Journal of Mathematical Extension
Vol. 16, No. 7, (2022) (10)1-18
URL: https://doi.org/10.30495/JME.2022.1907
ISSN: 1735-8299
Original Research Paper

Refinements of Numerical Radius Inequalities Via Specht's Ratio

Y. Khatib*
Mashhad Branch, Islamic Azad University

M. Hassani
Mashhad Branch, Islamic Azad University

M. Amyari
Mashhad Branch, Islamic Azad University

Abstract. We present some new numerical radius inequalities of Hilbert space operators. We improve and generalize some inequalities with respect to Specht's ratio. Let A and B be two positive invertible operators on a Hilbert space H and let X be a bounded operator on H. Then

$$\omega((A\natural B)X) \leq \frac{1}{2S(\sqrt{h})}\|X^*BX + A\|, \quad (h > 0,\, h \neq 1)$$

where $\|\cdot\|$, $\omega(\cdot)$, $S(\cdot)$, and $\natural$ denote the usual operator norm, numerical radius, the Specht's ratio, and the operator geometric mean, respectively.

AMS Subject Classification: 47A12; 47A30.
Keywords and Phrases: positive operators, normalized positive linear map, numerical radius, Specht's ratio.

Received: February 2021; Accepted: August 2021
*Corresponding Author

1 Introduction

Suppose that $(\mathcal{H}, \langle ., . \rangle)$ is a complex Hilbert space and that $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. We recall some definitions and concepts from [11].

An operator A in $B(\mathcal{H})$ is positive, denoted by $A \geq 0$, if A is self-adjoint ($A = A^*$) and $\langle Ax, x \rangle \geq 0$ for every $x \in \mathcal{H}$; equivalently, A is positive if and only if $A = B^*B$ for some operator $B \in B(\mathcal{H})$. In particular, for some scalars m and M, we write $mI \leq A \leq MI$ if $m \leq \langle Ax, x \rangle \leq M$ for every unit vector $x \in \mathcal{H}$, where I stands for the identity operator of $\mathcal{B}(\mathcal{H})$. The absolute value of A is denoted by $|A| = (A^*A)^{\frac{1}{2}}$. Note that for a self-adjoint operator A, $mI \leq A \leq MI$ if and only if $sp(A) \subset [m, M]$. Also the set of all positive invertible operators is denoted by $\mathcal{B}^+(\mathcal{H})$.

For an operator $A \in \mathcal{B}(\mathcal{H})$, the usual operator norm is defined by $\|A\| = \sup \|Ax\|$ for every unit vector $x \in \mathcal{H}$ and the numerical radius of A is given by $\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}$. The numerical radius satisfies

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1}$$

The second inequality in (1) has been improved in [9, Theorem 1] as follows:

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \tag{2}$$

for every operator $A \in \mathcal{B}(\mathcal{H})$. The left hand of inequality (2) was extended in [6, Theorem 1] as follows:

$$\omega^r(A) \leq \frac{1}{2}\left\||A|^{2r\nu} + |A^*|^{2r(1-\nu)}\right\|, \quad r \geq 1, 0 < \nu < 1. \tag{3}$$

Heydarbeygi and Amyari in [7, Theorem 2.2] improved the left hand of inequality (3) by an improvement of Hölder-McCarthy's inequality. Dragomir in [2, Theorem 1], proved the following inequality by the product of two operators

$$\omega^r(B^*A) \leq \frac{1}{2}\left\||A|^{2r} + |B|^{2r}\right\|, \quad r \geq 1. \tag{4}$$

Sababheh and Moradi in [12, Corollary 2.1] and [12, Proposition 2.2], respectively, proved the following inequalities:

$$\omega^r(A) \leq \Big\| \int_0^1 (t|A| + (1-t)|A^*|)^r dt \Big\| \leq \frac{1}{2}\big\| |A|^r + |A^*|^r \big\|. \tag{5}$$

and

$$\omega^r(B^*A) \leq \Big\| \int_0^1 (t|A|^2 + (1-t)|B|^2)^r dt \Big\| \leq \frac{1}{2}\big\| |A|^{2r} + |B|^{2r} \big\|. \tag{6}$$

In section 3, we improve the left hand of inequalities (5) and (6).

Let $A \in \mathcal{B}^+(\mathcal{H})$ and let B be a positive operator in $B(\mathcal{H})$. The operator ν-weighted geometric mean of A and B for $\nu \in [0,1]$ is defined by

$$A \natural_\nu B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}.$$

Recall that a linear map $\varphi : B(\mathcal{H}) \to B(\mathcal{K})$ is positive, if it preserves positivity. It is normalized if $\varphi(I_{\mathcal{H}}) = I_{\mathcal{K}}$. The Specht's ratio [4, 14] was defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \quad (h \neq 1)$$

for a positive real number h, and it has some properties as follows:

(i) $S(1) = 1$ and $S(h) = S(\frac{1}{h}) > 1$ for $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$.

(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

Lemma 1.1. *[5, Theorem 1] For $a, b > 0$ and $\nu \in [0,1]$, it follows that $(1-\nu)a + \nu b \geq S((\frac{b}{a})^r)a^{1-\nu}b^\nu$, where $r = \min\{\nu, 1-\nu\}$ and $S(.)$ is the Specht's ratio.*

Theorem 1.2. *[5, Theorem 2] Let A and B be two positive operators and let m, m', M, M' be positive real numbers satisfying the following conditions (i) or (ii):*

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$,

(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$,

with $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$\begin{aligned}(1-\nu)A + \nu B \geq S(h^r)A\natural_\nu B &\geq A\natural_\nu B \\ &\geq S(h^r)\{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1} \\ &\geq \{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1},\end{aligned}$$

where $\nu \in [0,1]$, $r = \min\{\nu, 1-\nu\}$, *and* $S(.)$ *is the Specht's ratio.*

Remark 1.3. Note that if $A = aI$, $B = bI$, $\nu = \frac{1}{2}$, and $r = \frac{1}{2}$ in Theorem 1.2, then

$$S(\sqrt{h})\sqrt{ab} \leq \tfrac{a+b}{2},$$

where $S(.)$ is the Specht's ratio.

2 Main Results

In this section, we state some useful lemmas that we need them for improving and generalizing some inequalities. The first lemma is a generalized form of the mixed Schwarz inequality, which was proved by Kittaneh [8, Theorem 1].

Lemma 2.1. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be an operator in* $B(\mathcal{H})$ *and let* f *and* g *be nonnegative continuous functions on* $[0, \infty)$ *satisfying* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|$$

for all $x, y \in \mathcal{H}$.

The well-known Hermite–Hadamard inequalities state that for a convex function $f : J \to \mathbb{R}$, it follows that

$$f\Big(\frac{a+b}{2}\Big) \leq \int_0^1 f(ta + (1-t)b)dt \leq \frac{f(a)+f(b)}{2}, \tag{7}$$

for every a, b in real interval J.

Let f be a convex function on a real interval J containing sp(A), where A is a self-adjoint operator. Then for every unit vector $x \in \mathcal{H}$, the inequality

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{8}$$

is an operator version of the Jensen's inequality due to Mond and Pečarić [10, Theorem 1].

The second lemma in this section is a direct result of [1, Theorem 2.3].

Lemma 2.2. *Let f be a nonnegative increasing convex function on $[0,\infty)$ and let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\|f((1-\nu)A+\nu B)\| \leq \|(1-\nu)f(A)+\nu f(B)\|$$

for every $0 \leq \nu \leq 1$.

The third lemma is useful at the end of section 2.

Lemma 2.3. *[12, Proposition 2.1] Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital positive linear map, let $A \in B(\mathcal{H})$, and let $f : [0,\infty) \to [0,\infty)$ be an increasing operator convex function. Then*

$$f(\omega^2(\varphi(A))) \leq \Big\|\varphi\Big(\int_0^1 f(t|A|^2+(1-t)|A^*|^2)dt\Big)\Big\|.$$

In particular, for any $1 \leq r \leq 2$, it follows that

$$\omega^{2r}(A) \leq \Big\|\int_0^1 (t|A|^2+(1-t)|A^*|^2)^r dt\Big\|.$$

Now, we want to improve inequality (2) with respect to Specht's ratio. We state the following lemma that contains a norm inequality for sums of positive operators, which is sharper than the triangle inequality.

Lemma 2.4. *[9, Lemma 3] Let $A, B \in B(\mathcal{H})$ be positive operators; then*

$$\|A+B\| \leq \frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{(\|A\|+\|B\|)^2+4\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|^2}\Big).$$

Let $A \in B(\mathcal{H})$. Kittaneh also proved that $|\langle Ax,y\rangle| \leq \langle |A|x,x\rangle^{\frac{1}{2}}\langle |A^*|y,y\rangle^{\frac{1}{2}}$ for all $x, y \in \mathcal{H}$ (see [9, Lemma 1]) and $\|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \|AB\|^{\frac{1}{2}}$ for positive operators $A, B \in B(\mathcal{H})$ (see [9, Lemma 2]).

Theorem 2.5. *Let A in $B(\mathcal{H})$ and let the positive real numbers m, m', M, M' satisfy one of the following conditions:*

(i) $0 < m'I \leq |A| \leq mI \leq MI \leq |A^*| \leq M'I$,

(ii) $0 < m'I \leq |A^*| \leq mI \leq MI \leq |A| \leq M'I$.

with $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$\omega(A) \leq \frac{1}{2S(\sqrt{h})}(\|A\| + \|A^2\|^{\frac{1}{2}}) \tag{9}$$

for an operator $A \in \mathcal{B}(\mathcal{H})$.

Proof. By using [9, Lemma 1] and Remark 1.3, we get

$$\begin{aligned} |\langle Ax, x\rangle| \leq \langle |A|x, x\rangle^{\frac{1}{2}} \langle |A^*|x, x\rangle^{\frac{1}{2}} &\leq \frac{1}{2S(\sqrt{h})}(\langle |A|x, x\rangle + \langle |A^*|x, x\rangle) \\ &= \frac{1}{2S(\sqrt{h})}\langle (|A| + |A^*|)x, x\rangle \end{aligned}$$

for each $x \in \mathcal{H}$. Then

$$\begin{aligned} \omega(A) &\leq \frac{1}{2S(\sqrt{h})} \sup\{|\langle (|A| + |A^*|)x, x\rangle| : x \in \mathcal{H}, \|x\| = 1\} \qquad (10) \\ &= \frac{1}{2S(\sqrt{h})} \||A| + |A^*|\|. \end{aligned}$$

Since $\||A|\| = \||A^*|\| = \|A\|$ and $\||A||A^*|\| = \|A^2\|$, from inequality (10) and [9, Lemma 2] for positive operators $|A|$ and $|A^*|$, we reach

$$\||A| + |A^*|\| \leq \|A\| + \|A^2\|^{\frac{1}{2}}. \tag{11}$$

Hence, inequalities (10) and (11) imply inequality (9). □

We use Lemma 2.2 to prove the following theorem.

Theorem 2.6. *Let* $A, B, X \in \mathcal{B}(\mathcal{H})$, *let the continuous functions* f *and* g *be non-negative functions on* $[0, \infty)$ *satisfying the relation* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$, *and let* k *be a non-negative increasing convex function on* $[0, \infty)$. *Also let the positive real numbers* m, m', M, M' *satisfy one of the following conditions:*

(i) $0 < m' \leq \langle B^* f^2(|X|)Bx, x\rangle \leq m < M \leq \langle A^* g^2(|X^*|)Ax, x\rangle \leq M'$

(ii) $0 < m' \leq \langle A^* f^2(|X|)Ax, x\rangle \leq m < M \leq \langle B^* g^2(|X^*|)Bx, x\rangle \leq M'$,

with $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$k(\omega(A^*XB)) \leq \frac{1}{2S(\sqrt{h})}\big\|k(B^*f^2(|X|)B) + k(A^*g^2(|X^*|)A)\big\|,$$

where $S(.)$ *is the Specht's ratio.*

Proof. Using Lemma 2.1, we get

$$|\langle A^*XBx, x\rangle| = |\langle XBx, Ax\rangle| \leq \sqrt{\langle B^*f^2(|X|)Bx, x\rangle\langle A^*g^2(|X^*|)Ax, x\rangle}. \quad (12)$$

Now Remark 1.3 implies that

$$\begin{aligned}
&\sqrt{\langle B^*f^2(|X|)Bx, x\rangle\langle A^*g^2(|X^*|)Ax, x\rangle} \\
&\quad \leq \frac{1}{2S(\sqrt{h}))}\Big(\langle B^*f^2(|X|)Bx, x\rangle + \langle A^*g^2(|X^*|)Ax, x\rangle\Big) \\
&\quad = \frac{1}{2S(\sqrt{h})}\Big(\langle (B^*f^2(|X|)B + A^*g^2(|X^*|)A)x, x\rangle\Big).
\end{aligned}$$

It follows from the last inequality and (12) that

$$|\langle A^*XBx, x\rangle| \leq \tfrac{1}{2S(\sqrt{h})}\Big(\big\langle (B^*f^2(|X|)B + A^*g^2(|X^*|)A)x, x\big\rangle\Big).$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|$=1, we reach

$$\omega(A^*XB) \leq \tfrac{1}{2S(\sqrt{h})}\|B^*f^2(|X|)B + A^*g^2(|X^*|)A\|.$$

Also,

$$\begin{aligned}
k(\omega(A^*XB)) &\leq k\Big(\frac{2}{2S(\sqrt{h})}\Big\|\frac{B^*f^2(|X|)B + A^*g^2(|X^*|)A}{2}\Big\|\Big) \\
&\leq \frac{1}{S(\sqrt{h})}k\Big(\Big\|\frac{B^*f^2(|X|)B + A^*g^2(|X^*|)A}{2}\Big\|\Big) \quad (13) \\
&\leq \frac{1}{S(\sqrt{h})}\Big\|k\Big(\frac{B^*f^2(|X|)B + A^*g^2(|X^*|)A}{2}\Big)\Big\| \quad (14) \\
&\leq \frac{1}{2S(\sqrt{h})}\Big\|k\Big(B^*f^2(|X|)B\Big) + k\Big(A^*g^2(|X^*|)A\Big)\Big\|. \\
&\qquad \text{(by Lemma 2.2)}
\end{aligned}$$

Note that inequalities (13) and (14) follow from that k is a non-negative increasing convex function and that $\frac{1}{S(\sqrt{h})} \leq 1$. Also by Jensen's inequality, we have $k(\|Y\|) = k(\sup_{\|x\|=1}\langle Yx, x\rangle) = \sup_{\|x\|=1} k(\langle Yx, x\rangle) \leq \sup_{\|x\|=1}\langle k(Y)x, x\rangle = \|k(Y\|$ for each positive operator $Y \in \mathcal{B}(\mathcal{H})$. □

Shebrawi and Albadawi [13, Remark 2.10], for each $A, B, X \in \mathcal{B}(\mathcal{H})$, proved the following general numerical radius inequality:

$$\omega^r(A^*XB) \leq \frac{1}{2}\left\|\left(A^*|X^*|A\right)^r + \left(B^*|X|B\right)^r\right\|, \quad r \geq 1. \tag{15}$$

From inequality (15) and Theorem 2.6, we obtain the following inequalities.

Corollary 2.7. *We know that $k(t) = t^r$, $r \geq 1$, is an increasing convex function on $[0, \infty)$. Let the assumption of Theorem 2.6 hold.*

(i) If $0 < m'I < B^*|X|B \leq mI < MI \leq A^*|X^*|A < M'I$ or $0 < A^*|X|A \leq mI < MI \leq B^*|X|B$, for positive real numbers m, m', M, M', then

$\omega^r(A^*XB) \leq \frac{1}{2S(\sqrt{h})}\left\|\left(A^*|X^*|A\right)^r + \left(B^*|X|B\right)^r\right\|, \quad r \geq 1,$

which improves inequality (15).

(ii) If $X = I$ holds in conditions (i), then

$$\omega^r(A^*B) \leq \frac{1}{2S(\sqrt{h})}\left\||A|^{2r} + |B|^{2r}\right\|,$$

which improves inequality (4).

(iii) If $A = B = I$ holds in conditions (i), then

$$\omega^r(X) \leq \frac{1}{2S(\sqrt{h})}\left\||X^*|^r + |X|^r\right\|,$$

where $S(.)$ is the Specht's ratio.

Definition 2.8. Let $A \in \mathcal{B}^+(\mathcal{H})$, let B be a self-adjoint operator, and let f be a continuous function on a real interval J containing $sp(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$. Then by using the continuous functional calculus, the f-connection is denoted by σ_f and defined as

$$A\sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

Note that the above definition for the functions t^ν and $(1-\nu)+\nu t$, where $\nu \in [0,1]$, leads to the operator ν-weighted geometric mean and the operator ν-weighted arithmetic mean, respectively. Now we introduce the following theorem relative to the numerical radius inequality concerning f-connection of operators.

Theorem 2.9. *Suppose that $A \in \mathcal{B}^+(\mathcal{H})$, that B is self-adjoint, that $X \in \mathcal{B}(\mathcal{H})$, and that f is a continuous function on the real interval J containing $sp(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions:*

(i) $0 < m'I \leq X^*A^{\frac{1}{2}}f^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \leq mI \leq MI \leq A \leq M'$,

(ii) $0 < mI' \leq A \leq mI \leq MI \leq X^*A^{\frac{1}{2}}f^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \leq M'I$,

with $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$\omega((A\sigma_f B)X) \leq \frac{1}{2S(\sqrt{h})}\left\|X^*A^{\frac{1}{2}}f^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}X + A\right\|, \tag{16}$$

where $S(.)$ is the Specht's ratio.

Proof. For every vector $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$\begin{aligned}
|\langle (A\sigma_f B)Xx, x\rangle| &= |\langle A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}Xx, x\rangle| \\
&= |\langle f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}Xx, A^{\frac{1}{2}}x\rangle| \\
&\leq \left\|f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}Xx\right\|\left\|A^{\frac{1}{2}}x\right\| \\
&= \sqrt{\langle f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}Xx, f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}Xx\rangle\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\rangle} \\
&= \sqrt{\langle X^*A^{\frac{1}{2}}f^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}Xx, x\rangle\langle Ax, x\rangle} \\
&\leq \frac{1}{2S(\sqrt{h})}\langle X^*A^{\frac{1}{2}}f^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}X + Ax, x\rangle.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|$=1, produces the desired inequality (16). □

Putting $f(t) = \sqrt{t}$ in Theorem 2.9, we get the following corollary.

Corollary 2.10. *Suppose that A, B, X are operators in $\mathcal{B}(\mathcal{H})$ such that $A \in \mathcal{B}^+(\mathcal{H})$ and B is positive. If one of the conditions (i) or (ii) of*

Theorem 2.9 is satisfied, then

$$\omega((A\natural B)X) \leq \frac{1}{2S(\sqrt{h})}\|X^*BX + A\|,$$

where $S(.)$ is the Specht's ratio.

Dragomir [3, section 2] introduced some inequalities for numerical radius inequality related to the product operators. In the following theorems, we improve some inequalities for numerical radius inequality related to the product operators by Specht's ratio.

Recall that if $X \in \mathcal{B}(\mathcal{H})$ is a positive operator, then $(z, y)_X = \langle Xz, y\rangle$ defines an inner product on the Hilbert space $\mathcal{H}$ and also it follows from [3, Corollary 1] that

$$\frac{1}{2}\Big(\|x\|_X\|z\|_X + |(x,z)_X|\Big)\|y\|_X^2 \geq |(x,y)_X(y,z)_X| \tag{17}$$

for each $x, y, z \in \mathcal{H}$ and $(x,x)_X = \|x\|_X^2$.

Theorem 2.11. *Suppose that $A, B, C, X \in \mathcal{B}(\mathcal{H})$, where X is a positive operator. Let the positive real numbers m, m', M, M' satisfy one of the following conditions:*

(i) $0 < m'I \leq A^*XA \leq mI \leq MI \leq B^*XB \leq M'I$

(ii) $0 < m'I \leq B^*XB \leq mI \leq MI \leq A^*XA \leq M'I$,

with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then for every $x \in \mathcal{H}$, it follows that

$$\begin{aligned}\big\|A^*XCx\big\|\big\|B^*XCx\big\| \leq \frac{1}{2}\big\|X^{\frac{1}{2}}Cx\big\|^2\Big[&\frac{1}{2S(\sqrt{h})}\Big(\big\|X^{\frac{1}{2}}A\big\|^2 + \big\|X^{\frac{1}{2}}B\big\|^2\Big)\\ &+\big\|B^*XA\big\|\Big]\end{aligned} \tag{18}$$

and

$$\omega(C^*XAB^*XC) \leq \frac{1}{2}\big\|X^{\frac{1}{2}}C\big\|^2\Big[\frac{1}{2S(\sqrt{h})}\Big(\|X^{\frac{1}{2}}A\|^2 + \|X^{\frac{1}{2}}B\|^2\Big) + \|B^*XA\|\Big], \tag{19}$$

where $S(.)$ is the Specht's ratio.

Proof. Since X is positive, from inequality (17), we have

$$\frac{1}{2}\big(\|z\|_X\|y\|_X+|(z,y)_X|\big)\|x\|_X^2 \geq |(z,x)_X(x,y)_X|,$$

where $x,y,z\in\mathcal{H}$. Therefore

$$\frac{1}{2}\Big[\langle Xz,z\rangle^{\frac{1}{2}}\langle Xy,y\rangle^{\frac{1}{2}}+|\langle Xz,y\rangle|\Big]\langle Xx,x\rangle \geq |\langle Xz,x\rangle\langle Xx,y\rangle|.$$

Replacing z by Az, y by By, and x by Cx in the previous inequality, we get that

$$\begin{aligned}&\frac{1}{2}\Big[\langle XAz,Az\rangle^{\frac{1}{2}}\langle XBy,By\rangle^{\frac{1}{2}}+|\langle XAz,By\rangle|\Big]\langle C^*XCx,x\rangle\\ &\geq |\langle z,A^*XCx\rangle\langle x,C^*XBy\rangle|.\end{aligned} \tag{20}$$

Taking the supremum over $z,y\in\mathcal{H}$ with $\|z\|=\|y\|=1$, leads to

$$\begin{aligned}&\big\|A^*XCx\big\|\big\|B^*XCx\big\| \qquad (21)\\
&=\sup_{\|z\|=1}|\langle z,A^*XCx\rangle|\sup_{\|y\|=1}|\langle y,B^*XCx\rangle|\\
&=\sup_{\|z\|=1}|\langle z,A^*XCx\rangle|\sup_{\|y\|=1}|\langle B^*XCx,y\rangle|\\
&=\sup_{\|z\|=1}|\langle z,A^*XCx\rangle|\sup_{\|y\|=1}|\langle x,C^*XBy\rangle|\\
&=\sup_{\|z\|=\|y\|=1}\Big\{|\langle z,A^*XCx\rangle\langle x,C^*XBy\rangle|\Big\}\\
&\leq \frac{1}{2}\langle C^*XCx,x\rangle\sup_{\|z\|=\|y\|=1}\Big(\langle A^*XAz,z\rangle^{\frac{1}{2}}\langle B^*XBy,y\rangle^{\frac{1}{2}}+|\langle B^*XAz,y\rangle|\Big)\\
&\leq \frac{1}{2}\langle C^*XCx,x\rangle\Big[\frac{1}{2S(\sqrt{h})}\Big(\langle A^*XAz,z\rangle+\langle B^*XBy,y\rangle\Big)+|\langle B^*XAz,y\rangle|\Big]\\
&\leq \frac{1}{2}\langle C^*XCx,x\rangle\Big(\sup_{\|z\|=1}\frac{1}{2S(\sqrt{h})}\langle A^*XAz,z\rangle+\sup_{\|y\|=1}\frac{1}{2S(\sqrt{h})}\langle B^*XBy,y\rangle\\
&\quad+\sup_{\|z\|=\|y\|=1}|\langle B^*XAz,y\rangle|\Big)\\
&=\frac{1}{2}\langle C^*XCx,x\rangle\Big[\frac{1}{2S(\sqrt{h})}\Big(\|A^*XA\|+\|B^*XB\|\Big)+\|B^*XA\|\Big]\end{aligned}$$

for every $x \in \mathcal{H}$. On the other hand, since

$$A^*XA = |X^{\frac{1}{2}}A|^2,\ B^*XB = |X^{\frac{1}{2}}B|^2, C^*XC = |X^{\frac{1}{2}}C|^2, \tag{22}$$

by utilizing (21), we reach inequality (18). Now, we have

$$|\langle C^*XBA^*XCx, x\rangle| \leq \left\|A^*XCx\right\|\left\|B^*XCx\right\| \text{ (Schwarz inequality).}$$

By using inequality (18), we get

$$\begin{aligned}|\langle C^*XBA^*XCx, x\rangle| \leq \frac{1}{2}\left\|X^{\frac{1}{2}}Cx\right\|^2\Big[\frac{1}{2S(\sqrt{h})}\Big(\|X^{\frac{1}{2}}A\|^2 \\ + \|X^{\frac{1}{2}}B\|^2\Big) + \|B^*XA\|\Big]\end{aligned} \tag{23}$$

for every $x \in \mathcal{H}$. Taking the supremum over $x \in \mathcal{H}$, with $\|x\|$=1 in (23) implies

$$\omega(C^*XBA^*XC) \leq \frac{1}{2}\left\|X^{\frac{1}{2}}C\right\|^2\Big[\frac{1}{2S(\sqrt{h})}\Big(\|X^{\frac{1}{2}}A\|^2+\|X^{\frac{1}{2}}B\|^2\Big)+\|B^*XA\|\Big]. \tag{24}$$

By inequality (24), since $\omega(C^*XBA^*XC) = \omega(C^*XAB^*XC)$, we reach the desired inequality (19). Note that $\|x\| = \sup\limits_{\|z\|=1} |\langle z, x\rangle|$ for all $x \in \mathcal{H}$.

□

Theorem 2.12. *Suppose that $A, B, C, X \in \mathcal{B}(\mathcal{H})$, where X is a positive operator such that $B^*XC = C^*XA$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions:*

(i) $0 < m'I \leq A^*XA \leq mI \leq MI \leq B^*XB \leq M'I$

(ii) $0 < m'I \leq B^*XB \leq mI \leq MI \leq A^*XA \leq M'I,$

with $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$\omega^2(C^*XA) \leq \frac{1}{2}\left\|X^{\frac{1}{2}}C\right\|^2\Big(\Big\|\frac{|X^{\frac{1}{2}}A|^2 + |X^{\frac{1}{2}}B|^2}{2S(\sqrt{h})}\Big\| + \omega(B^*XA)\Big),$$

where $S(.)$ is the Specht's ratio.

Proof. By using inequality (20) (see [3, Theorem 8, inequality (2.6)]), we have

$$\frac{1}{2}\Big(\langle A^*XAx,x\rangle^{\frac{1}{2}}\langle B^*XBx,x\rangle^{\frac{1}{2}}+|\langle B^*XAx,x\rangle|\Big)\langle C^*XCx,x\rangle \geq |\langle x,A^*XCx\rangle\langle x,B^*XCx\rangle| \tag{25}$$

for every $x\in\mathcal{H}$. We know that $B^*XC=C^*XA=(A^*XC)^*$. Then

$$\begin{aligned}|\langle x,A^*XCx\rangle\langle x,B^*XCx\rangle| &= |\langle x,A^*XCx\rangle\langle x,(A^*XC)^*x\rangle| \\ &= |\langle A^*XCx,x\rangle|^2=|\langle C^*XAx,x\rangle|^2\end{aligned} \tag{26}$$

for every $x\in\mathcal{H}$. Inequalities (25) and (26) imply that

$$|\langle C^*XAx,x\rangle|^2\leq\frac{1}{2}\Big(\langle A^*XAx,x\rangle^{\frac{1}{2}}\langle B^*XBx,x\rangle^{\frac{1}{2}}+|\langle B^*XAx,x\rangle|\Big)\langle C^*XCx,x\rangle \tag{27}$$

for every $x\in\mathcal{H}$. Now, by using Remark 1.3, we get

$$\begin{aligned}\langle A^*XAx,x\rangle^{\frac{1}{2}}\langle B^*XBx,x\rangle^{\frac{1}{2}} &\leq \frac{1}{2S(\sqrt{h})}\Big(\langle A^*XAx,x\rangle+\langle B^*XBx,x\rangle\Big) \\ &= \langle\frac{A^*XA+B^*XB}{2S(\sqrt{h})}x,x\rangle\end{aligned}$$

for every $\in\mathcal{H}$. Hence we can improve inequality (27), and we imply

$$|\langle C^*XAx,x\rangle|^2\leq\frac{1}{2}\Big(\langle\frac{A^*XA+B^*XB}{2S(\sqrt{h})}x,x\rangle+|\langle B^*XAx,x\rangle|\Big)\langle C^*XCx,x\rangle. \tag{28}$$

Equivalently, by using (22), we can write

$$|\langle C^*XAx,x\rangle|^2\leq\frac{1}{2}\Big(\langle\frac{|X^{\frac{1}{2}}A|^2+|X^{\frac{1}{2}}B|^2}{2S(\sqrt{h})}x,x\rangle+|\langle B^*XAx,x\rangle|\Big)\langle|X^{\frac{1}{2}}C|^2x,x\rangle$$

for every $x\in\mathcal{H}$. Now, by taking the supremum over $x\in\mathcal{H}$, with $\|x\|$=1 in (28), the following interest inequality is deduced:

$$\omega^2(C^*XA)\leq\frac{1}{2}\Big\|X^{\frac{1}{2}}C\Big\|^2\Big(\Big\|\frac{|X^{\frac{1}{2}}A|^2+|X^{\frac{1}{2}}B|^2}{2S(\sqrt{h})}\Big\|+\omega(B^*XA)\Big).$$

□

3 Some Improvement Inequalities Involving Integral

The purpose of this section is to establish a generalization of numerical radius inequalities based on Specht's ratio.
The following theorem is useful for improving inequality (5).

Theorem 3.1. *Let $A \in B(\mathcal{H})$ be an operator and let $f : [0, \infty) \to [0, \infty)$ be an increasing operator convex function. Also let the positive real numbers m, m', M, M' satisfy one of the following conditions:*

(i) $0 < m'I \leq |A| \leq mI \leq MI \leq |A^*| \leq M'$

(ii) $0 < m'I \leq |A^*| \leq mI \leq MI \leq |A| \leq MI'$,

with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then

$$f(\omega(A)) \leq \frac{1}{S(\sqrt{h})} \Big\| \int_0^1 f(t|A| + (1-t)|A^*|)dt \Big\|, \tag{29}$$

where $S(.)$ is the Specht's ratio.

Proof. Suppose that x is a unit vector in $\mathcal{H}$. Since f is a non-negative operator increasing convex function and $\frac{1}{S(\sqrt{h})} \leq 1$, then

$$\begin{aligned} f(|\langle Ax, x\rangle|) &\leq f\big(\sqrt{\langle |A|x, x\rangle \langle |A^*|x, x\rangle}\big) \quad \text{(by the Schwarz inequality)} \\ &\leq f\Big(\frac{\langle |A|x, x\rangle + \langle |A^*|x, x\rangle}{2S(\sqrt{h})}\Big) \quad \text{(by Remark 1.3)} \\ &\leq \frac{1}{S(\sqrt{h})} f\Big(\frac{\langle |A|x, x\rangle + \langle |A^*|x, x\rangle}{2}\Big) \\ &\leq \frac{1}{S(\sqrt{h})} \int_0^1 f(t\langle |A|x, x\rangle + (1-t)\langle |A^*|x, x\rangle)dt. \\ &\quad \text{(by inequality (7))} \end{aligned}$$

Moreover, we note that f is an operator convex function and we have

$$\begin{aligned} f(t\langle |A|x, x\rangle + (1-t)\langle |A^*|x, x\rangle) &= f(\langle (t|A| + (1-t)|A^*|)x, x\rangle) \\ &\leq \langle f(t|A| + (1-t)|A^*|)x, x\rangle. \\ &\quad \text{(by inequality (8))} \end{aligned}$$

Continuity of the inner product and integrating both sides of the above inequality from 0 to 1 give

$$\begin{aligned}
&\frac{1}{S(\sqrt{h})}\int_0^1 f(t\langle |A|x,x\rangle + (1-t)\langle |A^*|x,x\rangle)dt \\
&\quad\le \frac{1}{S(\sqrt{h})}\int_0^1 \langle f(t|A| + (1-t)|A^*|)x,x\rangle dt \\
&\quad= \frac{1}{S(\sqrt{h})}\Big\langle \int_0^1 (f(t|A| + (1-t)|A^*|)dtx,x\Big\rangle.
\end{aligned}$$

Therefore

$$f(|\langle Ax,x\rangle|) \le \frac{1}{S(\sqrt{h})}\Big\langle \int_0^1 \Big(f(t|A| + (1-t)|A^*|)dt\Big)x,x\Big\rangle.$$

By taking the supremum, we get the desired inequality (29). □

Recall that the function $f(t) = t^r,\ 1 \le r \le 2$ is an increasing operator convex function.

Corollary 3.2. *Under the assumption of Theorem 3.1, if* $f(t) = t^r,\ 1 \le r \le 2$, *then*

$$\omega^r(A) \le \tfrac{1}{S(\sqrt{h})}\Big\| \int_0^1 (t|A| + (1-t)|A^*|)^r dt\Big\|.$$

In particular,

$$\omega^2(A) \le \tfrac{1}{S(\sqrt{h})}\Big\| \int_0^1 (t|A| + (1-t)|A^*|)^2 dt\Big\|,$$

where $S(.)$ *is the Specht's ratio.*

In the next theorem, we try to improve inequality (6).

Theorem 3.3. *Let* $A,\ B \in B(\mathcal{H})$ *and let* $f : [0,\infty) \to [0,\infty)$ *be an increasing operator convex function. Also let the positive real numbers* m, m', M, M' *satisfy one of the following conditions:*

(i) $0 < m'I \le |A|^2 \le mI \le MI \le |B|^2 \le M'$

(ii) $0 < m'I \le |B|^2 \le m \le MI \le |A|^2 \le M'I$,

with $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$f(\omega(B^*A)) \leq \frac{1}{S(\sqrt{h})} \sup\Big(\int_0^1 f\Big(\Big\|(t|A|^2 + (1-t)|B|^2)^{\frac{1}{2}} x\Big\|^2\Big) dt\Big),$$

where $x \in \mathcal{H}$ *is a unit vector and* $S(.)$ *is the Specht's ratio.*

Proof. Remark 1.3 and the Schwarz inequality imply that

$$\begin{aligned} f(|\langle B^*Ax, x\rangle|) &= f(|\langle Ax, Bx\rangle|) \leq f(\|Ax\|\|Bx\|) \\ &= f\big(\sqrt{\langle |A|^2x, x\rangle\langle |B|^2x, x\rangle}\big) \\ &\leq f\Big(\frac{\langle |A|^2x, x\rangle + \langle |B|^2x, x\rangle}{2S(\sqrt{h})}\Big) \\ &\leq \frac{1}{S(\sqrt{h})} f\Big(\frac{\langle |A|^2x, x\rangle + \langle |B|^2x, x\rangle}{2}\Big). \end{aligned} \tag{30}$$

Take $a = \langle |A|^2x, x\rangle$ and $b = \langle |B|^2x, x\rangle$ in inequality (7), where $x \in \mathcal{H}$ is a unit vector. Hence

$$f\Big(\frac{\langle |A|^2x, x\rangle + \langle |B|^2x, x\rangle}{2}\Big) \leq \int_0^1 f\Big(\langle (t|A|^2 + (1-t)|B|^2)x, x\rangle\Big) dt. \tag{31}$$

Next, combining inequalities (30) and (31) leads to

$$\begin{aligned} f(|\langle B^*Ax, x\rangle|) &\leq \frac{1}{S(\sqrt{h})} \int_0^1 f\Big(\langle (t|A|^2 + (1-t)|B|^2)x, x\rangle\Big) dt \\ &\leq \frac{1}{S(\sqrt{h})} \int_0^1 f\Big(\Big\|(t|A|^2 + (1-t)|B|^2)x\Big\|\Big) dt \end{aligned}$$

for every unit vector $x \in \mathcal{H}$. Then taking the supremum over $x \in \mathcal{H}$ with $\|x\|$=1, we reach the desired inequality. □

We get the following result, by using Lemma 2.3 involving an improvement of inequality (6).

Corollary 3.4. *Under the assumption of Theorem 3.3, if we take* $\varphi = I$ *in Lemma 2.3, then*

$$f(\omega(B^*A)) \leq \frac{1}{S(\sqrt{h})} \Big\| \int_0^1 f(t|A|^2 + (1-t)|B|^2) dt \Big\|.$$

In particular, for each $1 \leq r \leq 2$*, it follows that*

$$\omega^r(B^*A)) \leq \tfrac{1}{S(\sqrt{h})}\Big\| \int_0^1 f(t|A|^2 + (1-t)|B|^2)^r dt\Big\|.$$

Acknowledgements
The authors would like to thank the referees for their useful comments.

References

[1] J. S. Aujla and F. C. Silva, Weak majorization inequalities and convex functions, *Linear Algebra Appl.*, 25 (2003), 217–233.

[2] S. S. Dragomir, Power inequalities for the numerical radius of a product of two operators in Hilbert spaces, *Sarajevo J Math.*, 5 (18) (2009), 269-278.

[3] S. S. Dragomir, On some inequalities for numerical radius of operators in Hilbert spaces, *Korean J. Math.*, 25 (2) (2017), 247–259.

[4] S. Izumino and Y. Seo, Determinant for positive operators and Specht's theorem, *Sci. Math.*, 1 (3) (1998), 307–310.

[5] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.*, 20 (1) (2012), 46–49.

[6] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *II, Studia Math.*, 182 (2) (2007), 133–140.

[7] Z. Heydarbeygi and M. Amyari, Some refinements of the numerical radius inequalities via Young inequality, *Kragujevac J. Math.*, 45 (2) (2021), 191–202.

[8] F. Kittaneh, Notes on some inequalities for Hilbert space operators, *Publ. Res. Inst. Math. Sci.*, 24 (2) (1988), 283–293.

[9] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.*, 158 (1) (2003), 11–17.

[10] B. Mond and J. E. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, 19 (3) (1993), 405–420.

[11] J. E. Pečarić, T. Furuta, J. Mićić Hot, and Y. Seo, *Mond Pečarić Method in Operator Inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space. Monographs in Inequalities*, 1. ELEMENT, Zagreb, (2005).

[12] M. Sababheh and H. R. Moradi, More accurate numerical radius inequalities, *Linear Multilinear Algebra*, 69(10) (2021), 1964–1973.

[13] K. Shebrawi and H. Albadawi, Numerical radius and operator norm inequalities, *J. Inequal. Appl.*, Art. ID 492154, (2009), 11 pp.

[14] W. Specht, Zur Theorie der elementaren Mittel, *Math. Z.*, 74 (1960), 91–98.

Yaser Khatib
Department of Mathematics
Assistant Professor of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran.
E-mail: yaserkhatibam@yahoo.com

Mahmoud Hassani
Department of Mathematics
Associate Professor of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran.
E-mail: mhassanimath@gmail.com

Maryam Amyari
Department of Mathematics
Associate Professor of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran.
E-mail: maryam_amyari@yahoo.com