ON SOME PROPERTIES of EDGE QUASI-DISTANCE-BALANCED GRAPHS

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Abstract

For an edge $e = uv$ in a graph G, $M_u^G(e)$ is introduced as the set all edges of G that are at shorter distance to u than to v. We say that G is an edge quasi-distance-balanced graph whenever for every arbitrary edge $e = uv$, there exists a constant $\lambda > 1$ such that $m_u^G(e) = \lambda^{\pm 1} m_v^G(e)$. We investigate that edge quasi-distance-balanced garphs are complete bipartite graphs $K_{m,n}$ with $m \neq n$. The aim of this paper is to investigate the notion of cycles in edge quasi-distance-balanced graphs, and expand some techniques generalizing new outcome that every edge quasi-distance-balanced graph is complete bipartite graph. As well as, it is demontrated that connected quasi-distance-balanced graph admitting a bridge is not edge quasi-distance-balanced graph.

Keywords: distance-balanced graphs, quasi-distance-balanced graphs, edge quasi-distance-balanced graphs, complete bipartite graphs, bridge.

2000 Mathematics Subject Classification: 05C10, 05C25.

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1 Introduction

Consider G be a finite, undirected and connected graph and has vertex set $V(G)$ and edge set $E(G)$. The distance between any two vertices $u, v \in V(G)$ in a graph G is defined the number of edges in a shortest path conneting them and it is shown by $d_G(u, v)$. For any two arbitrary edges $e = uv$, $é = úv́$, the distance between e and $é$ is defined by:

$$d_G(e, é) = min\{d_G(u, é), d_G(v, é)\} = min\{d_G(u, ú), d_G(u, v́), d_G(v, ú), d_G(v, v́)\}.$$

Set $\quad M_u(e) = \{é \in E(G) | d_G(u, é) < d_G(v, é)\},$

$\quad M_v(e) = \{é \in E(G) | d_G(v, é) < d_G(u, é)\},$

and $\quad M_0(e) = \{é \in E(G) | d_G(u, é) = d_G(v, é)\}.$

In this paper, we define $m_u(e)$, $m_v(e)$, $m_0(e)$ as follows:

$\quad m_u(e) = |\{é \in E(G) | d_G(u, é) < d_G(v, é)\}|,$

$\quad m_v(e) = |\{é \in E(G) | d_G(v, é) < d_G(u, é)\}|,$

and $\quad m_0(e) = |\{é \in E(G) | d_G(u, é) = d_G(v, é)\}|.$

For a given graph G, assume that $e = uv$ is an arbitrary edge of G. For any two integers i, j we let:

$$D́^i_j(e) = \{é \in E(G) | d_G(é, u) = i, d_G(é, v) = j\}.$$

The sets $D́^i_j(e)$ lead the way to a "distance partition" of $E(G)$ with respect to the edge $e = uv$. Only the sets $D́^{i-1}_i(e)$, $D́^i_i(e)$ and, $D́^i_{i-1}(e)$, for each $(1 \leqslant i \leqslant d)$ may be nonempty based on the triangle inequality (d is the diameter of the graph G). Also $D́^0_0(e) = \phi$.

For two adjacent vertices u, v of G we indicate $W^G_{u,v} = \{x \in V(G) | d_G(x, u) < d_G(x, v)\}$. Similarly, we can define $W^G_{u,v}$. We call G quasi-distance-balanced (QDB) whenever for two arbitrary adjacent vertices u and v of G there exists a constant $\lambda > 1$ such that $|W^G_{u,v}| = \lambda^{\pm 1} |W^G_{v,u}|$.

A graph G is defined as edge quasi-distance-balanced if there is a positive rational number $\lambda > 1$, in which for any edge $e = uv$ of G, either $m_u(e) = \lambda m_v(e)$ or $m_v(e) = \lambda m_u(e)$

We can orient edges of an edge quasi-distance-balanced graph G in the below way, we introduce $\vec{e} : u \to v$ for any edge $\vec{e} = \vec{uv} \in E(G)$, if and only if

$m_u(e) = \lambda m_v(e)$ or $m_v(e) = \lambda m_u(e)$. (an example is illustrated in Figure 1). Suppose the directed graph made in this way is denoted $Q(G)$. Assume that a cycle of length n in an edge quasi-DB graph G be as $C = e_1, ...e_n$. Consider C^+ is denoted as the set of indices $i \in \{1, ..., n\}$, where $e_i \to e_{i+1}$, that is, $C^+ = \{i \in \{1, ..., n\} | e_i \to e_{i+1}\}$ (here we liken e_{n+1} with e_1 and e_0 with e_n). In the same way suppose that $C^- = \{i \in \{1, ..., n\} | e_{i+1} \to e_i\}$.

The study of construction of cycle in edge quasi-distance-balanced graphs is principal aim of this paper. All examples of edge quasi-λ-DB graphs known to the authors are complete bipartite graphs. Therefore, at first the following theorem is presented.

Theorem 1.1 *Let G is a connected edge quasi-λ-distance-balanced graph. Then G is complete bipartite graph $K_{m,n}$ with $m \neq n$.*

Theorem 1.2 *If G be an edge quasi-DB-graph with a cycle $C = e_1, ..., e_n$, then*

$$\textstyle\sum_{i \in C^+} m_u(e_i) = \sum_{i \in C^-} m_v(e_i).$$

Theorem 1.3 *Suppose that G is an edge quasi-DB graph and $C = e_1, ..., e_n$ is a cycle in G. Then*

$$2 \leqslant |C^+| \leqslant n-2 \text{ and also } 2 \leqslant |C^-| \leqslant n-2.$$

.

There exist equalities between sizes of C^+ and C^-, for complete bipartite edge quasi-DB graph, as presented in the next theorem.

Theorem 1.4 *If G be a complete bipartite edge quasi-DB- graph with a cycle $C = e_1, ..., e_{2n}$ of length $2n$, then*

$$|C^+| = |C^-| = n.$$

a *bridge* is an edge of a graph whose deletion increases the graph's number of connected components. Below is the proof of Theorem 1.5.

In the next theorem, we characterize the relatrion between quasi-DB graphs having a bridge and edge quasi-DB graphs.

Theorem 1.5 *Every connected quasi-DB graphs admitting a bridge are not edge quasi-DB graphs.*

This article is arranged as follows. In a graph G some significant consequences concerning partitions of edge set are explained according to distance from some specified closed walk in Section 2 (notice Theorem 2.5). Theorem 1.1, 1.2, 1.3 and 1.4 are proved in section 3. In Section 4 we verify Theorem 1.5.

2 Preliminaries

Here, we are going to express some important concepts that will be applied in the succeeding sections. At first, we must describe useful terms.

Definition 2.1 *Asuume that G is a graph and $C = e_1, ..., e_n$ is a walk in G. We determine a mapping $\varphi_c = E(G) \to \mathbb{Z}^n$ with $\varphi_C(e) = (x_1, ..., x_n)$, where $x_i = d_G(\acute{e}, v) - d_G(\acute{e}, u)$, for every $i \in \{1, ..., n\}$. For $i \in \{1, ..., n\}$, consider $A_i^+ = \acute{D}{+}_2^1(e) = \{(x_1, ..., x_n) \in \{-1, 0, 1\}^n | x_i = 1\}$ and also $A_i^- = \acute{D}{-}_1^2(e) = \{(x_1, ..., x_n) \in \{-1, 0, 1\}^n | x_i = -1\}$.*

For every walk C in a graph G and every edge $e \in E(G)$, it follows that $\varphi_C(e) \in \{-1, 0, 1\}^n$, therefore the next remark.

Remark 2.2 *Suppose that G is a graph with a walk $C = e_1, ..., e_n$. A partition of $E(G)$ is the set $\{\varphi^{-1}((x_1, ...x_n))(x_1, ...x_n) \in \{-1, 0, 1\}^n\}$.*

In the following lemma is stated a relation between the sets $M_u(e)$, $M_v(e)$ and also the mapping φ_C.

Lemma 2.3 *If G be a graph and $C = e_1, e_2, ..., e_n$ be a walk, then $M_u(e) = \varphi_C^{-1}(\acute{D}_2^+(e))$ and $M_v(e) = \varphi_C^{-1}(\acute{D}_2^-(e))$ for every $i \in \{1, ..., n\}$.*

Proof. Suppose that $\acute{e} \in E(G)$, and at first assume that $\acute{e} \in M_u(e)$. Then $d(\acute{e}, u) < d(\acute{e}, v)$. Besides, since $\acute{e} \in E(G)$, it implies that $d(\acute{e}, v) = d(\acute{e}, u) + 1$. Therefore, $d(\acute{e}, v) - d(\acute{e}, u) = 1$, thus $\acute{e} \in M_u(e)$. This concludes that $M_u(e) = \varphi_C^{-1}(\acute{D}_2^{+1}(e))$. By analogy, $M_v(e) = \varphi_C^{-1}(\acute{D}_2^{-1}(e))$. □

Lemma 2.4 *Let G be a graph and let $C = e_1, ..., e_n$ be a walk of length n in G. Suppose that $\acute{e} \in E(G)$ and $\varphi_C(\acute{e}) = (x_1, ...x_n)$. Then*

$$\textstyle\sum_{i=1}^n x_i = d(\acute{e}, u_{n+1}) - d(\acute{e}, u_1).$$

Proof. By the definition of the cycle we have,

$$C = u_1, e_1, v_1 = u_2, e_2, v_2 = u_3, ..., v_{n-1} = u_n, e_n, v_n = u_{n+1},$$

and according to the defined mapping φ we have $x_i = d(é, v) - d(é, u)$. Therefore, $\Sigma_{i=1}^{n} x_i = \Sigma_{i=1}^{n} d(é, v_i) - d(é, u_i) = \Sigma_{i=1}^{n} d(é, u_{i+1}) - d(é, u_i) = \Sigma_{i=1}^{n} d(é, u_{n+1}) - d(é, u)$. $\square$

Theorem 2.5 *Consider in a graph G there exists a closed walk $e_1, ..., e_n$. Then*

$$\sum_{i=1}^{n} m_u(e_i) = \sum_{i=1}^{n} m_v(e_i).$$

Proof. Suppose that a closed walk in G be $C = u_1, e_1, ..., e_n, v_n, u_1$, and $é$ is an arbitrary edge of G. Let $\varphi_C(é) = (x_1, ..., x_n)$. Consider that $é$ contributes k to the sum $\sum_{i=1}^{n} m_u(e_i)$ such that $é \in M_u(e)$. Hence there exists exactly k coordinates of $\varphi_C(é)$ equal to 1 by Lemma 2.3. Lemma 2.4 concludes $\sum_{i=1}^{n} x_i = 0$. If $x_i \in \{-1, 0, 1\}$, then it can be shown that there are also exactly k coordinates of $\varphi_C(é)$ equal to -1. Thus, $é$ contributes k to the sum $\sum_{i=1}^{n} m_v(é)$. for every $é \in E(G)$, the proof completed. $\square$

3 The structure of cycles in edge quasi-distance-balanced graphs

Suppose that G is an edge quasi-DB graph. In the introduction, it was clarified a fundamental result that all examples of edge quasi-λ-DB graphs recognized to the authors are complete bipartite graphs. Hence, we initially prove Theorem 1.1.

Proof of Theorem1.1. Let G be a edge quasi-λ-DB graph with $d = diam(G)$, and the edge set $\{e_1, e_2, ..., e_{2l+1}\}$ form an odd circle with length $2l+1$ such that $e = uv \in E(G)$ and

$$A_{ij} = \{e \in E(G) | d(e, e_{i+k}) = m_{jk}, \;\; m_{jk} = \{1, 2, ..., d\}, k = 0, 1, ..., 2l\},$$
$$2 \leqslant j \leqslant r,$$

such that $M_u(e_i) = (\bigcup_{j=1}^{r} A_{ij}) \cup \{e_{i+2l}\}$ and $M_v(e_i) = (\bigcup_{j=1}^{r} A_{(i+1)j}) \cup \{e_{i+2}\}$, in which the computations in indexes i are performed modulo $2l+1$ and some $r \in \mathbb{N}$. Taking $|A_{ij}| = a_{ij}$ for $i = 0, 1, ..., 2l$ and $j = 1, 2, ..., r$ and following the hopothesis there exist $s_i \in \{\pm 1\}$, $i = 0, 1, ..., 2l$, such that,

$$\sum_{j=1}^{r} a_{0j} + 1 = \lambda^{s_0}(\sum_{j=1}^{r} a_{1j} + 1),$$
$$\sum_{j=1}^{r} a_{1j} + 1 = \lambda^{s_1}(\sum_{j=1}^{r} a_{2j} + 1),$$
$$\vdots$$
$$\sum_{j=1}^{r} a_{(2l-1)_j} + 1 = \lambda^{s_{2l-1}}(\sum_{j=1}^{r} a_{(2l)j} + 1),$$
$$\sum_{j=1}^{r} a_{(2l)j} + 1 = \lambda^{s_{2l}}(\sum_{j=1}^{r} a_{0j} + 1).$$

Now, combining all $(2l+1)$ equations above follows that $\lambda^{\Sigma_{i=0}^{2l} s_i} = 1$, that is, $\Sigma_{i=0}^{2l} s_i = 0$. On the other hand,

$$s_i \in \{\pm 1\} \Rightarrow 1 \leqslant |\Sigma_{i=0}^{2l} s_i|,$$

which is a contradiction and so G has no circle.
For being complete this graph, we must show that every vertex of the first set is connected to every vertex of the second set, that is, vertices degree in every set of graph is equal with opposite set size. Then, for vertices $u_i \in A$ and $v_i \in B$, $i \in \{1, 2, ..., n\}$ with $|A| = m$ and $|B| = n$, we have $deg(u_i) = n$ and $deg(v_i) = m$. Now, let G is not complete. So, there is at least an edge $e_i = u_i v_i$, so that, $deg(u_1) < n$ and $deg(v_1) < m$. Since, G is edge quasi-λ-DB, we conclude that

$$\frac{m_{u_1}(e_1)}{m_{v_1}(e_1)} \in \{\lambda_1, \frac{1}{\lambda_1}\}.$$

Let $E(G) - \{e_1\} = \{e_i\}$, $i \in \{2, 3, ..., n\}$ we have,

$$\frac{m_{u_i}(e_i)}{m_{v_i}(e_i)} \in \{\lambda_i, \frac{1}{\lambda_i}\}.$$

Therefore, we have found at least two distinct $\{\lambda_1, \lambda_i\}$ for G. Hence it contradicts with the definition of being edge quasi-λ-DB. This completes the proof. $\square$

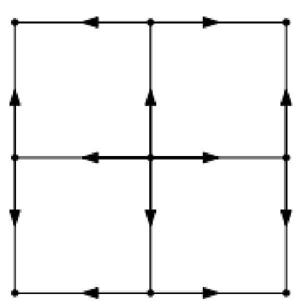

Figure 1: Oriented edges in $P_3 \square P_3$.

Let G be an edge quasi-λ-DB. Then, the edges of G are oriented naturally by $\vec{e}: u \to v$ if and only if $m_u^G(e) = \lambda m_v^G(e)$ or $m_v^G(e) = \lambda m_u^G(e)$. Bring in mind that we defined $C^+ = \{i \in \{1, ..., n\} | e_i \to e_{i+1}\}$ and $C^- = \{i \in \{1, ..., n\} | e_{i+1} \to e_i\}$, for a cycle $C = e_1, ..., e_n$, where $e_{n+1} = e_i$ and $e_0 = e_n$. We are now going to present the proof of Theorem 1.2.

Proof of Theorem 1.2. It is obvious to show that $C^+ \cap C^- = \phi$ and $C^+ \cup C^- = \{1, ..., n\}$. So Theorem 2.5 implies that

$$\sum_{i\in c^+} m_u(e_i) + \sum_{i\in c^-} m_u(e_i) = \sum_{i\in c^+} m_v(e_i) + \sum_{i\in c^-} m_v(e_i),$$

and accordingly

$$\sum_{i\in c^+} m_u(e_i) - \sum_{i\in c^-} m_v(e_i) = \sum_{i\in c^+} m_v(e_i) - \sum_{i\in c^-} m_u(e_i). \qquad (1)$$

It follows based on the definition of set c^+ and c^- that,

$$m_u(e_i) = \lambda m_v(e_i), \qquad (\forall i \in C^+) \qquad (2)$$

$$m_v(e_i) = \lambda m_u(e_i). \qquad (\forall i \in C^+) \qquad (3)$$

Combining equations (2) and (3) we have

$$\sum_{i\in C^+} m_u(e_i) - \sum_{i\in C^-} m_v(e_i) = \lambda(\sum_{i\in C^+} m_v(e_i) - \sum_{i\in C^-} m_u(e_i)). \qquad (4)$$

Applying equations $(1), (4)$ and the fact that $\lambda > 1$ the result follows. □

The following proof of the Theorem 1.3 gives us bounds on sizes of C^+ and C^-.

Proof of Theorem 1.3. At first, we suppose that $|C^-| = 0$. Thus, it is easy to see that $C^+ = \{1, ..., n\}$ and also $\sum_{i\in C^-} m_v(e_i) = 0$. Moreover, it follows that $\sum_{i\in C^+} m_u(e_i) = 0$ by Theorem 1.2. However, if $\acute{e} \in M_u(e)$, $\forall \acute{e} \in E(G)$, then we have the implication that $m_u(e) \geqslant 1$. Thus, it contradicts the fact that $\sum_{i\in C^+} m_u(e_i) = 0$. Therefore, it follows that $|C^-| \geqslant 1$. Now, let $|C^-| = 1$. Without loss of generality, let $C^- = \{n\}$ and $C^+ = \{1, ..., n-1\}$. We assert that

$$\sum_{i=1}^{n-1} m_u(e_i) - m_u(e_n) > 0. \qquad (5)$$

We are first going to prove that

$$M_u(e_n) \subseteq \bigcup_{i=1}^{n-1} M_u(e_i).$$

Assume that $\acute{e} \in M_u(e_1)$, and let $\varphi_c(\acute{e}) = (x_1, ..., x_n)$. By lemma 2.3, $x_n = -1$. Further, there are $j \in \{1, ..., n-1\}$, in which $x_j = 1$ by Lemma 2.4. We have $\acute{e}_j \in M_u(e_j)$ by Lemma 2.3. Therefore, $M_u(e_n) \subseteq \bigcup_{i=1}^{n-1} M_u(e_i)$. It is clear that

$\acute{e}_n \in M_u(e_n)$. It follows that a proper subset of $\bigcup_{i=1}^{n-1} M_u(e_i)$ is$M_u(e_n)$ which concludes equation (5). Hence, equation (5) contradicts Theorem 1.2. The recent contradiction yields that $|C^-| \geqslant 2$. Similarly, it implies that also $|C^+| \geqslant 2$. □

Corollary 3.1 *Suppose that G is an edge quasi-DB graph with a 4-cycle $C = e_1, e_2, e_3, e_4$. Then $|C^+| = |C^-| = 2$.*

It is indicated that for every cycle C in a complete bipartite quasi-DB graph, we have $|C^+| = |C^-|$ and it is stated in the proof of Theorem 1.4.

Proof of Theorem 1.4. To prove we suppose that G is a complete bipartite edge quasi-DB graph. Note that sets $M_u(e), M_v(e)$ and $M_0(e)$ form a partition of $E(G)$ for any edge $e = uv$ in G. Therefore, there exists constant T with $|E(G)| - m_0(e_i)/2 < T < |E(G)|$, such that

$$m_u(e_i) = T \ , \ m_v(e_i) = |E(G)| - (m_0(e_i) + T), \qquad (\forall i \in C^+),$$

and

$$m_u(e_i) = |E(G)| - (m_0(e_i) + T), \ m_v(e_i) = T, \qquad (\forall i \in C^-).$$

Inspired by Theorem 2.5 we obtain $\forall i \in \{1, ..., n\}$,

$$|C^+|.T + |C^-|.(|E(G)| - (m_0(e_i) + T)) = |C^+|.(|E(G)| - (m_0(e_i) + T)) + |C^-|.T,$$

which shows that

$$2(|C^+| - |C^-|).T = (|C^+| - |C^-|).|E(G)| - (|C^+| - |C^-|).m_0(e_i).$$

If $|C^+| - |C^-| \neq 0$, then it follows that $T = |E(G)|/2 - m_0(e_i)$, a contradiction. This contradiction implies that $|C^+| = |C^-|$. □

Remark 3.2 *If all edge quasi-DB graphs be complete bipartite, then we conclude that in all recognized edge quasi-DB graph, it holds $|C^+| = |C^-|$ for any cycle C.*

Now, the existance of cycles of length 5 in edge quasi-DB graphs will be investigated. A 5-cycle e_1, e_2, e_3, e_4, e_5 is called *central* if distance every edge in G be at most 2 from every edge on the 5-cycle, which means, $\forall e \in E(G)$, it holds $d(e, e_i) \leqslant 2, \forall i \in \{1, 2, 3, 4, 5\}$. We show that there does not any central 5-cycle in an edge quasi-DB graph in the following result.

Proposition 3.3 *Every graph G that has a central 5-cycle, is not edge quasi-DB.*

Proof. To prove we must suppose contrary. Hence G is edge quasi-DB and a central 5-cycle is induces by a cycle $C = e_1, e_2, e_3, e_4, e_5$ in G. To do this we assert that $M_v(e_i) \setminus \{e_{i+2}\} = M_u(e_{i+2}) \setminus \{e_i\}$, for every $\forall i \in \{1, 2, 3, 4, 5\}$. Consider $e \in M_v(e_i) \setminus \{e_{i+2}\}$. If $e = e_{i+1}$, then it is easy to see that $e \in M_u(e_{i+2}) \setminus \{e_i\}$. If $e \neq e_{i+1}$, such that there is central 5-cycle C in G, we have $d(e, e_{i+1}) = 1$. We know that G is complete bipartite hence, it implies that $d(e, e_i) = d(e, e_{i+2}) = 2$. It is now obvious that $e \in M_u(e_{i+2})\setminus\{e_i\}$ which follows that $M_v(e_i)\setminus\{e_{i+2}\} \subseteq M_u(e_{i+2})\setminus\{e_i\}$ and also the reverse inequality. Thus, we have

$$m_v(e_i) = m_u(e_{i+2}), \qquad \forall i \in (\{1, 2, 3, 4, 5\}). \tag{6}$$

If G is edge quasi-DB, then it yields that $m_u(e) = \lambda m_v(e)$, where $s_i \in \{\pm 1\}$. Multiplying these equalities and applying equation (6) this proves that

$$\prod_{i=1}^{5} m_u(e_i) = \lambda^{s_1+s_2+s_3+s_4+s_5} . \prod_{i=1}^{5} m_u(e_i).$$

We observe that $\lambda^{s_1+s_2+s_3+s_4+s_5} = 1$ since $m_u(e_i) \geqslant 1$ for each $\forall i \in \{1, 2, 3, 4, 5\}$. This is impossible, for $\lambda > 1$, and $s_1 + s_2 + s_3 + s_4 + s_5 \neq 0$. The proof is completed by the obtained contradiction. □

4 Bridges in quasi-DB graphs and edge quasi-DB graphs

The *minimum degree* of G is defined the degree of the vertex with the least number of edges incident to it and is denoted by $\delta(G)$. We would determine quasi-DB graphs with $\delta = 1$ in the next lemma.

Lemma 4.1 *Suppose that G is a connected quasi-DB graph. Every graph G with $\delta(G) = 1$ is isomorphic to a star.*

Proof. Assume that G is a connected quasi-DB graph and the degree of vertex u is 1 in G. Suppose that the only adjacent vertex of u is v. We know that $|W_{u,v}| = 1$ and $|W_{v,u}| = |V(G)| - 1$, which shows that $\lambda(G) = |V(G)| - 1$. Suppose that adjacent vertex of v is w different from u. If $|W_{v,w}| \geqslant 2$, then we observe that $|W_{v,w}| = |V(G)| - 1$ and also $|W_{w,v}| = 1$ which means that every adjacent vertex of v is a leaf in G, therefore G is isomorphic to star. □

We would determine the relation between quasi-DB graphs having a bridge and edge quasi-DB graphs. Recollect a *bridge* (or *cut-edge*) is an edge of a graph whose deletion increases the graph's number of connected components. Below is the proof of Theorem 1.5.

Proof of Theorem 1.5. Let G be a connected quasi-DB graph and $e_{12} = v_1v_2$ be a bridge in G. Let $\lambda = QDB(G)$. Consider the component containing v_i after removing bridge v_1v_2 be G_i for $i \in \{1, 2\}$. Without loss of generality we suppose that $|V(G_1)| \geqslant |V(G_2)|$. We have $W^G_{v_1,v_2} = V(G_1)$ and $W^G_{v_2,v_1} = V(G_2)$ which show that $\lambda = \frac{|V(G_1)|}{|V(G_2)|}$. Let $V(G_2) = \{v_2\}$. Then $\delta(G) = 1$, and based on Lemma 4.1 we can see that G is isomorphic to a star. If $x \in V(G_2)\setminus\{v_2\}$ then it is easily seen that $|W^G_{v_2,x}| \geqslant |V(G_1)| + 1$, and also $|W^G_{x,v_2}| \leqslant |V(G_2)| - 1$. Clearly, $|W^G_{v_2,x}| \geqslant |W^G_{x,v_2}|$, yielding that $|W^G_{v_2,x}| = \lambda|W^G_{x,v_2}|$, that is,

$$\lambda = \frac{|W^G_{v_2,x}|}{|W^G_{x,v_2}|} \geqslant \frac{|V(G_1)|+1}{|V(G_2)|+1} > \frac{|V(G_1)|}{|V(G_2)|} = \lambda,$$

a contradiction. A quasi-DB graph admitting a bridge is isomorphic to a star and it follows from the above contradiction. Now we are going to show that G is not edge quasi-DB graph. Let G be an edge quasi-DB graph. Let $e = uv$ be an edge in G such that v is a pendant vertex in G, so $deg(v) = 1$ but $deg(u) = |V(G)|-1 > 1$. Hence $0 = m_v(e) \neq \lambda m_u(e)$. It is a contradiction with $\lambda > 0$. Thus G is not an edge quasi-DB graph. □

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