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An Analytical Method for Solving Second-order Fuzzy Differential Equations

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Abstract. This paper is devoted to obtaining an analytical solution for the second-order fuzzy differential equations by considering the type of generalized Hukuhara differentiability of the solution. The effectiveness and efficiency of the approaches are illustrated by solving several practical examples such as the fuzzy Cauchy-Euler equation, the fuzzy Legendre equation, and the fuzzy Chebyshev differential equation.

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1 Introduction

Solving differential equations are studied by many researchers in that they are pervasive applications in varied scientific disciplines. Initial value problems are a well-known class of ordinary differential equations, which is utilized widely for modeling problems in several areas of science, such as biology, physics, and medicine [9, 10]. Most of these phenomena have some uncertainty and ambiguity in their initial measurements, and the fuzzy set theory is an executable tool for modeling systems with ambiguities. For this reason, fuzzy number have used to describe fuzzy parameters in initial value problems [5, 6, 12]. Zadeh's extension principle are used for solving the second-order fuzzy differential equations [5]. In recent years, many methods have been utilized for finding an analytical [4, 11] or numerical fuzzy solution for fuzzy differential equations [20, 1, 2, 3]

This paper examines the fuzzy solution of the following second-order fuzzy differential equation

$$\upsilon''_{gH}(t) \oplus q(t) \odot \upsilon'_{gH}(t) \oplus r(t) \odot \upsilon(t) = f(t), \quad t > t_0; \tag{1}$$

associated with

$$\upsilon(t_0) = \alpha, \quad \upsilon'_{gH}(t_0) = \beta. \tag{2}$$

The coefficient real-valued functions $q(t)$ and $r(t)$ are continuous for $t > t_0$ and $q(t)$ and $r(t)$ maintain the sign for $t > t_0$. Here, $f(t)$ is a fuzzy-valued continuous function, and α, β are two fuzzy numbers.

In this paper, we obtain an analytical solution of a class of second-order linear differential problem (1) under fuzzy initial values (2) by considering the type of generalized Hukuhara differentiability of solution. Next, we attain the solution for the fuzzy Cauchy-Euler equation, the fuzzy Legendre equation, and the fuzzy Chebyshev differential equation by this method.

The structure of this paper is organized as follows. Section 2 expresses some concepts associated with the background knowledge in fuzzy mathematics, with emphasis on the generalized Hukuhara partial differentiability. In Section 3, we obtain an analytical method for solving the second-order fuzzy differential equation under the coefficients' suitable conditions. The proposed algorithm is illustrated by solving some examples in Section 4, and some practical and essential equations such as the fuzzy Cauchy-Euler equation, the fuzzy Legendre equation, and the fuzzy Chebyshev differential equation by this method are solved. The conclusions are given in Section 5.

2 Fundamental Background

In this section, we recall some basic definitions and properties of fuzzy calculus theory which are useful in the next sections.

Let E be the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which defined over the real line. Let A is a fuzzy number on E, we define $[A]^r = \Big\{t \in \mathbb{R}\Big|A(t) \geq r\Big\}$ the $r-$level of A with $r \in (0,1]$ and $[A]^0 = cl\Big\{t \in \mathbb{R}\Big|A(t) > 0\Big\}$. We represent $[A^-(r),\ A^+(r)]$ the $r-$level of a fuzzy number A.

A triangular fuzzy number defined as a fuzzy set in E, that is specified by an ordered triple $A = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$. If $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ are two triangular fuzzy numbers. In this case, there are the following properties for these numbers (see [15, 14, 17]):

1. $[A]^r = [a_1 + (a_2 - a_1)r, a_3 - (a_3 - a_2)r]$ for all $r \in [0,1]$.

2. Two triangle fuzzy numbers A and B are said equal if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

3. $A \oplus B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

4. If there exists $C \in E$ such that $A = B \oplus C$, then C is called Hukuhara difference(H-difference) of A and B, and $A \ominus B = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.

5. The generalized Hukuhara difference of two fuzzy numbers

$$A, B \in E,$$

is the fuzzy number C, (if it exists), such that

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} (i).\ C = (a_1 - b_1, a_2 - b_2, a_3 - b_3); \\ Or, \\ (ii).\ C = (a_3 - b_3, a_2 - b_2, a_1 - b_1). \end{cases}$$

In this article, we assume that $A \ominus_{gH} B \in E$.

6. For all $\lambda \in \mathbb{R}$

$$\lambda A = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3), & \text{If } \lambda \geq 0; \\ \\ (\lambda a_3, \lambda a_2, \lambda a_1), & \text{If } \lambda < 0. \end{cases}$$

7. $A \odot B = \Big(\min\ Z, a_2 b_2, \max\ Z\Big)$, where

$$Z = \{a_1 b_1, a_1 b_3, a_3 b_1, a_3 b_3\}.$$

A function $\upsilon : \mathbb{I} := [a, b] \subseteq \mathbb{R} \to E$ is called fuzzy-valued function and $r-$levels of this function are $\upsilon(t; r) = [\upsilon^-(t; r)\ \ ,\ \ \upsilon^+(t; r)]$, for all $r \in [0, 1]$. This fuzzy-valued function is said to be generalized Hukuhara differentiable (gH-differentiable) at t, if the exists $\upsilon'_{gH}(t) \in E$ such that (see [18, 7])

$$\upsilon'_{gH}(t) = \lim_{h \to 0} \frac{\upsilon(t + h) \ominus_{gH} \upsilon(t)}{h}.$$

Here $\ominus_{gH}$ denotes the generalized Hukuhara difference [17]. If for any t, $\upsilon^-(t; r)$ and $\upsilon^+(t; r)$ are both differentiable, we say that

- υ is $[i - gH]$-differentiable if

$$\upsilon'_{i.gH}(t; r) := \upsilon'_{gH}(t; r) = [(\upsilon^-)'(t; r),\ (\upsilon^+)'(t; r)], \qquad 0 \leq r \leq 1,$$

- υ is $[ii - gH]$-differentiable if

$$\upsilon'_{ii.gH}(t; r) := \upsilon'_{gH}(t; r) = [(\upsilon^+)'(t; r),\ (\upsilon^-)'(t; r)], \qquad 0 \leq r \leq 1.$$

Remark 2.1. Throughout the rest of this paper $\upsilon(t)$ is gH-differentiable of order j, $j = 1, 2$ for all $t \in \mathbb{I}$ with no switching point on $\mathbb{I}$, and

i. If $\upsilon^{(j)}_{gH}(t;r) = \Big[(\upsilon^{-})^{(j)}(t;r), (\upsilon^{+})^{(j)}(t;r)\Big]$, we denote $\upsilon^{(j)}_{i.gH}(t) := \upsilon^{(j)}_{gH}(t)$,

ii. If $\upsilon^{(j)}_{gH}(t;r) = \Big[(\upsilon^{+})^{(j)}(t;r), (\upsilon^{-})^{(j)}(t;r)\Big]$, we set $\upsilon^{(j)}_{ii.gH}(t) := \upsilon^{(j)}_{gH}(t)$.

Moreover, based on different type of gH-differentiability we have the following cases:

(i-i-gH). If $\upsilon(t)$, $\upsilon'_{gH}(t)$ are $[i-gH]-$differentiable, we have

$$\upsilon'_{i.gH}(t;r) = \Big[(\upsilon^{-})'(t;r), (\upsilon^{+})'(t;r)\Big],$$
$$\upsilon''_{i.gH}(t;r) = \Big[(\upsilon^{-})''(t;r), (\upsilon^{+})''(t;r)\Big].$$

(i-ii-gH). If $\upsilon(t)$ is $[i-gH]-$differentiable and $\upsilon'_{gH}(t)$ is $[ii-gH]-$differentiable,

$$\upsilon'_{i.gH}(t;r) = \Big[(\upsilon^{-})'(t;r), (\upsilon^{+})'(t;r)\Big],$$
$$\upsilon''_{ii.gH}(t;r) = \Big[(\upsilon^{+})''(t;r), (\upsilon^{-})''(t;r)\Big].$$

(ii-i-gH). If $\upsilon(t)$ is $[ii-gH]-$differentiable and $\upsilon'_{gH}(t)$ is $[i-gH]-$differentiable,

$$\upsilon'_{ii.gH}(t;r) = \Big[(\upsilon^{+})'(t;r), (\upsilon^{-})'(t;r)\Big],$$
$$\upsilon''_{ii.gH}(t;r) = \Big[(\upsilon^{+})''(t;r), (\upsilon^{-})''(t;r)\Big].$$

(ii-ii-gH). If $\upsilon(t)$ and $\upsilon'_{gH}(t)$ are $[ii-gH]-$differentiable

$$\upsilon'_{ii.gH}(t;r) = \Big[(\upsilon^{+})'(t;r), (\upsilon^{-})'(t;r)\Big],$$
$$\upsilon''_{i.gH}(t;r) = \Big[(\upsilon^{-})''(t;r), (\upsilon^{+})''(t;r)\Big].$$

Theorem 2.2. *(see [13]) Let $\upsilon : \mathbb{I} \to E$ and $\mathcal{G} : \mathbb{I} \to \mathbb{R}$ be two differentiable functions (f is gH-differentiable). Then*

$$(\upsilon \odot \mathcal{G})'_{gH}(t) = \upsilon'_{gH}(t) \odot \mathcal{G}(t) \oplus \upsilon(t) \odot \mathcal{G}'_{gH}(t).$$

Definition 2.3. (See [6]) Let $f : \mathbb{I} \to E$ is a fuzzy-valued function and $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$, then

$$\int_a^b f(t)\mathtt{d}t = \Big(\int_a^b f_1(t)\mathtt{d}t, \int_a^b f_2(t)\mathtt{d}t, \int_a^b f_3(t)\mathtt{d}t\Big).$$

Lemma 2.4. *(See [19]) If $\upsilon : \mathbb{I} \to E$ be a fuzzy-valued function with no switching point in interval $\mathbb{I}$, then we have*

1. *If $\upsilon(t)$ is $[(i) - gH]-$differentiable, then*

$$\int_a^b \upsilon'_{i.gH}(t)\mathtt{d}t = \upsilon(b) \ominus \upsilon(a).$$

2. *If $\upsilon(t)$ is $[(ii) - gH]-$differentiable , then*

$$\int_a^b \upsilon'_{ii.gH}(t)\mathtt{d}t = (-1)\upsilon(a) \ominus (-1)\upsilon(b).$$

The significant contribution of this paper is to solve the second-order fuzzy differential equation under generalized Hukuhara differentiability. The next section will find an analytical solution for this problem based on [gH]-differentiability and will utilize the ideas presented in [8] for the second-order fuzzy differential equation.

3 Statement of the Method

The second-order fuzzy differential equation is defined as follows

$$\begin{cases} \upsilon''_{gH}(t) = F\Big(t, \upsilon(t), \upsilon'_{gH}(t)\Big), \quad t \in \mathbb{I}; \\ \\ \upsilon(t_0) = \alpha, \quad \upsilon'_{gH}(t_0) = \beta, \end{cases} \tag{3}$$

where $\alpha, \beta \in E$ and $\upsilon(t;r) = [\upsilon^-(t;r)\ ,\ \upsilon^+(t;r)]$ is a fuzzy function of t, $F : \mathbb{I} \times E \times E \to E$ is continuous fuzzy-valued function and $\upsilon'_{gH}(t)$, $\upsilon''_{gH}(t)$ are the the generalized Hukuhara derivative of $\upsilon(x)$ such that there is no switching points in $\mathbb{I}$.

Let $F : \mathbb{I} \times E \times E \to E$ is a continuous fuzzy-valued function and suppose that there exist $M_1, M_2 > 0$ such that

$$D\Big(F(t, \upsilon_1, \upsilon_2), F(t, w_1, w_2)\Big) \leq M_1 D(\upsilon_1, w_1) + M_2 D(\upsilon_2, w_2),$$

for all $t \in \mathbb{I}$, $\upsilon_1, \upsilon_2, w_1, w_2 \in E$. In this case, it can be proved that the initial-value problem (3) has a unique solution on $\mathbb{I}$ for each case [4].

Now, we are going to obtain an analytical solution for the following initial fuzzy valued problem

$$\begin{cases} \upsilon''_{gH}(t) \oplus q(t) \odot \upsilon'_{gH}(t) \oplus r(t) \odot \upsilon(t) = f(t), \quad t > t_0; \\ \\ \upsilon(t_0) = \alpha, \quad \upsilon'_{gH}(t_0) = \beta. \end{cases} \tag{4}$$

The coefficient real-valued functions $q(t)$ and $r(t)$ are continuous for $t > t_0$ and $q(t)$ and $r(t)$ maintain the sign for $t > t_0$. Here, $f(t)$ is a fuzzy-valued continuous function, and α, β are two fuzzy numbers.

Theorem 3.1. *For the given second-order fuzzy differential equation (4), if there exists a real constant c such that*

$$cq(t) + r(t)e^{\int \frac{r(t)}{q(t)} \mathtt{d}t} = 0,$$

then the fuzzy solution is given by

$$\upsilon(t) = \frac{1}{\xi_1(t)} z(t),$$

where $z(t)$, depending on the type of gH-differentiability is obtained in one of the following cases:

(i.i.gH). *If $\upsilon(t)$ and $\upsilon'_{gH}(t)$ are [(i)-gH]-differentiable, then*

$$z(t) \quad = \quad z(t_0) \oplus h(t_0) z'_{i.gH}(t_0) \int_{t_0}^{t} \frac{1}{h(x)} \mathtt{d}x \oplus L_{tt}^{-1}(g(t)).$$

(ii.ii.gH). *If $\upsilon(t)$ and $\upsilon'_{gH}(t)$ are [(ii)-gH]-differentiable, then*

$$z(t) \quad = \quad z(t_0) \ominus (-1) h(t_0) z'_{ii.gH}(t_0) \int_{t_0}^{t} \frac{1}{h(x)} \mathtt{d}x \oplus L_{tt}^{-1}(g(t)).$$

(i.ii.gH). *If $\upsilon(t)$ is [(i)-gH]-differentiable and $\upsilon'_{gH}(t)$ is [(ii)-gH]- differentiable, then*

$$z(t) = z(t_0) \oplus h(t_0) z'_{ii.gH}(t_0) \int_{t_0}^{t} \frac{1}{h(x)} \mathtt{d}x \ominus (-1) L_{tt}^{-1}(g(t)).$$

(ii.i.gH). *If $\upsilon(t)$ is [(ii)-gH]-differentiable and $\upsilon'_{gH}(t)$ is [(i)-gH]- differentiable, then*

$$z(t) = z(t_0) \ominus (-1) h(t_0) z'_{ii.gH}(t_0) \int_{t_0}^{t} \frac{1}{h(x)} \mathtt{d}x \ominus (-1) L_{tt}^{-1}(g(t)).$$

Where

$$\begin{aligned} \xi_1(t) &= e^{\int \frac{r(t)}{q(t)} \mathtt{d}t}, \quad S(t) = \frac{1}{\xi_1(t)}, \quad T(t) = \frac{q(t)}{\xi_1(t)}, \\ \xi_2(t) &= e^{\int \frac{T(t)}{S(t)} \mathtt{d}t}, \quad h(t) = \xi_2(t) S(t), \quad g(t) = \xi_2(t) f(t), \\ z(t_0) &= \alpha \xi_1(t_0), \quad z'(t_0) = \alpha \xi_1'(t_0) \oplus \beta \xi_1(t_0). \end{aligned}$$

Proof.

Multiplying both side of Equation (4) by $\xi_1(t) = e^{\int \frac{r(t)}{q(t)} \mathtt{d}t}$

$$\xi_1(t) \odot \upsilon''_{gH}(t) \oplus \xi_1(t) q(t) \odot \upsilon'_{gH}(t) \oplus \xi_1(t) r(t) \odot \upsilon(t) = \xi_1(t) \odot f(t).$$

Given that $\xi_1'(t) q(t) = \xi_1(t) r(t)$, it can be concluded that

$$\xi_1(t) \odot \upsilon''_{gH}(t) \oplus \xi_1(t) q(t) \odot \upsilon'_{gH}(t) \oplus \xi_1'(t) q(t) \odot \upsilon(t) = \xi_1(t) \odot f(t) \quad (5)$$

Using Theorem 2.2, we can rewrite Equation (5) as follows

$$\xi_1(t) \odot \upsilon''_{gH}(t) \oplus q(t) \odot \Big(\xi_1(t) \odot \upsilon(t)\Big)'_{gH} = \xi_1(t) \odot f(t).$$

Therefore

$$\upsilon''_{gH}(t) \oplus \frac{q(t)}{\xi_1(t)} \odot \Big(\xi_1(t) \odot \upsilon(t)\Big)'_{gH} = f(t). \quad (6)$$

Assuming $\xi_1(t) \odot \upsilon(t) = z(t)$, Equation (6) is rewritten as follows

$$\left(\left(\frac{1}{\xi_1(t)}\right)' z(t) \oplus \frac{1}{\xi_1(t)} z'_{gH}(t)\right)'_{gH} \oplus \frac{q(t)}{\xi_1(t)} \odot z'_{gH}(t) = f(t),$$

using Theorem 2.2 we obtain

$$\left(\frac{1}{\xi_1(t)} \odot z'_{gH}(t)\right)'_{gH} \oplus \left(\left(\frac{1}{\xi_1(t)}\right)' \odot z(t)\right)'_{gH} \oplus \frac{q(t)}{\xi_1(t)} \odot z'_{gH}(t) = f(t). \quad (7)$$

Now, if we assume that $\left(\frac{1}{\xi_1(t)}\right)' = c$ or in other word

$$cq(t) + r(t)e^{-\int \frac{r(t)}{q(t)}\mathrm{d}t} = 0,$$

where c is a real constant, then Equation (7) becomes

$$\left(S(t) \odot z'_{gH}(t)\right)'_{gH} \oplus T(t) \odot z'_{gH}(t) = f(t). \quad (8)$$

Where $S(t) = \frac{1}{\xi_1(t)}$ and $T(t) = \frac{q(t)}{\xi_1(t)} + c$.

Now, multiplying both side of Equation (8) by $\xi_2(t) = e^{\int \frac{T(t)}{S(t)}\mathrm{d}t}$,

$$\xi_2(t) \odot \left(S(t) \odot z'_{gH}(t)\right)'_{gH} \oplus \xi_2(t)T(t) \odot z'_{gH}(t) = \xi_2(t) \odot f(t).$$

According to $\xi_2'(t)S(t) = \xi_2(t)T(t)$ and Theorem 2.2, we have

$$\left(\xi_2(t)S(t) \odot z'_{gH}(t)\right)'_{gH} = \xi_2(t) \odot f(t). \quad (9)$$

Now, Equation (9) can be write in the following form

$$L_{tt}z = g(t), \quad (10)$$

where

$$\begin{aligned} L_{tt}z(t) &\equiv \left(h(t) \odot z'_{gH}(t)\right)'_{gH}, \quad (11) \\ h(t) &= \xi_2(t)S(t), \quad g(t) = \xi_2(t) \odot f(t). \end{aligned}$$

The inverse of Equation (11) can be considered as follows

$$L_{tt}^{-1}z(t)=\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x\int_{t_0}^{x}z(\tau)\mathtt{d}\tau.$$

Note that $L_{tt}^{-1}L_{tt}\neq L_{tt}L_{tt}^{-1}$.

Operating with inverse operator L_{tt}^{-1} on Equation (10) yields

$$\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x\int_{t_0}^{x}\Big(h(\tau)\odot z'_{gH}(\tau)\Big)'_{gH}\mathtt{d}\tau=L_{tt}^{-1}(g(t)). \tag{12}$$

By considering the type of gH-differentiability, we have

(i.i.gH). In this case $z(t)$ and z'_{gH} are [i-gH]-differentiable. Then Equation (12) can rewrite as follows

$$\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x\int_{t_0}^{x}\Big(h(\tau)z'_{i.gH}(\tau)\Big)'_{i.gH}\mathtt{d}\tau=L_{tt}^{-1}(g(t)).$$

Using Lemma 2.4 yields

$$\int_{t_0}^{t}\frac{1}{h(x)}\Big(h(x)z'_{i.gH}(x)\ominus h(t_0)z'_{i.gH}(t_0)\Big)\mathtt{d}x=L_{tt}^{-1}(g(t)) \quad\Rightarrow$$

$$z(t)\ominus z(t_0)\ominus h(t_0)z'_{i.gH}(t_0)\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x=L_{tt}^{-1}(g(t)),$$

and finally the analytical **(i-i-gH)**$-$solution of the fuzzy problem (4) is obtained as follows

$$z(t)=z(t_0)\oplus h(t_0)z'_{i.gH}(t_0)\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x\oplus L_{tt}^{-1}(g(t)).$$

In the following, we also prove **(ii.i.gH)**. The other cases can be proved in the same way.

(ii.i.gH). Suppose that $z(t)$ is [ii-gH]-differentiable and z'_{gH} is [i-gH]-differentiable. Then using Equation (12) and Lemma 2.4 we obtain

$$\int_{t_0}^{t}\frac{1}{h(x)}\Big(h(x)z'_{i.gH}(x)\ominus h(t_0)z'_{i.gH}(t_0)\Big)\mathtt{d}x=L_{tt}^{-1}(g(t)).$$

Using Lemma 2.4 again yields

$$(-1)z(t_0) \ominus (-1)z(t) \ominus h(t_0)z'_{i.gH}(t_0) \int_{t_0}^{t} \frac{1}{h(x)} \mathtt{d}x = L_{tt}^{-1}(g(t)),$$

and finally

$$z(t) \quad = \quad z(t_0) \ominus (-1)h(t_0)z'_{ii.gH}(t_0) \int_{t_0}^{t} \frac{1}{h(x)} \mathtt{d}x \ominus (-1)L_{tt}^{-1}(g(t)).$$

□

4 Application

To illustrate the efficiency and accuracy of the method for solving the second-order fuzzy differential equations, some different examples will be solved in this section, some practical and essential equations such as the fuzzy Cauchy-Euler equation, the fuzzy Legendre equation, and the fuzzy Chebyshev differential equation. All calculations were performed on a PC running Mathematica software.

4.1 Fuzzy Cauchy-Euler Equation

The fuzzy differential equation

$$at^2\upsilon''_{gH}(t) \oplus bt\upsilon'_{gH}(t) \oplus c\upsilon(t) = f_1(t),$$

is called the fuzzy Cauchy-Euler equation of order two.

We first write the Cauchy-Euler equation in the standard form

$$\upsilon''_{gH}(t) \oplus \frac{b}{at}\upsilon'_{gH}(t) \oplus \frac{c}{at^2}\upsilon(t) = f(t),$$

where $f(t) = \frac{c}{at^2} f_1(t)$.

Example 4.1. Consider the following second-order fuzzy Cauchy-Euler equation

$$\begin{cases} 9t^2\upsilon''_{gH}(t)\ominus_{gH} t\upsilon'_{gH}(t)\oplus \upsilon(t) = \\ \Big(9t^2+t+1, 18t^2+2t+2, 27t^2+3t+3\Big)e^{-t}, \quad t>1; \\ \\ \upsilon(1) = \Big(e^{-1}, 2e^{-1}, 3e^{-1}\Big), \\ \upsilon'_{gH}(1) = \Big(-3e^{-1}, -2e^{-1}, -e^{-1}\Big). \end{cases}$$

This fuzzy Cauchy-Euler equation in the standard form is

$$\upsilon''_{gH}(t)\ominus_{gH}\frac{1}{9t}\upsilon'_{gH}(t)\oplus\frac{1}{9t^2}\upsilon(t) = \Big(e^{-t}+\frac{e^{-t}}{9t}+\frac{e^{-t}}{9t^2}, \\ 2e^{-t}+\frac{2e^{-t}}{9t}+\frac{2e^{-t}}{9t^2}, 3e^{-t}+\frac{e^{-t}}{3t}+\frac{e^{-t}}{3t^2}\Big).$$

Here, $q(t)=\frac{-1}{9t}$ and $r(t)=\frac{1}{9t^2}$ and

$$f(t) = \Big(e^{-t}+\frac{e^{-t}}{9t}+\frac{e^{-t}}{9t^2}, 2e^{-t}+\frac{2e^{-t}}{9t}+\frac{2e^{-t}}{9t^2}, 3e^{-t}+\frac{e^{-t}}{3t}+\frac{e^{-t}}{3t^2}\Big).$$

The condition of Theorem 3.1 is fulfilled and straightforward computation yields $c=1$. Therefore

$$\xi_1(t)=\frac{1}{t}, \quad S(t)=t, \quad T(t)=\frac{8}{9}, \quad \xi_2(t)=t^{\frac{8}{9}}, \quad h(t)=t^{\frac{17}{9}},$$

$$g(t) = \Big((e^{-t}+\frac{e^{-t}}{9t}+\frac{e^{-t}}{9t^2})t^{\frac{8}{9}}, (2e^{-t}+\frac{2e^{-t}}{9t}+\frac{2e^{-t}}{9t^2})t^{\frac{8}{9}}, \\ (3e^{-t}+\frac{e^{-t}}{3t}+\frac{e^{-t}}{3t^2})t^{\frac{8}{9}}\Big),$$

$$z(1) = \Big(e^{-1}, 2e^{-1}, 3e^{-1}\Big),$$

$$z'(1) = -1(\Big(e^{-1}, 2e^{-1}, 3e^{-1}\Big)\oplus\Big(-3e^{-1}, -2e^{-1}, -e^{-1}\Big) = \\ \Big(-6e^{-1}, -4e^{-1}, -2e^{-1}\Big).$$

We want to obtain a **(ii.ii-gH)**-differentiable solution for this fuzzy Cauchy-Euler equation

$$\begin{aligned} z(t) &= z(t_0) \ominus (-1)h(t_0)z'_{ii.gH}(t_0)\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x \oplus L_{tt}^{-1}(g(t)) \\ &= \Big(\frac{e^{-t}}{t}, \frac{2e^{-t}}{t}, \frac{3e^{-t}}{t}\Big). \end{aligned}$$

The **(ii.ii-gH)**-differentiable solution is

$$\upsilon(t) = \Big(e^{-t}, 2e^{-t}, 3e^{-t}\Big).$$

Example 4.2. Consider the following second-order fuzzy Cauchy-Euler equation

$$\begin{cases} \upsilon''_{gH}(t) \oplus \frac{1}{3t}\upsilon'_{gH}(t) \ominus_{gH} \frac{1}{3t^2}\upsilon(t) = \\ \Big(-\frac{104}{3} - \frac{13(2-t^2)}{3t^2}, -\frac{40}{3} - \frac{5(2-t^2)}{3t^2}, -8 - \frac{2-t^2}{t^2}\Big), \qquad t \in [1, \sqrt{2}]; \\ \\ \upsilon(1) = \Big(3, 5, 13\Big), \qquad \upsilon'_{gH}(1) = \Big(-26, -10, -6\Big). \end{cases}$$

The condition of Theorem 3.1 is fulfilled and straightforward computation yields $c = 1$. Therefore

$$\xi_1(t) = \frac{1}{t}, \quad S(t) = t, \quad T(t) = \frac{4}{3}, \quad \xi_2(t) = t^{\frac{4}{3}}, \quad h(t) = t^{\frac{7}{3}},$$

$$g(t) = \Big(t^{\frac{4}{3}}\Big(-\frac{104}{3} - \frac{13(2-t^2)}{3t^2}\Big),$$
$$t^{\frac{4}{3}}\Big(-\frac{40}{3} - \frac{5(2-t^2)}{3t^2}\Big), t^{\frac{4}{3}}\Big(-8 - \frac{2-t^2}{t^2}\Big)\Big),$$

$$z(1) = \Big(3, 5, 13\Big),$$

$$z'(1) = -1\Big(3, 5, 13\Big) \oplus \Big(-26, -10, -6\Big) = \Big(-39, -15, -9\Big).$$

A **(ii.i-gH)**-differentiable solution is obtained by the following formula

$$\begin{aligned} z(t) &= z(t_0) \ominus (-1)h(t_0)z'_{ii.gH}(t_0)\int_{t_0}^{t}\frac{1}{h(x)}\mathtt{d}x \ominus (-1)L_{tt}^{-1}(g(t)) \\ &= \Big(\frac{6}{t} - 3t, \frac{10}{t} - 5t, \frac{26}{t} - 13t\Big). \end{aligned}$$

The **(ii.i-gH)**-differentiable solution is

$$\upsilon(t) = \Big(6 - 3t^2, 10 - 5t^2, 26 - `3t^2\Big).$$

4.2 Fuzzy Legendre Equation

The equation $(1-t^2)\upsilon''_{gH}(t) \ominus 2t\upsilon'_{gH}(t) \oplus p(p+1)\upsilon(t) = f(t)$, where $-1 \le t \le 1$ and $f(t) \in E$, for any real number p, is called fuzzy Legendre's equation of degree p. Sabzi et.al [16] solved this equation by using the fuzzy generalized power series method.

In this section, we intend to obtain an analytical solution for the fuzzy Legendre equation using the method described in this paper.

Example 4.3. Consider the following fuzzy Legendre equation of degree 1

$$\begin{cases} (1-t^2)\upsilon''_{gH}(t) \ominus 2t\upsilon'_{gH}(t) \oplus 2\upsilon(t) = f(t), \quad t \in [-1,1]; \\ \upsilon(-1) = \Big(2.5, 5.9, 11.7\Big), \quad \upsilon'_{gH}(-1) = \Big(-23.4, -11.8, -5\Big). \end{cases}$$

Where

$$f(t) = \Big(-5t^2 + 5(1-t^2), -11.8t^2 + 11.8(1-t^2), -23.4t^2 + 23.4(1-t^2)\Big).$$

For $q(t) = \frac{-2t}{1-t^2}$ and $r(t) = \frac{2}{1-t^2}$ the condition of Theorem 3.1 is fulfilled and direct calculation concludes $c = 1$. So, by applying the method which is discussed in detail in the Section 3, we obtain the following **(i.i-gH)**-differentiable solution

$$\upsilon(t) = \Big(2.5t^2, 5.9t^2, 11.7t^2\Big).$$

Example 4.4. Consider the following fuzzy Legendre equation of degree 2

$$\begin{cases} \upsilon''_{gH}(t) \ominus \frac{2t}{(1-t^2)}\upsilon'_{gH}(t) \oplus \frac{2}{(1-t^2)}\upsilon(t) = f(t), \quad t \in [-1,1]; \\ \upsilon(-1) = \Big(3e, 5e, 15e\Big), \qquad \upsilon'_{gH}(-1) = \Big(-15e, -5e, -3e\Big). \end{cases}$$

Where

$$f(t)=\Big(3e^{-t}+\frac{6e^{-t}}{1-t^2}+\frac{6te^{-t}}{1-t^2}5e^{-t}+\frac{10e^{-t}}{1-t^2}+\frac{10te^{-t}}{1-t^2},15e^{-t}+\frac{30e^{-t}}{1-t^2}+\frac{30te^{-t}}{1-t^2}\Big).$$

Using Theorem 3.1 we obtain $c=1$. Therefore

$$\xi_1(t)=\frac{1}{t},\quad S(t)=t,\quad T(t)=1-\frac{2t^2}{1-t^2},$$
$$\xi_2(t)=t(1-t^2),\quad h(t)=t^2(1-t^2).$$

A **(ii.ii-gH)**-differentiable solution is obtained by the following formula

$$z(t)=\Big(\frac{3e^{-t}}{t},\frac{5e^{-t}}{t},\frac{15e^{-t}}{t}\Big).$$

The **(ii.ii-gH)**-differentiable solution is

$$\upsilon(t)=\Big(3e^{-t},5e^{-t},15e^{-t}\Big).$$

4.3 Fuzzy Chebyshev Differential Equation

The second-order fuzzy differential equation

$$(1-t^2)\odot\upsilon''_{gH}(t)\ominus_{gH}t\upsilon'_{gH}(t)\oplus n^2\upsilon(t)=f(t),$$

is called fuzzy Chebyshev differential equation.

Example 4.5. We are going to obtain an **(i.i.gH)-**differentiable solution for the following fuzzy Chebyshev differential equation by using the method described in Section 3.

$$\begin{cases}(1-t^2)\odot\upsilon''_{gH}(t)\ominus_{gH}t\upsilon'_{gH}(t)\oplus 2\upsilon(t)=f(t), & t\in[-1,1];\\ \upsilon(-1)=\Big(0,0,0\Big),\qquad \upsilon'_{gH}(-1)=\Big(12,21,24\Big).\end{cases}$$

Where

$$f(t)=\Big(\frac{4(8t^3-6t-1)}{t^2-1},\frac{7(8t^3-6t-1)}{t^2-1},\frac{12(8t^3-6t-1)}{t^2-1}\Big).$$

Using Theorem 3.1 we obtain $c = 1$. Therefore

$$\xi_1(t) = \frac{1}{t},\ S(t) = t,\ T(t) = 1 - \frac{t^2}{1-t^2},$$
$$\xi_2(t) = t(\sqrt{1-t^2}),\ h(t) = t^2(\sqrt{1-t^2}).$$

And the **(i.i.gH)-**differentiable solution for this problem is

$$v(t) = \Big(4 + 4t^3, 7 + 7t^3, 12 + 12t^3\Big).$$

5 Conclusion

This paper obtained an analytical solution for the second-order fuzzy differential equations by considering the type of generalized Hukuhara differentiability of solution. The idea of this method was to change the problem (1) to the general operator form $L_{tt}z(t) \equiv \Big(h(t)z'_{gH}(t)\Big)'_{gH}$ and obtained the inverse differential operator L_{tt}^{-1}. Therefore, the exact solutions of the problem (1)-(2) can be obtained from operating with L_{tt}^{-1}. Finally, to show the approaches' effectiveness and efficiency, several practical examples such as the fuzzy Cauchy-Euler equation, the fuzzy Legendre equation, and the fuzzy Chebyshev differential equation were solved.

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