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Pseudo-Derivative and Pseudo-Integral of Fractional Order

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Abstract. Integral and differential of fractional order are important notion that is often used in dealing with Frechet geometry. Based on pseudo-operations given by monotone and continuous function g, we study pseudo-derivative and pseudo-integral of fractional order through this paper.

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1. Introduction

Motivation for the research presented here lies both in capability of the pseudo-analysis, generalization of the classical analysis, to extend the range of the possible applications. The pseudo-analysis is based, instead of the usual field of real numbers, on a semiring acting on the real interval $[a, b] \subset [-\infty, +\infty]$, denoting the corresponding operations as $\oplus$ (pseudo-addition) and $\odot$ (pseudo-multiplication) of the following form:

$$x \oplus y = g^{-1}(g(x) + g(y)), \quad x \odot y = g^{-1}(g(x)g(y)),$$

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where g is a strictly monotone and continuous generating function ([1, 7, 9, 11, 13, 14]).
In recent years fractional calculus has received much interest due to its new applications ([2, 3, 4, 8]). A recent set of applications in mechanical engineering and an electrical engineering control theory approach may be found in ([5, 6]). The fractional calculus is a field of mathematics study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. In this paper, in order to broaden the area of possible applications of fractional derivative and fractional integral, we shall give some notions and theorems from g-calculus which are analogous to the classical theorems of the usual calculus of fractional derivative and fractional integral.

2. Preliminaries

In this section we collect the necessary definitions and basic notions.
Let $[a,b]$ be a closed subinterval of $[-\infty,+\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be a total order on $[a,b]$. A semiring is the structure $([a,b],\oplus,\odot)$ when the following holds:

- $\oplus$ is pseudo-addition, i.e., a function $\oplus : [a,b]\times[a,b] \longrightarrow [a,b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by 0;
- $\oplus$ is pseudo-multiplication, i.e., a function $\odot : [a,b]\times[a,b] \longrightarrow [a,b]$ which is commutative, positively non-decreasing ($x \preceq y$ implies $x\odot z \preceq y\odot z$,$z \in [a,b]_+ = \{x : x \in [a,b], 0 \preceq x\}$), associative and for which there exists a unit element denoted by 1;
- $0 \odot x = 0$
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$

Definition 2.1. ([13]) *The pseudo-operations are defined by a monotone and continuous function* $g : [a,b] \longrightarrow [0,\infty]$*, i.e., pseudo-operations are* *given with*

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad , \quad x \odot y = g^{-1}(g(x)g(y)). \tag{1}$$

Theorem 2.2. ([7]) *Let m be a sup-measure on $([0,\infty], B([0,\infty]))$, where $B([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A) = esssup_\mu(\psi(x)|x \in A)$, and $\psi(x) : [0,\infty] \longrightarrow [0,\infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0,\infty), B)$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0,\infty)$, such that $lim_{\lambda\longrightarrow\infty} m_\lambda = m$.*

Remark 2.3. *Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [7]. For any continuous function $f : [0,\infty] \longrightarrow [0,\infty]$ the integral $\int^\oplus f \odot m$ can be obtained as a limit of g-integrals ([7]).*

Proposition 2.4. *For any pseudo-addition $\oplus$ with a generator g there exists $\oplus_\lambda$ and $\odot_\lambda$ which generated by g^λ (the function g of the power λ), such that*

$$x \oplus_\lambda y = (g^\lambda)^{-1}(g^\lambda(x) + g^\lambda(y)) \quad , \quad x \odot_\lambda y = (g^\lambda)^{-1}(g^\lambda(x)g^\lambda(y)), \tag{2}$$

For more details see ([7, 12]).

Definition 2.5. ([4])*Let function f be defined on the interval $[c,d]$ and with values in $[a,b]$. If f is differentiable on (c,d), and has same monotonicity as the function g then we define the g-derivative of f at the point $x \in (c,d)$ as*

$$\frac{d^\oplus f(x)}{dx} = g^{-1}\Big(\frac{d}{dx} g(f(x))\Big). \tag{3}$$

Definition 2.6. ([2]) *Let g be a generating function defined on the interval $[a,b]$ and suppose that $\oplus$ and $\odot$ are pseudo-operations given by (1). The pseudo-integral for a function $f : [c,d] \longrightarrow [a,b]$ reduces on the g-integral as follows*

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1} \int_c^d g(f(x))dx. \tag{4}$$

Theorem 2.7. ([2])*Suppose that f is continuous on $[c,d]$. Then we have*

$$\frac{d^{\oplus}}{dx}\Big(\int_{[c,x]}^{\oplus} f dx\Big) = f(x), \tag{5}$$

for each $x \in (c,d)$.

Definition 2.8. *The gamma function is a continuous extension to the factorial function, which is only defined for the nonnegative integers. The gamma function is defined by*

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad forall \quad x \in R. \tag{6}$$

While there are other continuous extensions to the factorial function, the gamma function is the only one that is convex for positive real numbers ([1, 2, 4, 5]). We have

$$\Gamma(1) = \int_0^{\infty} e^{-t}dt = -e^{-t}]_0^{\infty} = 0 - (-1) = 1.$$

Also, we can see that $\Gamma(2) = \Gamma(1) = 1$.
The beauty of the gamma function can be found in its properties. For example, this function is unique in that the value of any quantity x. Also we have:

$$\Gamma(x+1) = x\Gamma(x) \quad x > 0, \tag{7}$$

and

$$\Gamma(x+1) = x! \quad or \quad \Gamma(x) = (x-1)!; \quad x > 0, \tag{8}$$

Example 2.9. For non integer value we have

$$\begin{aligned}(\frac{7}{2})! &= \Gamma(\frac{7}{2}+1) = \frac{7}{2}\Gamma(\frac{7}{2}) = \frac{7}{2} \times \frac{5}{2}\Gamma(\frac{5}{2}) \\ &= \cdots = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{105}{16}\Gamma(\frac{1}{2})\end{aligned}$$

and

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} t^{-1/2}e^{-t}dt = 2\int_0^{\infty} e^{-u^2}du = \sqrt{\pi}.$$

Hence, we have

$$(\frac{7}{2})! = \frac{105}{16}\Gamma(\frac{1}{2}) = \frac{105\sqrt{\pi}}{16},$$

see ([4,5]).

Now we can define derivative and integral from fractional order.

3. Fractional Derivative and Fractional Integral

Assuming a function $f(x)$ defined on $(0, \infty)$. The definite integral of $f(x)$ from 0 to x, is denoted by:

$$(If)(x) = \int_0^x f(t)dt.$$

Repeating this process gives:

$$(I^2 f)(x) = \int_0^x \Big(\int_0^t f(s)ds \Big) dt,$$

and this can be extended arbitrarily.
The Cauchy formula for repeated integration, namely

$$(I^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt, \tag{9}$$

leads to a straightforward way to a generalization for real n.
Simply using the Gamma function to remove the discrete nature of the factorial function (recalling that $\Gamma(n+1) = n!$)) gives us a natural candidate for fractional applications of the integral operator.

$$(I^a f)(x) = \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t)dt, \qquad a > 0. \tag{10}$$

This is in fact a well-defined operator.
Let us assume that $f(x)$ is a monomial of the form $f(x) = x^k$. We have

$$(If)(x) = \frac{x^{k+1}}{k+1},$$

$$(I^2 f)(x) = \frac{x^{k+2}}{(k+1)(k+2)},$$
$$\vdots$$
$$(I^n f)(x) = \frac{k!}{(n+k)!} x^{n+k}.$$

After replacing the factorials with the Gamma function, We lead to

$$(I^a f)(x) = \frac{\Gamma(k+1)}{\Gamma(a+k+1)} x^{a+k}, \tag{11}$$

At the same way, for fractional derivative of function f we have

$$\frac{d^a}{dx^a} x^k = \frac{\Gamma(k+1)}{\Gamma(k-a+1)} x^{k-a} \qquad x > 0 \quad , \quad a > 0. \tag{12}$$

For more details see ([4, 6]).
Another important issue is relation between fractional derivative and fractional integral where we have used the fundamental theorem of the usual calculus.

Theorem 3.1. ([2])*Let $a > 0$, there exists an integer $p > a$ such that*

$$\frac{d^a}{dx^a} f(x) = \frac{d^p}{dx^p} (I^{p-a} f)(x). \tag{13}$$

Example 3.2. For $k = 1$ and $a = 1/2$, we obtain the half-derivative of the function $f(x) = x$, where $p = 1$ as

$$\begin{aligned}
\frac{d^{1/2}}{dx^{1/2}} x = \frac{d}{dx}(I^{1-1/2} x) = \frac{d}{dx}(I^{1/2} x) &= \frac{d}{dx}\Big(\frac{\Gamma(1+1)}{\Gamma(1/2+1+1)} x^{1/2+1}\Big) \\
&= \frac{d}{dx}\Big(\frac{\Gamma(2)}{\Gamma(5/2)} x^{3/2}\Big) \\
&= \frac{d}{dx}\Big(\frac{1}{\frac{3}{2} \times \frac{1}{2}\Gamma(\frac{1}{2})} x^{3/2}\Big) \\
&= \frac{d}{dx}\Big(\frac{4}{3\sqrt{\pi}} x^{3/2}\Big) \\
&= \frac{4}{3\sqrt{\pi}} \times \frac{3}{2} x^{1/2} = \frac{2\sqrt{x}}{\sqrt{\pi}}.
\end{aligned}$$

4. Main Results

Now we define pseudo-derivative of fractional order as generalization of both fractional order and pseudo-derivative.

4.1 Pseudo-derivative of fractional order

Definition 4.1.1. *Let the function $f(x) = x^k$ be defined on the interval $[c, d]$ and with values in $[a, b]$. If f is differentiable on (c, d), and has same monotonicity as the function g then we define the fractional order of pseudo-derivative for function f at the point $x \in (c, d)$ as*

$$\frac{d^{(a)\oplus} f(x)}{dx^a} = g^{-1}\Big(\frac{d^a}{dx^a} g(f(x))\Big), \tag{14}$$

where $x > 0$ and $a > 0$.

Example 4.1.2. Taking $g(x) = \sqrt{x}$, for function $f(x) = x$, using the ralation (5) of (3) for $p = 1$, the pseudo half-derivative (means $a = \frac{1}{2}$) is given by

$$\begin{aligned}
\frac{d^{(1/2)\oplus}}{dx^{1/2}} x &= g^{-1}\Big(\frac{d^{1/2}}{dx^{1/2}} g(x)\Big) = g^{-1}\Big(\frac{d^{1/2}}{dx^{1/2}} x^{1/2}\Big) \\
&= g^{-1} \frac{d}{dx}(I^{(1-1/2)} x^{1/2}) \\
&= g^{-1}\Big(\frac{d}{dx}(\frac{\Gamma(1/2+1)}{\Gamma(1/2+1/2+1)} x^{1/2+1/2})\Big) \\
&= g^{-1}\Big(\frac{d}{dx}(\frac{\Gamma(3/2)}{\Gamma(2)} x)\Big) \\
&= g^{-1}\Big(\frac{d}{dx}(\frac{\sqrt{\pi}}{2} x)\Big) \\
&= g^{-1}(\frac{\sqrt{\pi}}{2}) = (\frac{\sqrt{\pi}}{2})^2 \\
&= \frac{\pi}{4}.
\end{aligned}$$

4.2 Pseudo-integral of fractional order

Definition 4.2.1. *Let g be a generating function and $\oplus$ and $\odot$ pseudo-operations given by (1) of (2). The pseudo-integral of fractional order for a function $f : [c, d] \longrightarrow [a, b]$ reduces on the fractional g-integral as follows*

$$(I^{(a)\oplus} f)(x) = g^{-1}\Big((I^a g(f))(x)\Big), \tag{15}$$

where $f(x) = x^k$, $x > 0$ and $a > 0$.

Example 4.2.2. Let $g(x) = \sqrt{x}$ be continuous and strictly monotone function and $f(x) = x$ be the identity function. Then $g^{-1}(x) = x^2$ and for pseudo integration of order $a = \frac{1}{2}$ we have

$$\begin{aligned}
(I^{\oplus(a)} f)(x) &= g^{-1}\Big((I^a g(f))(x)\Big) \\
&= g^{-1}\Big((I^{1/2}(x^{1/2}))\Big) \\
&= g^{-1}\Big(\frac{\Gamma(k+1)}{\Gamma(a+k+1)} x^{a+k}\Big) \\
&= g^{-1}\Big(\frac{\Gamma(3/2)}{\Gamma(2)} x\Big) = g^{-1}\Big(\Gamma(3/2)x\Big) \\
&= g^{-1}\Big(1/2\Gamma(1/2)x\Big) = g^{-1}\Big(\frac{\sqrt{\pi}}{2} x\Big) \\
&= \Big(\frac{\sqrt{\pi}}{2} x\Big)^2,
\end{aligned}$$

where $a = \frac{1}{2}$, $\quad k = \frac{1}{2}$ and $p = 1$.
Following theorem analogous to the classical theorem of the usual calculus.

Theorem 4.2.3. *If generating function g is monotone bijection, then*

$$\Gamma^{\oplus}(x \oplus 1) = x \odot \Gamma^{\oplus}(x). \tag{16}$$

Proof. We have

$$\begin{aligned}\Gamma^{\oplus}(x\oplus 1) &= \Gamma^{\oplus}\Big(g^{-1}(g(x)+g(1))\Big)\\ &= g^{-1}\Big(\Gamma(g(g^{-1}(g(x)+g(1))))\Big)\\ &= g^{-1}\Big(\Gamma(g(x)+g(1))\Big)\\ &= g^{-1}\Big(\Gamma(\sqrt{x}+1)\Big).\end{aligned}$$

On the other hand we have

$$\begin{aligned}x\odot\Gamma^{\oplus}(x) &= g^{-1}\Big(g(x)g(\Gamma^{\oplus}(x)\Big)\\ &= g^{-1}\Big(g(x)g(g^{-1}(\Gamma(g(x))))\Big)\\ &= g^{-1}\Big(\sqrt{x}\Gamma(\sqrt{x})\Big).\end{aligned}$$

Now, from (7) we have

$$\Gamma(\sqrt{x}+1)=\sqrt{x}\Gamma(\sqrt{x}).$$

Hence

$$g^{-1}\Big(\Gamma(\sqrt{x}+1)\Big)=g^{-1}\Big(\sqrt{x}\Gamma(\sqrt{x})\Big),$$

which completes the proof. □

Theorem 4.2.4. *Let $g(u)=u^k$ be a strictly monotone and continuous function and suppose that $\oplus_{1/k}$ and $\odot_{1/k}$ be operations satisfying (2)of(2). Then for $g^{-1}(u)=u^{1/k}$ and $f(x)=x$ we have*

$$(I^{\oplus_{1/k}(a)}f)(x)=\frac{\Gamma^{\oplus_{1/k}}(k\oplus_{1/k}1)}{\Gamma^{\oplus}(a\oplus_{1/k}k\oplus_{1/k}1)}\odot_{1/k}x^{(a\oplus_{1/k}k)}. \qquad (17)$$

Proof. Note that

$$(I^{\oplus_{1/k}(a)}f)(x)=g^{-1/k}\Big(I^a x^k\Big)=g^{-1/k}\Big(\frac{\Gamma(k+1)}{\Gamma(a+k+1)}x^{a+k}\Big).$$

On the other hand we have

$$\frac{\Gamma^{\oplus_{1/k}}(k\oplus_{1/k}1)}{\Gamma^{\oplus_{1/k}}(a\oplus_{1/a}k\oplus_{1/k}1)}\odot_{1/k}x^{(a\oplus_{1/k}k)}$$
$$=\frac{\Gamma^{\oplus_{1/k}}\Big(g^{-1/k}(g^{1/k}(k)+g(1))\Big)}{\Gamma^{\oplus_{1/k}}\Big(g^{-1/k}(g^{1/a}(a)+g^{1/k}(k)+g(1)\Big)}\odot_{1/k}x^{\Big(g^{-1/k}(g^{1/k}(k)+g^{1/a}(a))\Big)}$$
$$=\frac{g^{-1/k}\Big(\Gamma(g^{1/k}(k)+g(1))\Big)}{g^{-1/k}\Big(\Gamma(g^{1/a}(a)+g^{1/k}(k)+g(1)\Big)}\odot_{1/k}x^{g^{-1/k}(a+k)}$$
$$=\frac{g^{-1/k}\Big(\Gamma(k+1)\Big)}{g^{-1/k}\Big(\Gamma(a+k+1)\Big)}\odot_{1/k}x^{g^{-1/k}(a+k)}$$
$$=g^{-1/k}\Big(\frac{(\Gamma(k+1)}{(\Gamma(a+k+1)\Big)}g^{1/k}(x^{g^{-1/k}(a+k)})\Big)$$
$$=g^{-1/k}\Big(\frac{\Gamma(k+1)}{\Gamma(a+k+1)}x^{a+k}\Big),$$

which proves the claim.
One of the aim of this paper, is to provide estimation for pseudo-analysis, counterpart of relation $(5) of (3)$. □

Theorem 4.2.5. *Suppose that f is continuous on $[c,d]$ and differentiable on (c,d). For every $a>0$, and integer value $p>a$, we have*

$$\frac{d^{(a)\oplus}}{dx^a}f(x)=\frac{d^{(p)\oplus}}{dx^p}(I^{(p-a)\oplus}f)(x). \tag{18}$$

Proof. We note that

$$\begin{aligned}\frac{d^{(p)\oplus}}{dx^p}(I^{(p-a)\oplus}f)(x)&=\frac{d^{(p)\oplus}}{dx^p}\Big(g^{-1}(I^{(p-a)}g(f(x)))\Big)\\&=g^{-1}\Big(\frac{d^p}{dx^p}g(g^{-1}(I^{(p-a)}g(f(x)))\Big)\\&=g^{-1}\Big(\frac{d^p}{dx^p}(I^{(p-a)}g(f(x))\Big)\\&=g^{-1}\Big(\frac{d^a}{dx^a}g(f(x))\Big)\\&=\frac{d^{(a)\oplus}}{dx^a}f(x),\end{aligned}$$

this completes the proof. □

Example 4.2.6. Let $g(x) = \sqrt{x}$, for $f(x) = x$, pseudo half-derivative of f is given as

$$\begin{aligned}
\frac{d^{(1/2)\oplus}}{dx^{1/2}} x = \frac{d^{\oplus}}{dx}(I^{(1-1/2)\oplus} x) &= \frac{d^{\oplus}}{dx}\Big(g^{-1}(I^{1/2} g(x))\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}(I^{1/2} x^{1/2})\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}(\frac{\Gamma(1/2+1)}{\Gamma(1/2+1/2+1)} x^{1/2+1/2})\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}(\frac{\Gamma(3/2)}{\Gamma(2)} x)\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}(\frac{\sqrt{\pi}}{2} x)\Big) \\
&= g^{-1}\Big(\frac{d}{dx} g(\frac{\sqrt{\pi}}{2} x)^2\Big) \\
&= g^{-1}\Big(\frac{d}{dx}(\frac{\sqrt{\pi}}{2} x)\Big) \\
&= g^{-1}(\frac{\sqrt{\pi}}{2}) = (\frac{\sqrt{\pi}}{2})^2 \\
&= \frac{\pi}{4},
\end{aligned}$$

where $p = 1$, $k = \frac{1}{2}$ and $a = \frac{1}{2}$.

Example 4.2.7. Let $g(x) = \sqrt{x}$. For $f(x) = x^{-1}$, the pseudo half-derivative of f is given as

$$\begin{aligned}
\frac{d^{(1/2)\oplus}}{dx^{1/2}} x^{-1} &= \frac{d^{\oplus}}{dx}(I^{(1-1/2)\oplus} x^{-1}) = \frac{d^{\oplus}}{dx}\Big(g^{-1}(I^{1/2} g(x^{-1}))\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}(I^{1/2} x^{-1/2})\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}\Big(\frac{\Gamma(-1/2+1)}{\Gamma(-1/2+1/2+1)} x^{-1/2+1/2}\Big)\Big) \\
&= \frac{d^{\oplus}}{dx}\Big(g^{-1}\Big(\frac{\Gamma(1/2)}{\Gamma(1)}\Big)\Big) \\
&= g^{-1}\Big(\frac{d}{dx} g(g^{-1}(\sqrt{\pi}))\Big) \\
&= g^{-1}(\frac{d}{dx}\sqrt{\pi}) = g^{-1}(0) = 0,
\end{aligned}$$

where $p = 1$, $k = \frac{-1}{2}$ and $a = \frac{1}{2}$.

5. Conclusion

In this paper, we have introduced pseudo-derivative and pseudo-integral of fractional order. Using pseudo-analysis, the relations between the integral of fractional order and pseudo-integral of fractional order was shown. We plan to work further on equations of fractional order in the pseudo-analysis framework.

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