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Identifying Exact Pairs of Zero-divisors from Zero-divisor Graphs of Commutative Rings

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Abstract. We provide criteria for identifying exact pairs of zero-divisors from zero-divisor graphs of commutative rings, and extend these criteria to compressed zero-divisor graphs. Finally, our results are translated as constructions for exact zero-divisor subgraphs.

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1 Introduction

Let R be a commutative, Noetherian ring with identity and let $x \in R$. If there exists $y \in R$ such that $\operatorname{ann}(x) = yR$ and $\operatorname{ann}(y) = xR$ then x is an *exact zero-divisor*; x and y are an *exact pair of zero-divisors.* Beginning with [8], exact zero-divisors have found applications to commutative and homological algebra; see [5] and [10], for example. Interest in these elements is partly due to the ascent and descent of properties between R and R/xR; the complete intersection, Gorenstein, Cohen-Macaulay, and Koszul properties, for example.

On the other hand, zero-divisor graph of a ring, denoted by $\Gamma(R)$, have also generated a large collection of literature; see [4] and [7], for example. The vertex set of the zero-divisor graph of R consists of nonzero

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zero-divisors with edges between vertices if and only if their product is zero.

Our results provide criteria for identifying exact pairs of zero-divisors from zero-divisor graphs. In many cases, a quick inspection of $\Gamma(R)$ is sufficient to recognize at least some of the exact pairs of zero-divisors. For some classes of rings, the zero-divisor graph provides enough information to ascertain the complete list of exact pairs of zero-divisors. Central to our development is the following property: We say that R has *principal annihilator symmetry* (PAS) if, for any pair of elements $x, y \in R$ with the property that $\operatorname{ann}(x) = yR$, then it follows that $\operatorname{ann}(y) = xR$. For example, Artinian rings have this property. The following results are given in Sections 2 and 3.

Theorem. *Let R have principal annihilator symmetry, with vertices x, y, and z.*

1. (a) *If x is adjacent to a leaf y then x and y are an exact pair of zero-divisors. In addition, if x is also adjacent to a non-leaf z, then x and z do not form an exact pair of zero-divisors.*
   (b) *If x is adjacent to exactly two distinct vertices y and z, then x and y are an exact pair of zero-divisors (as are x and z).*
2. *If R is Artinian Gorenstein then every exact pair of zero-divisors can be read from $\Gamma(R)$ using the following property: x and y are an exact pair of zero-divisors if and only if $\operatorname{ann}(x) = \operatorname{ann}(\operatorname{ann}(y))$.*

Zero-divisor graphs can be infinite and, in some cases, too crowded to be clearly displayed. It is helpful to pass to the *compressed* zero-divisor graph, introduced in [14], where the vertices are classes of zero-divisors that have equal annihilators. One particular advantage of the compressed zero-divisor graph is that the associated primes are embedded within and, in some cases, can be read directly off of the graph. Given the compressed zero-divisor graph, we would like to provide criteria for identifying exact pairs of zero-divisors. In general, we cannot achieve this goal: it is possible to have exact and non-exact zero-divisors in the same class. However, this obstruction does not arise for some classes of rings. In Section 4, we extend the theorem above to the compressed zero-divisor graph of Artinian Gorenstein rings.

The exact pairs of zero-divisors of a ring induce a natural subgraph of the zero-divisor graph. Our results translate easily as tools for constructing these graphs. However, constructing subgraphs of compressed zero-divisor graphs that display exact zero-divisors is more nuanced. We explore this topic further in Section 5.

2 Preliminaries

In this section we review zero-divisor graphs and exact zero-divisors. We also define principal annihilator symmetry, a property that will play a reoccurring role throughout.

Let R be a commutative Noetherian ring with identity throughout. Let $Z(R)$ be the set of zero-divisors of R and $Z(R)^*$ be the set of nonzero zero-divisors. For a graph G, let $V(G)$ denote the set of vertices in G and $E(G)$ denote the set of edges in G. If $a, b \in V(G)$, $a - b$ denotes an edge between a and b in $E(G)$. These elements are called adjacent in G. (Note, that we include loops as edges.) If a has only one adjacent vertex b, then a is a *leaf* and we say that b has a leaf a. The *zero-divisor graph of R*, denoted $\Gamma(R)$, has the vertex set $V(\Gamma(R)) = Z(R)^*$ with $a - b \in E(\Gamma(R))$ if and only if $ab = 0$ where $a, b \in Z(R)^*$.

Example 2.1. The zero-divisor graph of $\mathbb{Z}_{18}$ is given in Figure 1.

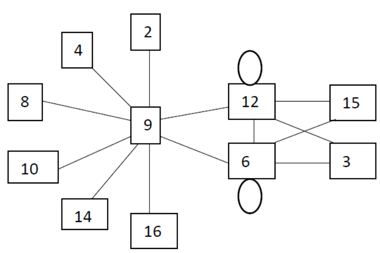

Figure 1: $\Gamma(\mathbb{Z}_{18})$

Example 2.2. Let $R = \mathbb{Z}_2[X, Y]/(X^2, Y^2, XY)$ and let x and y denote the images of X and Y, respectively. Figure 2 illustrates $\Gamma(R)$.

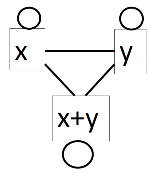

Figure 2: $\Gamma(\mathbb{Z}_2[X,Y]/(X^2,Y^2,XY))$

Exact zero divisors were first introduced in [9]. They appeared later in [13] under the name *morphic* from a noncommutative context. More recently, their application to homological properties of commutative rings were first considered in [8].

Definition 2.3. *An element $x \in R$ is an exact zero-divisor if and only if there exists $y \in R$ such that $\operatorname{ann}(x) = yR$ and $\operatorname{ann}(y) = xR$, in which case x and y are an exact pair of zero-divisors. If every zero-divisor of R is an exact zero-divisor, then R is an exact zero-divisor ring.*

Remark 2.4. Equivalent characterizations of an exact zero-divisor x were given in [8] and [13]. For example, x is an exact zero-divisor if and only if $R/xR \cong \operatorname{ann}(x)$. Exact zero-divisors have proved useful in commutative and homological algebra. Their utility is partly due to the transferability of properties between R and R/xR. For example, the Gorenstein, complete intersection, Cohen-Macaulay, or Koszul [8] properties hold in R if and only if they hold in R/xR, respectively.

Example 2.5. Let $R = \mathbb{Z}_2[X,Y]/(X^2,Y^2,XY)$ as in Example 2.1 and let x and y denote the images of X and Y, respectively. No zero-divisors in R are exact zero-divisors. Indeed, $\operatorname{ann}(x) = \operatorname{ann}(y) = \operatorname{ann}(x+y) = (x,y)$.

Example 2.6. Example 2 in [13] shows that the direct products of exact zero-divisor rings is again an exact zero-divisor ring.

Both conditions in Definition 2.3 are required. That is, if $\operatorname{ann}(x) = yR$, it is not necessarily true that $\operatorname{ann}(y) = xR$:

Example 2.7. Let $R = \mathbb{Z}[X, Y]/(X^3, XY)$ and let x and y denote the images of X and Y, respectively. Then $\operatorname{ann}(y) = xR$. However, $\operatorname{ann}(x) = (x^2, y)R$

Definition 2.8. *We say that the ring R is principal annihilator symmetric (PAS) if for all $x \in R$ such that $\operatorname{ann}(x) = yR$ for some $y \in R$, we have that $\operatorname{ann}(y) = xR$.*

Remark 2.9. Artinian rings are PAS rings, as follows from the more general [2, Lemma 3.2].

The next proposition follows immediately from the definitions.

Proposition 2.10. *Every principal ideal PAS ring is an exact zero-divisor ring. Conversely, exact zero-divisor rings are PAS rings.*

Example 2.11. From Example 12 in [13], we have that $\mathbb{Z}_n$ is an exact zero-divisor ring for all $n \geq 2$. Hence, $\mathbb{Z}_n$ is PAS from Proposition 2.10.

Example 2.12. Let $R = k[x]/(x^n)$ where k is a field and $n \geq 2$. Since R is principal Artinian, it is also a PAS ring. Thus R is an exact zero-divisor ring.

3 Identifying exact zero-divisors from graphs of rings with principal annihilator symmetry

In this section we provide criteria for identifying exact zero-divisors when given a zero-divisor graph of a PAS ring. In particular, the results can be applied to finite graphs; from [1, Theorem 2.2], we have that $\Gamma(R) \neq \emptyset$ and finite implies R is finite, and thus PAS by Remark 2.9.

Proposition 3.1. *Let R be a PAS ring and $x, y \in V(\Gamma(R))$ such that x is a leaf to y. Then x and y are an exact pair of zero-divisors.*

Proof. Since x is a leaf, we may set $\operatorname{ann}(x) = \{0, y\}$ where $y \in V(\Gamma(R))$ is nonzero. Since $y \in \operatorname{ann}(x)$, we have that $yR \subseteq \operatorname{ann}(x)$. Clearly, $\operatorname{ann}(x) \subseteq yR$. Hence, $yR = \operatorname{ann}(x)$. Since R is PAS, it follows that x and y are an exact pair of zero-divisors. $\square$

Example 3.2. If $\Gamma(R)$ is a star graph, then the central node with any of the leaves forms an exact pair of zero-divisors.

Proposition 3.3. *Let $a, b, c, d \in V(\Gamma(R))$, with $a - b$, $c - d \in E(\Gamma(R))$, and $a - d \notin E(\Gamma(R))$. Then b and c do not form an exact pair of zero-divisors.*

Proof. For the sake of contradiction, suppose that $\operatorname{ann}(b) = cR$. Then $a \in cR$ since $a - b \in E(\Gamma(R))$. Hence, $\operatorname{ann}(c) \subseteq \operatorname{ann}(a)$. Since c-$- d \in E(\Gamma(R))$, we have that $d \in \operatorname{ann}(a)$, a contradiction. $\square$

Corollary 3.4. *Let $y \in V(\Gamma(R))$ be adjacent to a leaf and to a non-leaf. Then y and the adjacent non-leaf do not form an exact pair of zero-divisors.*

Lemma 3.5. *Let $x, y, z \in R$ be distinct and nonzero. If $\operatorname{ann}(x) = \{0, y, z\}$, then $\operatorname{ann}(x) = yR = zR$.*

Proof. We show that $\operatorname{ann}(x) = yR$. Clearly, $yR \subseteq \operatorname{ann}(x)$. We show $\operatorname{ann}(x) \subseteq yR$. Obviously, 0 and y are in yR. It remains to show $z \in yR$. We have that $x(y-z) = 0$. Thus, $y-z = 0$ or $y-z = y$ or $y-z = z$. The first two cases contradict the hypothesis. Hence, $y - z = z$. A similar argument shows that $z - y = y$. Adding these two equations shows that $y + z = 0$. Since $z = -y$, we have that $z \in yR$. $\square$

Corollary 3.6. *Let R be PAS and x, y and z be distinct vertices in $\Gamma(R)$. If y and z are the only vertices adjacent to x then x and y are an exact pair of zero-divisors (in which case x and z are also an exact pair).*

Example 3.7. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$ and let x denote the image of X. Figure 3 illustrates $\Gamma(R)$. The complete list of exact pairs of zero-divisors can be read from the $\Gamma(R)$ using Proposition 3.3 and Corollary 3.6. For example, Corollary 3.6 implies that $(0, 1 + x)$ and $(1, 0)$ are an exact pair of zero-divisors, while Proposition 3.3 implies that $(1, 0)$ and $(0, x)$ do not form an exact pair. The other vertices can be checked similarly.

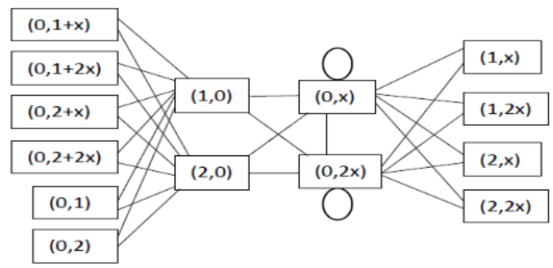

Figure 3: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2))$

4 Artinian Gorenstein and exact zero-divisor rings

In this section, we provide a characterization of the exact zero-divisor property for Artinian Gorenstein rings and exact zero-divisor rings that enables one to identify all of the exact pairs of zero-divisors for these rings from their zero-divisor graph.

Remark 4.1. An Artinian ring R is Gorenstein if and only if $\operatorname{ann}(\operatorname{ann}(I)) = I$ for every ideal I of R; see [6, Exercise 3.2.15], for example.

Remark 4.2. Let R be an exact zero-divisor ring. Then for any nonzero zero-divisor $y \in R$, we have that $yR = \operatorname{ann}(\operatorname{ann}(y))$.

Theorem 4.3. *Let R be an Artinian Gorenstein ring or an exact zero-divisor ring. Let x and y be nonzero zero-divisors in R. Then x and y are an exact pair of zero-divisors if and only if* $\operatorname{ann}(x) = \operatorname{ann}(\operatorname{ann}(y))$.

Proof. Sufficiency follows easily from the definitions. Assume $\operatorname{ann}(x) = \operatorname{ann}(\operatorname{ann}(y))$. By Remarks 4.1 and 4.2 we have that $yR = \operatorname{ann}(\operatorname{ann}(y))$. Hence, $yR = \operatorname{ann}(x)$. Since these rings are PAS, the conclusion follows. $\square$

Note, Theorem 4.3 recovers the results from Section 2 for Artinian Gorenstein and exact zero-divisor rings.

Since $\Gamma(R)$ displays all the annhilators of the ring, finite $\Gamma(R)$ is sufficient for identifying the exact pairs of zero-divisors for Artinian Gorenstein and exact zero-divisor rings.

Example 4.4. Let $R = \mathbb{Z}_{18}$. The zero-divisor graph is given in Example 2.1. To find the exact zero-divisors, choose any two adjacent vertices, read off their annihilators from the graph, and determine if they meet the criteria of Theorem 4.3. For example, to determine if 9 and 12 are an exact pair of zero-divisors, let $X = \{3, 6, 9, 12, 15\}$ be the vertices adjacent to 12 and $Y = \{2, 4, 6, 8, 10, 12, 14, 16\}$ be the vertices adjacent to 9. The only vertex adjacent to each vertex in Y is 9. Hence, $\operatorname{ann}(12) \neq \operatorname{ann}(\operatorname{ann}(9))$. Thus, 9 and 12 do not form an exact pair of zero-divisors. On the other hand, 12 and 15 are an exact pair of zero-divisors. Indeed, let $Z = \{6, 12\}$ be the vertices adjacent to 15. The set of vertices adjacent to each vertex in Z is X. In other words, $\operatorname{ann}(12) = \operatorname{ann}(\operatorname{ann}(15))$.

5 Compressed graphs

Compressed zero-divisor graphs were defined in [14], as follows. Let $a, b \in Z(R)^*$ and define the relation $a \sim b$ if and only if $\operatorname{ann}(a) = \operatorname{ann}(b)$. It is routine to check that $\sim$ defines an equivalence relation on $Z(R)^*$. Set $[a] = \{b \in Z(R)^* : a \sim b\}$ to denote the equivalence class of a. Denoted $\overline{\Gamma(R)}$, the *compressed* zero-divisor graph of R has as a vertex set $V(\overline{\Gamma(R)}) = \{[a] : a \in Z(R)^*\}$. An edge $[a]-[b]$ is in $E(\overline{\Gamma(R)})$ if and only if $a - b \in E(\Gamma(R))$. (We include loops as edges.) The product $[a][b] = [ab]$ is well-defined, as shown in [3, Lemma 3.1]. Compressed zero-divisor graphs are useful for many reasons: 1) they are often finite, even when $\Gamma(R)$ is not, 2) they streamline the presentation of the zero-divisor graph, while preserving ring-theoretic properties, and 3) the associated primes of the ring are embedded within.

In this section we provide criteria for identifying exact pairs of zero-divisors when given $\overline{\Gamma(R)}$. Unfortunately, such criteria are only possible for some classes of rings; there are multiple representations for $\overline{\Gamma(R)}$ and it is possible to have exact zero-divisors and non-exact zero-divisors in the same class.

Example 5.1. Let $R = \mathbb{Z}[X, Y, Z]/(XYZ)$ and let x, y, and z denote the images of X, Y, and Z, respectively. Then xz is an exact zero-divisor with $\operatorname{ann}(x^2z^3) = \operatorname{ann}(xz)$, but x^2z^3 is not an exact zero-divisor. Indeed, in Figure 4, we have two representations of $\overline{\Gamma(R)}$: In the left graph, we have vertices that correspond to exact zero-divisors where pairs are connected with dashed lines, while the right graph has no visible exact zero-divisors.

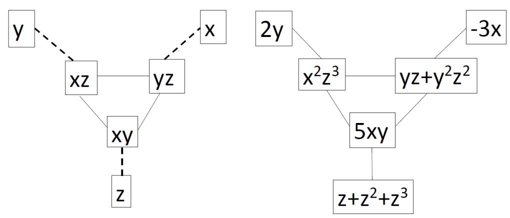

Figure 4: $R = \mathbb{Z}[X, Y, Z]/(XYZ)$ Left: $\overline{\Gamma(R)}$ is represented using only exact zero-divisor vertices with exact pairs indicated with a dashed line. Right:$\overline{\Gamma(R)}$ is represented using only non-exact zero-divisor vertices.

If R is PAS, then the obstruction described above does not occur.

Lemma 5.2. *Let R be a PAS ring and $x \in R$. If $[x]$ contains an exact zero-divisor, then $[x]$ contains only exact zero-divisors. Further more, for any exact pair of zero-divisors x^* and y^* in R, we have that x and y are an exact pair of zero-divisors for all $x \in [x^*]$ and $y \in [y^*]$.*

Proof. Let $x_1, x_2 \in [x^*]$. If x_1 and y are an exact pair of zero-divisors, we show that x_2 and y are an exact pair of zero-divisors. Indeed, we have that $\text{ann}(x_2) = \text{ann}(x_1)$ since x_1 and x_2 are in the same congruence class. Also, $\text{ann}(x_1) = yR$ since x_1 and y are an exact pair of zero-divisors. Therefore $\text{ann}(x_2) = yR$, and $\text{ann}(y) = x_2R$ since R is PAS. The rest of the lemma follows. $\square$

We now prove criteria for identifying exact zero-divisors from compressed zero-divisor graphs analogous to those given in Section 2 and Section 3.

Proposition 5.3. *Let $[a], [b^*], [c^*], [d] \in V(\overline{\Gamma(R)})$, with $[a] - [b^*]$, $[c^*]$-$- [d] \in E(\overline{\Gamma(R)})$ and $[a] - [d] \notin E(\overline{\Gamma(R)})$. Then b and c do not form an exact pair of zero-divisors for any $b \in [b^*]$ and $c \in [c^*]$.*

Proof. Taking $b \in [b^*]$ and $c \in [c^*]$, makes the proof identical to the proof of Proposition 3.3. $\square$

Proposition 5.4. *Let R be Artinian Gorenstein and $[x^*] \in V(\overline{\Gamma(R)})$. If $[x^*]$ is a leaf then x is an exact zero-divisor for all $x \in [x^*]$. Moreover, if $[y^*] \in V(\overline{\Gamma(R)})$ has a leaf then any element from $[y^*]$ forms an exact pair of zero-divisors with any element in the leaf's congruence class.*

Proof. Assume $[x^*] - [y^*] \in E(\overline{\Gamma(R)})$ and $[x^*]$ is a leaf. Let $x \in [x^*]$ and $y \in [y^*]$. We show x and y are an exact pair of zero-divisors and the rest of the proposition follows. Since $[x^*]$ is a leaf, $\operatorname{ann}(x) = [y^*]R$. (Where $[y^*]R$ denotes the ideal in R generated by the elements of $[y^*]$.)

Claim: $[y^*]R = yR$ for all $y \in [y^*]$. Indeed, suppose there exist y' and y'' such that $y'R \neq y''R$. Then $\operatorname{ann}(y') \neq \operatorname{ann}(y'')$ since R is Artinian Gorenstein (see Remark 4.1), which is a contradiction, and the claim follows.

Hence, $\operatorname{ann}(x) = yR$. Since R is PAS, $\operatorname{ann}(y) = xR$. $\square$

Corollary 5.5. *Let R be Artinian Gorenstein and $[y^*] \in V(\overline{\Gamma(R)})$ be adjacent to a leaf, and to a non-leaf $[z]$. If $y \in [y^*]$ then y does not form an exact pair of zero-divisors with any element of $[z]$.*

Theorem 4.3 can also be used for the compressed case. To emphasize this application, we introduce notation: Let $\operatorname{ann}([x]) = \{y \in R \text{ such that } yx' = 0 \text{ for all } x' \in [x]\}$. However, this does not generate new content; it is clear that $\operatorname{ann}([x]) = \operatorname{ann}(x)$. Under this notation, Theorem 4.3 reads as the following corollary:

Corollary 5.6. *Let R be an Artinian Gorenstein ring or an exact zero-divisor ring. Let x^* and y^* be nonzero zero-divisors in R with $x \in [x^*]$ and $y \in [y^*]$. Then x and y are an exact pair of zero-divisors if and only if $\operatorname{ann}([x^*]) = \operatorname{ann}(\operatorname{ann}([y^*]))$.*

Example 5.7. Consider the Artinian Gorenstein ring $R = k[X]/(X^6)$, where k is a field, and let x be the image of X in R. Then $\Gamma(R)$ is infinite if k is infinite. Below, we present the finite graph $\overline{\Gamma(R)}$. Using Corollary 5.6, we can identify the exact pairs of zero-divisors. For example, denote the set of vertices adjacent to $[x^4]$ as A and the set of vertices adjacent to $[x^2]$ as B. Hence, $\operatorname{ann}(\operatorname{ann}([x^4])) = \operatorname{ann}(A)$. The only vertices adjacent to each vertex in A are $[x^4]$ and $[x^5]$, that is, $\mathrm{ann}(A) = B = \mathrm{ann}([x^2])$. Therefore, for any nonzero $u, v \in k$, we have that ux^4 and vx^2 are an exact pair of zero-divisors. Similarly, we have that $\mathrm{ann}(\mathrm{ann}([x]) = \mathrm{ann}([x^5])$ and $\mathrm{ann}(\mathrm{ann}([x^3])) = \mathrm{ann}([x^3])$.

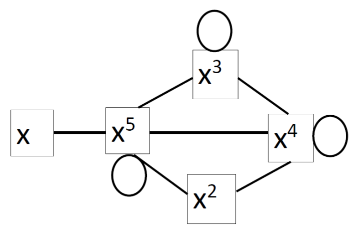

Figure 5: $\overline{\Gamma(R)}$

6 Exact zero-divisor subgraphs

Exact zero-divisor graphs were introduced by Lalchandani in [11], and further studied in [12]. (These are the only known studies of exact zero-divisor graphs at the time of this writing.) We refer to them as subgraphs since they are always subgraphs of the zero-divisor graph.

Definition 6.1. *We define an exact zero-divisor subgraph associated to a ring R as the vertex set consisting of the exact zero-divisors of R where vertices x and y are adjacent if and only if they are an exact pair of zero-divisors. We denote the exact zero-divisor subgraph of R by $\mathcal{G}(R)$.*

Remark 6.2. The above results that identify exact zero-divisors can be translated in terms of constructing exact zero-divisor subgraphs. For instance, Theorem 4.3 says that $\Gamma(R)$ is sufficient for constructing $\mathcal{G}(R)$ when R is Artinian Gorenstein or an exact zero-divisor ring.

Example 6.3. From the discussion in Example 4.4, we have the exact zero-divisor graph of $\mathbb{Z}_{18}$. (Figure 6)

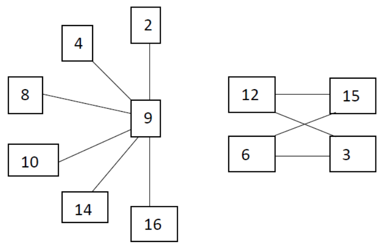

Figure 6: $\mathcal{G}(\mathbb{Z}_{18})$

Example 6.4. Let $R = \mathbb{Z}_2[X,Y]/(X^2,Y^2,XY)$ as in Examples 2.2 and 2.5. Then $\mathcal{G}(R)$ is empty.

Many properties enjoyed by $\Gamma(R)$ do not transfer to $\mathcal{G}(R)$. For example, $\Gamma(R)$ is connected [1, Theorem 2.3]. As Examples 2.1 and 6.3 illustrate, $\mathcal{G}(R)$ is not necessarily connected.

As in the case of $\Gamma(R)$, ring-theoretic properties can be read off $\mathcal{G}(R)$. The following proposition uses distance between vertices to identify such a property; the distance between vertices x and y in a graph G is the length of the shortest path, and is denoted $d_G(x,y)$.

Proposition 6.5. *Let $x, y \in V(\mathcal{G}(R))$ and $d_{\mathcal{G}(R)}(x,y) = 2$. Then $R/xR \cong R/yR$.*

Proof. Since $d_{\mathcal{G}(R)}(x,y) = 2$, there exists $z \in V(\mathcal{G}(R))$ such that $x - z$ and $y - z$ are in $E(\mathcal{G}(R))$. Hence, $\operatorname{ann}(x) = zR = \operatorname{ann}(y)$. Since x and y are exact zero-divisors, we have that $R/xR \cong \operatorname{ann}(x)$ and $R/yR \cong \operatorname{ann}(y)$, see Remark 2.4. $\square$

We now turn our attention to the compressed case. The following definition was given in [11].

Definition 6.6. *The compressed exact zero-divisor graph of a ring R, denoted $\overline{\mathcal{G}(R)}$, is the graph whose vertices are classes of elements $[a]$ where $a \in V(\mathcal{G}(R))$ and $[a] - [b]$ is an edge if and only if a and b are an exact pair of zero-divisors.*

Example 6.7. In this example, we translate the computations from Example 5.7 into a compressed exact zero-divisor graph. Recall, $R =$

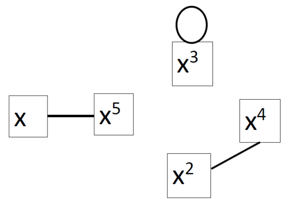

Figure 7: $\overline{\mathcal{G}(R)}$.

$k[X]/(X^6)$ is PAS. Let x be the image of X in R (see Figure 7). More generally, let $R = k[X]/(X^n)$. Then we have the following:

1. If n is odd then $\overline{\mathcal{G}(R)}$ is the disjoint union of $\frac{n-1}{2}$ components of the form K_2.

2. If n is even then $\overline{\mathcal{G}(R)}$ is the disjoint union of $\frac{n-2}{2}$ components of the form K_2 and one K_1 component.

Example 6.8. Figure 8 displays $\overline{\mathcal{G}(\mathbb{Z}_{12})}$. We consider two methods of its construction: 1) Extract the exact zero-divisors from $\Gamma(\mathbb{Z}_{12})$, obtaining $\mathcal{G}(\mathbb{Z}_{12})$. Then, identify equal annihilators and collapse $\mathcal{G}(\mathbb{Z}_{12})$ to $\overline{\mathcal{G}(\mathbb{Z}_{12})}$. 2) Collapse $\Gamma(\mathbb{Z}_{12})$ to $\overline{\Gamma(\mathbb{Z}_{12})}$, then use Corollary 5.6 to obtain $\overline{\mathcal{G}(\mathbb{Z}_{12})}$, see Figure 8. The second method of construction has the advantage of using the compressed graph.

Figure 8: Left: $\overline{\Gamma(\mathbb{Z}_{12})}$ Right: $\overline{\mathcal{G}(\mathbb{Z}_{12})}$

In Example 6.8, we described two methods of constructing $\overline{\mathcal{G}(R)}$ from $\Gamma(R)$. The following remark expands this discussion.

Remark 6.9. Some representations of vertices of $\overline{\Gamma(R)}$ can lead to an obstruction in identifying $\overline{\mathcal{G}(R)}$ as a subgraph of $\overline{\Gamma(R)}$, (See Example

5.1). However, from Lemma 5.2, if R is PAS then $\overline{\mathcal{G}(R)}$ is a subgraph of $\overline{\Gamma(R)}$ for any representation of $\overline{\Gamma(R)}$.

Consider the following commutative diagram of graph homomorphisms:

$$\begin{tikzcd} \mathcal{G}(R) \arrow[r, hook, "i"] \arrow[d, two heads, "\Pi_{\mathcal{G}}"] & \Gamma(R) \arrow[d, two heads, "\Pi_{\Gamma}"] \\ \overline{\mathcal{G}(R)} \arrow[r, hook, "\bar{i}"] & \overline{\Gamma(R)} \end{tikzcd}$$

The maps in the diagram are naturally induced. The horizantal maps are injections and the vertical maps are surjections. Under this notation, the conclusion of Lemma 5.2 tells us that for an exact zero-divisor $a \in R$, $\Pi_\Gamma^{-1}([a]) \subseteq i(\mathcal{G}(R))$, which implies the following containment: $\Pi_\Gamma^{-1}(\bar{i}(\Pi_{\mathcal{G}}(\mathcal{G}(R))) \subseteq i(\mathcal{G}(R))$. The reverse containment is clear from a diagram chase in the commutative diagram. In other words, from Lemma 5.2, we have that if R is PAS, then $\Pi_\Gamma^{-1}(\bar{i}(\Pi_{\mathcal{G}}(\mathcal{G}(R))) = i(\mathcal{G}(R))$.

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