the distinct work on the fuzzy integral equations is carried out by Mordeson and Newman (see [22]) based on Zadeh's extension principle. Afterwards, Diamond in [12] studied the NFVIE with applications in optimal control theory. The existence and uniqueness results of the solution of fuzzy Volterra integral equations have been investigated by many researchers using Banach fixed-point theorem (see [5, 6, 16, 27, 29]). In [24], are provided conditions for having the bounded solutions of fuzzy integral equations. In the literature, we can find different approaches for solving fuzzy integral equation such as Picard's approximations scheme and quadrature rules which are exploited in [2, 7, 8, 9, 10, 14, 16, 17, 36, 37] and [26]. In several papers are used direct method for solving fuzzy integral equations such as the Adomian decomposition technique (see [3]), Nystrom method ([1]), Bernstein polynomials and Haar wavelets (see [13, 35]). In some papers to solution of these equations are applied successive approximations scheme with interpolation of Lagrange(see [15]), finite differences(see [25]) and block-pulse functions (see [37, 40]). Bica and Popescu, in [10] presented an iterative scheme by using successive approximation scheme and Trapezoidal quadrature formula for solving NFVIEs. An iterative numerical procedure has been presented recently in [39] which is solved NFVIE via three-point quadrature rule.

In the present study, we provide an iterative procedure to the NFVIE of the following form:

$$y(t) = g(t) \oplus (FR)\int_a^t f(t,s,y(s))ds, \qquad t \in [a,b] \tag{1}$$

where $f : [a,b] \times [a,b] \times \mathbb{R}_F \to \mathbb{R}_F$ and $g : [a,b] \to \mathbb{R}_F$ are fuzzy continuous. The purpose of the present work is to present an iterative numerical scheme to gain the approximate solution of (1) based on block pulse functions.

2 Preliminaries

Herein, the essential notions of fuzzy mathematics employed in the subsequent sections will be stated.

Definition 2.1. (see [33].) A fuzzy number is a function u from $\mathbb{R}$ to $[0,1]$ satisfying the following conditions:
(i) u is normal,
(ii) u is fuzzy convex set,
(iii) u is upper semi-continuous on $\mathbb{R}$,
(iv) $\overline{supp(u)}$ is a compact subset of $\mathbb{R}$, where $supp(u) = \{ x \in \mathbb{R} : u(x) > 0\}$ is the support of u and $\overline{supp(u)}$ denotes the closure of $supp(u)$.

We denote by $\mathbb{R}_F$ the set of all fuzzy numbers. For $0 < r \leq 1$, the $r-$level sets of u is defined as $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$.Then, we represent the parametric form of u as:

$$[u]^r = [\underline{u}(r), \overline{u}(r)], \quad 0 < r \leq 1,$$

where $\underline{u}(r), \overline{u}(r)$ are two real functions $\underline{u}, \overline{u} : [0,1] \to \mathbb{R}$, that $\underline{u}$ is increasing and $\overline{u}$ is decreasing (see [19]). Furthermore, $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$. So, the $r-$cuts of a fuzzy number $[u]^r$, $r \in [0,1]$, are compact subset of $\mathbb{R}$. For $u_1, u_2 \in \mathbb{R}_F$, $k \in \mathbb{R}$, the addition and the scalar multiplication based on levelsetwise are defined as follows

$$[u_1 + u_2]^r = [\underline{u}_1(r) + \underline{u}_2(r), \overline{u}_1(r) + \overline{u}_2(r)], \forall 0 \leq r \leq 1$$

$$[k \odot u_1]^r = \begin{cases} [k\underline{u}_1(r), k\overline{u}_1(r)], & \text{if } \ k \geq 0 \\ [k\overline{u}_1(r), k\underline{u}_1(r)], & \text{if } \ k < 0. \end{cases}$$

Definition 2.2. (see [18].) Suppose that $u_1, u_2 \in \mathbb{R}_F$ the value $D(u_1, u_2) = \sup\limits_{r \in [0,1]} \max\{|\underline{u}_1(r) - \underline{u}_2(r)|, |\overline{u}_1(r) - \overline{u}_2(r)| \}$ is the distance between u_1 and u_2.

One can notice that $(\mathbb{R}_F, D)$ is a complete metric space.

Theorem 2.3. *([18].) For any $u_1,\ u_2,\ u_3,\ u \in \mathbb{R}_F,\ k_1 \geq 0,\ k_2 \geq 0,\ k \in \mathbb{R}$, the following properties hold:*
(i) $D(u_1 \oplus u_3, u_2 \oplus u_3) = D(u_1, u_2)$,
(ii) $D(k \odot u_1, k \odot u_2) = |k|\, D(u_1, u_2)$,

(iii) $D(u_1,u_2) \leq D(u_1,u_3) + D(u_3,u_2)$,
(iv) $D(u_1 \oplus u_2, u_3) \leq D(u_1,u_3) + D(u_2,u_3)$,
(v) $D(k_1 \cdot u, k_2 \cdot u) = |k_1 - k_2| D\left(u, \widetilde{0}\right)$.

Definition 2.4. (see [21]). Let $f : [a,b] \to \mathbb{R}_F$:
(i) f is fuzzy continuous at $x_0 \in [a,b]$, if $\forall\, \varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $D(f(x), f(x_0)) < \varepsilon$.
(ii) f is bounded iff there exists $M > 0$ such that $D\left(f(x), \widetilde{0}\right) \leq M$, $\forall\, x \in [a,b]$.

We note that f is fuzzy continuous on $[a,b]$ if f is continuous at each $x_0 \in [a,b]$, and the space of all such functions is denoted by $C([a,b],\mathbb{R}_F)$ and any fuzzy continuous function is bounded. For any fuzzy-valued function (or fuzzy mapping) $f : [a,b] \to \mathbb{R}_F$, the left and right $r-$level functions of f are specified respectively as $\underline{f}(\cdot,r)$, $\overline{f}(\cdot,r) : [a,b] \to \mathbb{R}$, $r \in [0,1]$.

The uniform distance between $f,\ g \in C([a,b],\mathbb{R}_F)$ is denoted by the supremum metric $D^*(f,g) = \sup\limits_{a \leq x \leq b} D(f(x),g(x))$ on the set $C([a,b],\mathbb{R}_F)$. The space $C([a,b],\mathbb{R}_F)$ equipped with the uniform distance is a complete metric space.

Definition 2.5. (see [18].) Let $f : [a,b] \to \mathbb{R}_F$ be a fuzzy-valued function. For every partition $P = \{[x_{i-1},x_i] :\ i = \overline{1,n}\}$ of $[a,b$ and arbitrary point $\xi_i \in [x_{i-1},x_i]$, $i = \overline{1,n}$, we say that f is fuzzy-Riemann integrable to $I_f \in \mathbb{R}_F$ if for $\forall\, \varepsilon > 0$, there exists $\delta > 0$ such that $\max\{|x_i - x_{i-1}| : i = \overline{1,n}\} < \delta$ implies that

$$D\left(\sum_{i=1}^{n} (x_i - x_{i-1}) \odot f(\xi_i), I_f\right) < \varepsilon.$$

The quantity I_f is written as

$$I_f = (FR)\int_a^b f(x)dx.$$

If $f \in C([a,b],\mathbb{R}_F)$, then fuzzy-Riemann integral of f exists, and

$$\left[(FR)\int_a^b f(x)\,dx\right]^r = \left[\int_a^b \underline{f}(x,r)\,dx, \int_a^b \overline{f}(x,r)\,dx\right],\quad \forall\, 0 \leq r \leq 1.$$

Lemma 2.6. *(see [18]). If $f, g \in C([a,b], \mathbb{R}_F)$, then the function $F : [a,b] \to \mathbb{R}_+$ given by $F(x) = D(f(x), g(x))$ is continuous on $[a,b]$, and*

$$D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx.$$

Definition 2.7. (see [7].) Let $f : [a,b] \to \mathbb{R}_F$ is L-Lipschitz. Then

$$D(f(x_1), f(x_2)) \leq L|x_1 - x_2|, \quad \forall\ x_1, x_2 \in [a,b].$$

A function $F : \mathbb{R}_F \to \mathbb{R}_F$ is L-Lipschitz if there exists $L_F \geq 0$ such that $D(F(u_1), F(u_2)) \leq L_F D(u_1, u_2)$, $\forall\ u_1, u_2 \in \mathbb{R}_F$.

Definition 2.8. (see [20]). Block-pulse functions on the unit interval $[a,b)$ is defined as follows:

$$\varphi_i(t) = \begin{cases} 1 & t \in [a + (i-1)h, a + ih), \\ 0 & otherwise, \end{cases}$$

where $i = \overline{1,n}$ with a positive integer value for n and $h = \frac{b-a}{n}$. Also, φ_i is called i^{th} block-pulse function (BPF).

Some properties of the BPFs are disjointness, orthogonality and completeness (see [20]).
It is clear that $\varphi_i(t) \geq 0$, for all $t \in [a,b)$, $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ are linearly independent, and

$$\sum_{i=1}^{n} \varphi_i(t) = 1.$$

Now, we employ the following notation:

$$B_i(t) = \int_0^t \varphi_i(\tau)d\tau, \quad t \in [a,b]. \tag{2}$$

By using the definition of block-pulse function the integral given in Eq. (2) can be computed as follows:

$$B_i(t) = \begin{cases} 0 & t \in [a, a + (i-1)h), \\ t - ih & t \in [a + (i-1)h, a + ih), \\ h & t \in [a + ih, b) \end{cases}$$

where $h=\frac{b-a}{n}$. Now, we can evaluate the approximate value of the fuzzy integral of $f\in C([a,b],\mathbb{R}_F)$ which satisfy in Lipschitz condition from a to $t=t_i$, based on block-pulse functions as follows:

$$(FR)\int_a^{t_i} f(s)ds \simeq I(f) = (FR)\int_a^{t_i}\sum_{j=1}^{n} f(z_j)\odot\varphi_j(s)ds,$$

where $z_j=\frac{t_{j-1}+t_j}{2}$ for $j=\overline{1,n}$. The error estimate of this approximation can be obtained as:

$$\begin{aligned}
&D\left((FR)\int_a^{t_i} f(s)ds, I(f)\right)=\\
&=D\left((FR)\int_a^{t_i}\sum_{j=1}^{n}\varphi_j(s)\odot f(s)ds,(FR)\int_a^{t_i}\sum_{j=1}^{n} f(z_j)\odot\varphi_j(s)ds\right)\\
&\leq\int_a^{t_i}\sum_{j=1}^{n}\varphi_j(s)D(f(s),f(z_j))ds\leq\int_a^{t_i}\sum_{j=1}^{n}\varphi_j(s)L|s-z_j|ds.
\end{aligned}$$

By similar way in [40], we compute the above integral as follows:

$$\begin{aligned}
&\int_a^{t_i}\sum_{j=1}^{n}\phi_j(s)\left(L\left|s-z_j\right|\right)ds=\\
&=L\sum_{j=1}^{n}\int_a^{t_i}\phi_j(s)\left|s-z_j\right|ds=L\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left|s-t_{j-1}\right|ds\\
&=L\sum_{j=1}^{n}\left(\int_{t_{j-1}}^{z_j}\left(z_j-s\right)ds+\int_{z_j}^{t_j}\left(s-z_j\right)ds\right)=\frac{L(b-a)^2}{4n}.
\end{aligned}$$

Finally, we have:

$$D\left((FR)\int_a^{t_i} f(s)ds,(FR)\int_a^{t_i}\sum_{j=1}^{n} f(z_j)\odot\varphi_j(s)ds\right)\leq\frac{L(b-a)^2}{4n}.$$

3 Successive approximations method

In this section, problem (1) will be studied according to the following hypotheses:

(i) $g \in C([a,b],\mathbb{R}_F)$ and $f \in C\left([a,b]\times[a,b]\times\mathbb{R}_F,\mathbb{R}_F\right)$;
(ii) there exist α and $\eta > 0$ such that

$$D\left(f\left(t,s_1,u_1\right),f\left(t,s_2,u_2\right)\right) \le \eta|s_1-s_2|+\alpha D\left(u_1,u_2\right),\ \forall\ t,s_1,s_2\in[a,b],\ u_1,u_2\in\mathbb{R}_F;$$

(iii) $\gamma = \alpha(b-a) < 1$;
(iv) there exists $\beta > 0$ such that

$$D(g(t_1),g(t_2)) \le \beta|t_1-t_2|,\ \forall\ t_1,t_2\in[a,b].$$

Lemma 3.1. *Suppose that the following assumptions are satisfied:*

(i) $f \in C\left([a,b]\times[a,b]\times\mathbb{R}_F,\mathbb{R}_F\right)$,
(ii) there exists $\mu > 0$ *such that*

$$D\left(f\left(t_1,s,u\right),f\left(t_2,s,u\right)\right) \le \mu|t_1-t_2|,\ \forall\ t_1,t_2,s\in[a,b],\ u\in\mathbb{R}_F,$$

then the function $F:[a,b]\to\mathbb{R}_F$ *formulated by*

$$F(t) = (FR)\int_a^t f(t,s,y(s))ds,$$

is uniformly continuous.

Proof. By the hypothesis (i), there exist constants $M_y,\ M_f > 0$ such that $D\left(y(s),\tilde{0}\right) \le M_y,\ D\left(f(t,s,u),\tilde{0}\right) \le M_f$ for all $t,s\in[a,b],\ u\in\mathbb{R}_F$. Taking $t_1,t_2\in[a,b]$ with $a\le t_1\le t_2\le b$, we have: and regarding to hypothesis (ii) we have:

$$\begin{aligned}
D(F(t_1),F(t_2)) &= D\left((FR)\int_a^{t_1} f(t_1,s,y(s))ds,(FR)\int_a^{t_2} f(t_2,s,y(s))ds\right)\\
&\le D\left((FR)\int_a^{t_1} f(t_1,s,y(s))ds,(FR)\int_a^{t_1} f(t_2,s,y(s))ds\right)\\
&\quad + D\left((FR)\int_{t_1}^{t_2} f(t_2,s,y(s))ds,\tilde{0}\right)\\
&\le \int_a^{t_1} D\left(f(t_1,s,y(s)),f(t_2,s,y(s))\right)ds\\
&\quad + \int_{t_1}^{t_2} D\left(f(t_2,s,y(s)),\tilde{0}\right)ds.
\end{aligned}$$

According to hypothesis (ii), we obtain:

$$D(F(t_1), F(t_2)) \leq \int_a^{t_1} \mu|t_1-t_2|ds+\int_{t_1}^{t_2} M_f ds \leq \mu(b-a)|t_1-t_2|+M_f(t_2-t_1).$$

For any $\epsilon > 0$, there exists δ,

$$0 < \delta \leq \min\left(1, \frac{\epsilon}{\mu(b-a)+M_f}\right)$$

such that $|t_1 - t_2| < \delta$ concludes that

$$D(F(t_1), F(t_2)) \leq (\mu(b-a)+M_f)\,\delta \leq \epsilon.$$

This proves that F is uniformly continuous. $\square$

To express the existence and uniqueness of Eq. (1), we consider the associated sequence of successive approximations:

$$y_0(t) = g(t),$$
$$y_m(t) = g(t) \oplus (FR)\int_a^t f(t, s, y_{m-1}(s))ds, \quad t \in [a, b],\ m \in \mathbb{N}$$

and we construct the sequence $(F_m(t))_{m\in\mathbb{N}}$ such that $F_m : [a, b]\times[a, b] \to \mathbb{R}_F$, by $F_m(t, s) = f(t, s, y_m(s))$, $m \in \mathbb{N}$.

Theorem 3.2. *(see [10].) Considering hypotheses $(i)-(iii)$, the equation (1) has unique solution in $C([a, b], \mathbb{R}_F), y^* \in C([a, b], \mathbb{R}_F)$ and sequence of Picard's approximations $(y_m)_{m\in\mathbb{N}} \subset C([a, b], \mathbb{R}_F)$, depicted in (3) tends to y^* in $C([a, b], \mathbb{R}_F)$ for any election of $y_0 \in C([a, b], \mathbb{R}_F)$. Furthermore, the below error bounds satisfied:*

$$D(y^*(t), y_m(t)) \leq \frac{\gamma^m}{1-\gamma}.D(y_1(t), y_0(t), \quad \forall t \in [a, b], m \in \mathbb{N}^*, \tag{3}$$

and

$$D(y^*(t), y_m(t)) \leq \frac{\gamma}{1-\gamma}.D(y_m(t), y_{m-1}(t), \quad \forall t \in [a, b], m \in \mathbb{N}^*.$$

Choosing $y_0 \in C([a, b], \mathbb{R}_F)$, $y_0 = g$ the inequality (3) converts

$$D(y^*(t), y_m(t)) \leq \frac{\gamma^m}{1-\gamma}M_0(b-a), \quad \forall t \in [a, b], m \in \mathbb{N}^*, \tag{4}$$

where $M_0 \geq 0$ is such that $D(f(t, s, g(s)), \tilde{0}) \leq M_0$, $\forall\ t, s \in [a, b]$, Furthermore, conditions (i)-(iv) provide uniformly bounded and uniform Lipschitz of the sequences $(y_m)_{m\in\mathbb{N}^*}$ and $(F_m)_{m\in\mathbb{N}^*}$ respectively.

4 Description of the method and error analysis

Now, we present the iterative method using block-pulse functions to approximate the solution of (1). Taking into account the uniform partition of the interval $[a, b]$,

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b,$$

with $t_i = a + \frac{i(b-a)}{n}$, $\forall\, i = \overline{0, n}$. For $i = \overline{0, n}$, we have:

$$\begin{aligned} &y_0(t_i) = g(t_i), \\ &y_m(t_i) = g(t_i) \oplus (FR) \int_a^{t_i} f(t, s, y_{m-1}(s))ds, \quad m \in \mathbb{N}. \end{aligned} \tag{5}$$

Now, we approximate the integral in (5) as follows:

$$(FR) \int_a^{t_i} f(t, s, y_{m-1}(s))ds \simeq (FR) \int_a^{t_i} \sum_{j=1}^{n} f(t_i, z_j, y_{m-1}(z_j)) \odot \varphi_j(s)ds,$$

where $z_j = \frac{t_{j-1}+t_j}{2}$ for $j = \overline{1, n}$.
Then, the following approximations are gained:

$$y_m(t_i) = g(t_i) \oplus \sum_{j=1}^{n} f(t_i, z_j, y_{m-1}(z_j)) \odot B_j(t_i) \oplus R_{m,i}, \tag{6}$$

where

$$B_j(t_i) = \int_a^{t_i} \varphi_j(s)ds \quad \text{and} \quad D(R_{m,i}, \tilde{0}) \leq \frac{L(b-a)^2}{4n}. \tag{7}$$

In this way, we get the following iterative procedure:

$$\begin{aligned} &\tilde{y}_0(t_i) = y_0(t_i) = g(t_i), \\ &\tilde{y}_m(t_i) = g(t_i) \oplus \sum_{j=1}^{n} f(t_i, z_j, \tilde{y}_{m-1}(z_j)) \odot B_j(t_i). \end{aligned} \tag{8}$$

The following theorem proves the convergence of the presented method.

Theorem 4.1. *By the hypothesis (i)-(iv), the iterative scheme (8) converges to the exact solution y^* of (1), and the following error bound is true on the knots $t_i = \frac{i(b-a)}{n}$, $i = \overline{0,n}$:*

$$D(y^*(t_i), \tilde{y}_m(t_i)) \leq \frac{\gamma^m}{1-\gamma} M_0(b-a) + \frac{L(b-a)^2}{4n(1-\gamma)}.$$

Proof. According to Eq. (4), we know:

$$D(y^*(t_i), y_m(t_i)) \leq \frac{\gamma^m}{1-\gamma} M_0(b-a), \quad \forall i = \overline{0,n}, \ m \in \mathbb{N}^*. \tag{9}$$

Since

$$D(y^*(t_i), \tilde{y}_m(t_i)) \leq D(y^*(t_i), y_m(t_i)) + D(y_m(t_i), \tilde{y}_m(t_i)), \tag{10}$$

so, we try to gain the estimates for $D(y_m(t_i), \tilde{y}_m(t_i))$, $\forall\, i = \overline{0,n},\ m \in \mathbb{N}^*$. Since

$$y_1(t_i) = g(t_i) \oplus \sum_{j=1}^{n} f(t_i, z_j, g(z_j)) \oplus R_{1,i}, \ i = \overline{0,n},$$

Hence

$$\begin{aligned}
&D(y_1(t_i), \tilde{y}_1(t_i)) \leq D(g(t_i), g(t_i)) \\
&+ D\left(\sum_{j=1}^{n} f(t_i, z_j, g(z_j)) \odot B_j(t_i) \oplus R_{1,i}, \sum_{j=1}^{n} f(t_i, z_j, g(z_j)) \odot B_j(t_i)\right) \\
&\leq D\left(R_{1,i}, \tilde{0}\right) \leq \frac{L(b-a)^2}{4n}.
\end{aligned}$$

Now, using Eqs. (6)-(8) we get:

$$\begin{aligned}
&D(y_2(t_i),\tilde{x}_2(t_i)) \leq D(g(t_i),g(t_i))+\\
&+D\left(\sum_{j=1}^{n} f(t_i,z_j,y_1(t_j))\odot B_j(t_i)\oplus R_{2,i},\sum_{j=1}^{n} f(t_i,z_j,\tilde{y}_1(t_j))\odot B_j(t_i)\right)\\
&=D\left(\sum_{j=1}^{n} f(t_i,z_j,y_1(t_j))\odot\int_a^{t_i}\varphi_j(s)ds\oplus R_{2,i},\sum_{j=1}^{n} f(t_i,z_j,\tilde{y}_1(z_j))\odot\int_a^{t_i}\varphi_j(s)ds\right)\\
&\leq\int_a^{t_i}\sum_{j=1}^{n}\varphi_j(s)D\left(f(t_i,z_j,y_1(z_j)),f(t_i,z_j,\tilde{y}_1(z_j))\right)ds+D\left(R_{2,i},\tilde{0}\right)\\
&\leq\int_a^{t_i}\sum_{j=1}^{n}\varphi_j(s)\alpha D\left(y_1(z_j),\tilde{y}_1(z_j)\right)ds+D\left(R_{2,i},\tilde{0}\right)\\
&\leq\frac{\alpha L(b-a)^3}{4n}+\frac{L(b-a)^2}{4n}=(1+\gamma)\frac{L(b-a)^2}{4n}.
\end{aligned}$$

By mathematical induction, for $m\in\mathbb{N}^*,\ m\geq 3$, we get:

$$\begin{aligned}
D(y_m(t_i),\tilde{y}_m(t_i)) &\leq \left(1+\gamma\cdots+\gamma^{m-1}\right)\frac{L(b-a)^2}{4n}\\
&=\left(\frac{1-\gamma^m}{1-\gamma}\right)\frac{L(b-a)^2}{4n}.
\end{aligned}$$

By considering condition (iii), we get:

$$D(y_m(t_i),\tilde{y}_m(t_i))\leq\frac{L(b-a)^2}{4n(1-\gamma)},\quad\forall\ i=\overline{0,n},\ m\in\mathbb{N}^*. \tag{11}$$

So, the assertion follows from (9), (10) and (11). $\square$

5 Numerical experiment

To establish the theoretical results in the previous section, a numerical example has been demonstrated by iterative procedure (8). The results have been provided by Mathematica.

Example 5.1. Let the NFVIE

$$y(t) = g(t) \oplus (FR) \int_0^t te^{-s} \odot [y(s)]^2 ds, \quad s \leq t, \quad t \in [0,1]$$

with

$$\underline{g}\,(t,r) = rt - tr^2 + r^2 e^{-t}\left(\frac{t^3}{2} + t^2 + t\right),$$

$$\overline{g}\,(t,r) = (2-r)t + (2-r)^2\left(\frac{t^3 e^{-t}}{2} + t^2 e^{-t} + te^{-t} - t\right).$$

The exact solution is as follows

$$\left[\underline{y}(t,r), \overline{y}(t,r)\right] = \left[rt, (2-r)t\right].$$

To solve the above stated equation, we apply the iterative scheme (8) with $n = 10$ and $m = 5$, to determine r-levels of the results $\underline{e}(r) = |\underline{\tilde{y}_m}(t,r) - \underline{y}(t,r)|$, $\overline{e}(r) = |\overline{\tilde{y}_m}(t,r) - \overline{y}(t,r)|$, at point $t = 0.5$, for $r \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ are put in Table 1. To peruse the convergence of the presented method is considered $n = 100$ and $m = 5$ too and the obtained results are given in Table 1.

Table 1: Error on the r-level sets for Example 5.1 at $t = 0.5$

	$n = 10, \quad m = 5$		$n = 100, \quad m = 5$	
r-level	$\underline{e}(r)$	$\overline{e}(r)$	$\underline{e}(r)$	$\overline{e}(r)$
0.25	1.252e-004	6.392e-003	1.196e-005	6.031e-004
0.50	5.042e-004	4.664e-003	4.808e-005	4.410e-004
0.75	1.142e-003	3.216e-003	1.087e-004	3.048e-004
1	2.045e-003	2.045e-003	1.941e-004	1.941e-004

According to Table 1, the convergence of presented method is confirmed.

6 Conclusions

The current paper presented an iterative numerical scheme via block-pulse functions to solve NFVIEs. The proposed method is computationally efficient with respect to the other existing methods. Moreover, We have given the error estimate of the iterative technique by taking into account the Lipschitz condition. The numerical results confirmed the validity of the results in theoretical aspect.

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