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Original Research Paper

A Note On The Maximal Numerical Range Of The Bimultiplication $M_{2,A,B}$

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Abstract. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. For $A, B \in \mathcal{B}(\mathcal{H})$, define the bimultiplication operator $M_{2,A,B}$ on the class of Hilbert-Schmidt operators by $M_{2,A,B}(X) = AXB$. In this paper, we show that if B^*, the adjoint operator of B, is hyponormal, then

$$co\big(W_0(A)\,W_0(B)\big) \subseteq W_0(M_{2,A,B}),$$

where co stands for the convex hull and $W_0(.)$ denotes the maximal numerical range. If in addition, A is hyponormal, we show that

$$co\big(W_0(A)\,W_0(B)\big) = W_0(M_{2,A,B}).$$

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1 Introduction

Before stating the results, we set some notations and recall some results from the literature.
If L is a subset of the complex plane $\mathbb{C}$, we shall write $\overline{L}$ and $co(L)$ for the closure and the convex hull of L, respectively.
Let $\mathcal{A}$ be a C^*-algebra with unit I and let $\mathcal{A}'$ be its dual space. Define the state space of $\mathcal{A}$ by

$$\mathcal{S}(\mathcal{A}) = \{f \in \mathcal{A}' : f(I) = \|f\| = 1\}.$$

For $A \in \mathcal{A}$, the algebraic numerical range of A is given by

$$V(A) = \{f(A) : f \in \mathcal{S}(\mathcal{A})\}.$$

It is well known that $V(A)$ $(A \in \mathcal{A})$ is a bounded convex compact set and contains the convex hull of the spectrum $\sigma(A)$ of A; that is, $co\big(\sigma(A)\big) \subseteq V(A)$, this result follows at once from the corresponding properties of the set $\mathcal{S}(\mathcal{A})$. For more details, see [17].

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. For $A \in \mathcal{B}(\mathcal{H})$, the numerical range of A is defined as follows

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \ \|x\| = 1\}.$$

The most important properties of the numerical range are that it is convex and that its closure includes the spectrum of the operator. It is closed if $\dim(\mathcal{H}) < \infty$, but it is not always closed if $\dim(\mathcal{H}) = \infty$. The interested reader is referred to [3, 4, 10, 11] and references therein for a comprehensive account of the theory of the numerical range. There is another set which is close to the numerical range; that is the maximal numerical range. It was introduced by Stampfli [18] and is given as follows.

Definition 1.1. *For $A \in \mathcal{B}(\mathcal{H})$, the maximal numerical range $W_0(A)$ of A is given by*

$$W_0(A) = \{\lim_n \langle Ax_n, x_n\rangle : x_n \in \mathcal{H}, \ \|x_n\| = 1, \ \lim_n \|Ax_n\| = \|A\|\}.$$

It was shown in [18] that $W_0(A)$ is nonempty, closed, convex, and contained in the closure of the numerical range; $W_0(A) \subseteq \overline{W(A)}$. Let us recall some results we will need in the sequel. First, recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive, we often write $A \geq 0$ for brevity, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ (i.e, $W(A)$ is formed by non-negative real numbers). It is called hyponormal if $A^*A - AA^* \geq 0$ (i.e., $\|Ax\| \geq \|A^*x\|$ for all $x \in \mathcal{H}$). Familiar examples of hyponormal operators are normal operators, those A for which $A^*A = AA^*$. Recall also that if A is hyponormal, then $r(A) = \|A\|$ (see, [10]). Here, $r(A)$ is the spectral radius of A given by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. It is known (see for example, [16]), that for any operator $A \in \mathcal{B}(\mathcal{H})$

$$W_0(A) \cap C_A = \sigma_n(A),$$

where $C_A = \{z : |z| = \|A\|\}$ and $\sigma_n(A) = \{\lambda \in \sigma(A) : |\lambda| = \|A\|\}$. Therefore, since $W_0(A)$ is convex,

$$co(\sigma_n(A)) \subseteq W_0(A), \tag{1}$$

for any $A \in \mathcal{B}(\mathcal{H})$. In the case where A is hyponormal, we proved in [2] the following.

Theorem 1.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then*

$$W_0(A) = co\big(\sigma_n(A)\big).$$

Let $A \in \mathcal{A}$, we say that a linear functional $f \in \mathcal{S}(\mathcal{A})$ is maximal for A if $f(A^*A) = \|A\|^2$. For any $A \in \mathcal{A}$, we set

$$\mathcal{S}_{max}(A) := \{f \in \mathcal{S}(\mathcal{A}) : f \text{ is maximal for } A\}.$$

According to [1, Theorem 6.2.17], for every $A \in \mathcal{A}$ there exists a state s on $\mathcal{A}$ such that $s(A^*A) = \|A\|^2$. Therefore, $\mathcal{S}_{max}(A)$ is a nonempty subset of $\mathcal{S}(\mathcal{A})$ for all $A \in \mathcal{A}$.
Define the algebraic maximal numerical range of A as follows.

Definition 1.3. *Let $A \in \mathcal{A}$. The algebraic maximal numerical range of A is the set*

$$V_0(A) = \{f(A) : f \in \mathcal{S}_{max}(A)\}.$$

In [8], Fong established that $V_0(A)$ is a non-empty convex compact subset of $V(A)$ and in the case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$, he proved the following.

Theorem 1.4. *For any operator $A \in \mathcal{B}(\mathcal{H})$, $V_0(A) = W_0(A)$.*

Let $\mathcal{C}_2(\mathcal{H})$ denote the class of Hilbert-Schmidt operators on $\mathcal{H}$. Recall that $\mathcal{C}_2(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) : \|X\|_2 < \infty\}$, where $\|X\|_2^2 = tr(X^*X)$, $(X \in \mathcal{C}_2(\mathcal{H}))$, and tr stands for the usual trace functional. Recall also that $\mathcal{C}_2(\mathcal{H})$ is a Hilbert space with respect to the inner product $\langle X, Y\rangle_2 = tr(XY^*)$, $(X, Y \in \mathcal{C}_2(\mathcal{H})$ associated with the norm $\|.\|_2$. For $A \in \mathcal{B}(\mathcal{H})$, the left and right multiplications $L_{2,A}$ and $R_{2,A}$ are defined on $\mathcal{C}_2(\mathcal{H})$ by $L_{2,A}(X) = AX$ and $R_{2,A}(X) = XA$, respectively. For $A, B \in \mathcal{B}(\mathcal{H})$, the bimultiplication $M_{2,A,B}$ is defined on $\mathcal{C}_2(\mathcal{H})$ by $M_{2,A,B}(X) = (L_{2,A}R_{2,B})X = AXB$. The operators $L_{2,A}$ and $R_{2,A}$ are then particular bimultiplications since $L_{2,A} = M_{2,A,I}$ and $R_{2,A} = M_{2,I,A}$. Some results concerning the norm and the spectrum of $M_{2,A,B}$ are proved in [7] and [14]. Let $A, B \in \mathcal{B}(\mathcal{H})$. In [7], it is proved that

$$\|M_{2,A,B}\| = \|A\| \, \|B\| \,, \tag{2}$$

in particular, $\|L_{2,A}\| = \|R_{2,A}\| = \|A\|$. In [14], it is proved that

$$\sigma(M_{2,A,B}) = \sigma(A)\sigma(B). \tag{3}$$

Note that

$$\sigma_n(M_{2,A,B}) = \sigma_n(A)\sigma_n(B). \tag{4}$$

Our motivation stems from the following theorem which is proved in [13] for the subnormality case and is generalized in [5] for the hypnormality case.

Theorem 1.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If either A or B is hyponormal, then*

$$\overline{W(M_{2,A,B})} = \overline{co}\big(W(A)\,W(B)\big). \tag{5}$$

Our purpose is studying Equality (5) when replacing the numerical range by the maximal numerical range. In Section 2, we begin by showing that for any $A, B \in \mathcal{B}(\mathcal{H})$ with B^* is hyponormal

$$co\big(W_0(A)\,W_0(B)\big) \subseteq W_0(M_{2,A,B}). \tag{6}$$

Next, we show that if B^* is hyponormal and A has a normal dilation N on some complex Hilbert space $\mathcal{K}$ with $\sigma(N) \subseteq \sigma(A)$, then

$$W_0(M_{2,A,B}) = co\big(W_0(A)\, W_0(B)\big). \tag{7}$$

Recall that if A and B are bounded linear operators on complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, the operator B is said to be a dilation of the operator A (or A is dilated to B) if there is an isometry V from $\mathcal{H}$ to $\mathcal{K}$ such that $A = V^*BV$. Using the fact that any hyponormal operator $A \in \mathcal{B}(\mathcal{H})$ has a normal dilation N on some complex Hilbert space $\mathcal{K}$ with $\sigma(N) \subseteq \sigma(A)$ (see, [9]), we deduce that Equality (7) remains true if A and and B^* are hyponormal. At the end of Section 2, we give some remarks about the maximal numerical range of the generalized derivation $\delta_{2,A,B}$ on $\mathcal{C}_2(\mathcal{H})$, defined by $\delta_{2,A,B}(X) = AX - XB$ ($X \in \mathcal{C}_2(\mathcal{H})$).
Before closing this introduction, note that in the definition of a maximal linear functional for an element $A \in \mathcal{A}$ intervenes the adjoint A^* of A. Then, for $A, B \in \mathcal{B}(\mathcal{H})$, in the sequel we will need the adjoint $M^*_{2,A,B}$ of $M_{2,A,B}$. For the sake of completeness and for the convenience of the reader, we shall show here that $M^*_{2,A,B} = M_{2,A^*,B^*}$. Indeed, for any operators $A, B \in \mathcal{B}(\mathcal{H})$ and $X, Y \in \mathcal{C}_2(\mathcal{H})$, we have

$$\begin{aligned}
\langle M_{2,A,B}X, Y\rangle_2 &= \langle AXB, Y\rangle_2 \\
&= tr(AXBY^*) \\
&= tr(XBY^*A) \quad \text{(because } X,\ BY^* \in \mathcal{C}_2(\mathcal{H})) \\
&= tr(X(A^*YB^*)^*) \\
&= \langle X, A^*YB^*\rangle_2 \\
&= \langle X, M_{2,A^*,B^*}Y\rangle_2.
\end{aligned}$$

Note that in the third equality we used the fact that if $S, T \in \mathcal{C}_2(\mathcal{H})$, then for any $A \in \mathcal{B}(\mathcal{H})$, $tr(AST) = tr(STA)$, see, [6, Proposition 18.8 and Theorem 18.11].
From now on, all operators are bounded. We shall denote the set of all bounded operators on a Banach space E by $\mathcal{L}(E)$.

2 Main results

In this section, we consider Equality (7) when the operators A and B^* are hyponormal. For the moment, we establish Inclusion (6) whenever B^* is hyponormal.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If B^* is hyponormal, then*

$$co\big(W_0(A)\,W_0(B)\big) \subseteq W_0(M_{2,A,B}).$$

Proof. Let $(\lambda, \mu) \in W_0(A) \times W_0(B)$. There is two sequences (x_n) and (y_n) of unit vectors in $\mathcal{H}$ such that $\lambda = \lim_n \langle Ax_n, x_n\rangle$, $\lim_n \|Ax_n\| = \|A\|$, $\mu = \lim_n \langle Bx_n, y_n\rangle$ and $\lim_n \|By_n\| = \|B\|$. Recall that $x_n \otimes y_n \in \mathcal{C}_2(\mathcal{H})$, where $x_n \otimes y_n$ is the tensor product of x_n and y_n and define the map $f_{\lambda,\mu}$ on $\mathcal{L}(\mathcal{C}_2(\mathcal{H}))$ by

$$f_{\lambda,\mu}(T) = \lim_n \langle T(x_n \otimes y_n), x_n \otimes y_n\rangle_2.$$

It is clear that $f_{\lambda,\mu}$ is a linear functional on $\mathcal{L}(\mathcal{C}_2(\mathcal{H}))$. For any $T \in \mathcal{L}(\mathcal{C}_2(\mathcal{H}))$, since $\|x_n \otimes y_n\|_2 = \|x_n\|\,\|y_n\| = 1$, we have

$$\begin{aligned} |f_{\lambda,\mu}(T)| &= |\lim_n \langle T(x_n \otimes y_n), x_n \otimes y_n\rangle_2| \\ &\leq \|T\| \lim_n \|x_n \otimes y_n\|^2 \\ &= \|T\|, \end{aligned}$$

so, $\|f_{\lambda,\mu}\| \leq 1$. Since, $f_{\lambda,\mu}(I) = 1$, then $f_{\lambda,\mu} \in \mathcal{S}(\mathcal{C}_2(\mathcal{H}))$. On the other hand

$$\begin{aligned} f_{\lambda,\mu}(M^*_{2,A,B} M_{2,A,B}) &= f_{\lambda,\mu}(M_{2,A^*A,BB^*}) \\ &= \lim_n \langle M_{2,A^*A,BB^*}(x_n \otimes y_n), x_n \otimes y_n\rangle_2 \\ &= \lim_n \langle A^*A(x_n \otimes y_n)BB^*, x_n \otimes y_n\rangle_2 \\ &= \lim_n \langle A^*Ax_n \otimes BB^*y_n, x_n \otimes y_n\rangle_2 \\ &= \lim_n tr\,[(A^*Ax_n \otimes BB^*y_n)(y_n \otimes x_n)] \\ &= \lim_n \langle A^*Ax_n, x_n\rangle \langle y_n, BB^*y_n\rangle \\ &= \lim_n \|Ax_n\|^2 \|B^*y_n\|^2. \end{aligned}$$

Since B^* is hyponormal, then

$$\|B\|^2 = \|B^*\|^2 \geq \|B^* y_n\|^2 \geq \|B y_n\|^2,$$

and hence, $\lim_n \|B^* y_n\|^2 = \|B\|^2$. So, $f_{\lambda,\mu}(M^*_{2,A,B} M_{2,A,B}) = \|A\|^2 \|B\|^2$. By Equation (2), $\|M_{2,A,B}\| = \|A\| \|B\|$, then $f_{\lambda,\mu}(M^*_{2,A,B} M_{2,A,B}) = \|M_{2,A,B}\|^2$, that is $f_{\lambda,\mu} \in \mathcal{S}_{max}(M_{2,A,B})$. A similar calculation as above gives $f_{\lambda,\mu}(M_{2,A,B}) = \lambda\mu$. Consequently, $\lambda\mu \in W_0(M_{2,A,B})$ and it follows that $W_0(A)\, W_0(B) \subseteq W_0(M_{2,A,B})$. The convexity of $W_0(M_{2,A,B})$ gives us the desired result. $\square$

Corollary 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then $W_0(M_{2,A,A^*})$ contains a real segment (which may be a point).*

Proof. Using Equation (1) and Theorem 2.1 (taking $B = A^*$), we get

$$co\big(\sigma_n(A)\sigma_n(A^*)\big) \subseteq co\big(W_0(A)W_0(A^*)\big) \subseteq W_0(M_{2,A,A^*}).$$

Note that $\sigma_n(A)$ is non-empty because A is hyponormal. On the other hand, $\sigma_n(A^*) = \{\overline{\lambda} : \lambda \in \sigma_n(A)\}$, here $\overline{\lambda}$ denotes the conjugate of λ. Therefore, $W_0(M_{2,A,A^*})$ contains real numbers and the convexity then yields the result. $\square$

Corollary 2.3. *For any $A \in \mathcal{B}(\mathcal{H})$, $W_0(L_{2,A}) = W_0(A)$ and if A is normal, then $W_0(R_{2,A}) = W_0(A)$.*

Proof. From the previous theorem, we have $W_0(A) \subseteq W_0(M_{2,A,I}) = W_0(L_{2,A})$. Now, we show that $W_0(L_{2,A}) \subseteq W_0(A)$. Therefore, let $\lambda \in W_0(L_{2,A})$, then there is $f \in \mathcal{S}_{max}(L_{2,A})$ such that $\lambda = f(L_{2,A})$. Define the map h on $\mathcal{L}(\mathcal{C}_2(\mathcal{H})))$ by $h(T) = f(L_{2,T})$. We claim that $h \in \mathcal{S}_{max}(A)$. Everything but $h(A^*A) = \|A\|^2$ is obvious. So, $h(A^*A) = f(L_{2,A^*A}) = f(L_{2,A^*}L_{2,A}) = f(L^*_{2,A}L_{2,A}) = \|L_{2,A}\|^2 = \|A\|^2$. Since $\lambda = f(L_{2,A}) = h(A)$, it follows that $\lambda \in W_0(A)$ and hence $W_0(L_{2,A}) \subseteq W_0(A)$. The proof of the second part is similar taking into account that in this case A is normal. $\square$

Theorem 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that B^* is hyponormal. If A has a normal dilation N on some complex Hilbert space $\mathcal{K}$ with $\sigma(N) \subseteq \sigma(A)$, then*

$$W_0(M_{2,A,B}) = co\big(W_0(A)\, W_0(B)\big).$$

For the proof, we need the following auxiliary lemmas.

Lemma 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If A and B^* are hyponormal, then so is $M_{2,A,B}$.*

Proof. We first show that for any operators $C, D \in \mathcal{B}(\mathcal{H})$ such that $C \geq 0$ and $D \geq 0$, we have $M_{2,C,D} \geq 0$. Indeed, by hypothesis, $W(C)$ and $W(D)$ are positive, then so is $\overline{co}\big(W(C)W(D)\big)$. Since C is hyponormal we can apply Theorem 1.5 to obtain the result. Now, assume that A and B^* are hyponormal. We have

$$\begin{aligned} M_{2,A,B}^* M_{2,A,B} - M_{2,A,B} M_{2,A,B}^* &= M_{2,A^*A,BB^*} - M_{2,AA^*,B^*B} \\ &= M_{2,A^*A,BB^*} - M_{2,A^*A,B^*B} \\ &\quad + M_{2,A^*A,B^*B} - M_{2,AA^*,B^*B} \\ &= M_{2,A^*A,BB^*-B^*B} + M_{2,A^*A-AA^*,B^*B}. \end{aligned}$$

Since $A^*A \geq 0$, $BB^* - BB^* \geq 0$, $A^*A - AA^* \geq 0$ and $B^*B \geq 0$, the first case states that $M_{2,A,B}^* M_{2,A,B} - M_{2,A,B} M_{2,A,B}^* \geq 0$, that is $M_{2,A,B}$ is hyponormal. $\square$

Lemma 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$. If there exists a hyponormal operator H on some complex Hilbert space $\mathcal{K}$ and an isometry V from $\mathcal{H}$ to $\mathcal{K}$ such that $A = V^*HV$ and $\sigma(H) \subseteq \sigma(A)$, then*

$$W_0(A) = W_0(H).$$

Proof. We have $A = V^*HV$, then $\|A\| \leq \|V^*\| \, \|H\| \, \|V\| = \|H\|$. Since H is hyponormal and $\sigma(H) \subseteq \sigma(A)$, we derive that $\|H\| = r(H) \leq r(A) \leq \|A\|$. Consequently, $\|A\| = \|H\|$. Since H is hyponormal, by Theorem 1.2, $W_0(H) = co\big(\sigma_n(H)\big) \subseteq co\big(\sigma_n(A)\big) \subseteq W_0(A)$, (the first inclusion is due to the fact that $\sigma(H) \subseteq \sigma(A)$ and $\|H\| = \|A\|$). The proof of the inclusion $W_0(A) \subseteq W_0(H)$ is analogous to that of [2, Lemma 3.1]. $\square$

Proof of Theorem 2.4. By hypothesis, there is an isometry V from $\mathcal{H}$ to $\mathcal{K}$ such that $A = V^*NV$. A simple calculation gives $M_{2,A,B} = L_{2,V}^* M_{2,N,B} L_{2,V}$. Moreover, $L_{2,V}$ is an isometry and by Lemma 2.5, $M_{2,N,B}$ is hyponormal. On the other hand, since $\sigma(N) \subseteq \sigma(A)$, by Equation (3), we have $\sigma(M_{2,N,B}) = \sigma(N)\sigma(B) \subseteq \sigma(A)\sigma(B) = \sigma(M_{2,A,B})$.

Then, according to Lemma 2.6, we obtain $W_0(M_{2,A,B}) = W_0(M_{2,N,B})$. Therefore,

$$\begin{aligned} W_0(M_{2,A,B}) &= W_0(M_{2,N,B}) \\ &= co\big(\sigma_n(M_{2,N,B})\big) \text{ (by Theorem 1.2)} \\ &= co\big(\sigma_n(N)\sigma_n(B)\big) \text{ (by Equation (4))} \\ &\subseteq co\big(W_0(N)\,W_0(B)\big) \text{ (by Equation (1))} \\ &= co\big(W_0(A)\,W_0(B)\big) \text{ (by Lemma 2.6).} \end{aligned}$$

We conclude by Theorem 2.1 that $W_0(M_{2,A,B}) = co\big(W_0(A)\,W_0(B)\big)$.

Since any hyponormal operator $A \in \mathcal{B}(\mathcal{H})$ has a normal dilation N on some complex Hilbert space $\mathcal{K}$ with $\sigma(N) \subseteq \sigma(A)$ (see, [9]), we have the following corollary.

Corollary 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If A and B^* are hyponormal, then*

$$W_0(M_{2,A,B}) = co\big(W_0(A)\,W_0(B)\big).$$

Corollary 2.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If A is hyponormal and B is normal, then*

$$W_0(M_{2,A,B}) = co\big(W_0(A)\,W_0(B)\big).$$

Proof. Just notice that if B is normal, so is B^*. Then, B^* is hyponormal and the result follows from the previous corollary. □

Remark 2.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$. There is no condition on A and B for the inclusion $\overline{co}\big(W(A)\,W(B)\big) \subseteq \overline{W(M_{2,A,B})}$, see [15]. However, there is one condition (that is, B^* is hyponormal) for having Inclusion (6). On the other hand there is one condition (that is, either A or B is hyponormal) for Equality (5) while there are two conditions for Equality (7). In other word, one more condition for the case of the maximal numerical range. This is due to the fact that the numerical range is defined by only one condition while the maximal numerical range is defined by two conditions. We may see this in the proof of Theorem 2.1, when we show the second condition concerning the maximal linear functional for $M_{2,A,B}$. According to this point of view, we therefore estimate that two conditions but not one will be necessary to have Equality (7).*

We end this paper by the following remark concerning the maximal numerical range of $\delta_{2,A,B}$.

Remark 2.10. *It is proved in [12] that for any operators $A, B \in \mathcal{B}(\mathcal{H})$ we have*

$$W(\delta_{2,A,B}) = W(A) - W(B).$$

Unfortunately, this identity does not hold in general for the maximal numerical range. Moreover, there may be no inclusion relationship between the subsets $W_0(\delta_{2,A,B})$ and $W_0(A) - W_0(B)$ as is shown in the following example. Let A be the operator on the complex Hilbert space $\mathcal{H} = \mathbb{C}^2$ represented by

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $\delta_{2,A,I} = L_{2,A-I}$, then by Corollary 2.3, $W_0(\delta_{2,A,I}) = W_0(A-I)$. Since $A-I$ is hyponormal, then by Theorem 1.2, $W_0(A-I) = co\big(\sigma_n(A-I)\big) = \{-1\}$. For the same reason as above, $W_0(A) = co\big(\sigma_n(A)\big) = \{1\}$ and $W_0(A) - W_0(I) = \{0\}$.

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