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Iterative approximation for generalized asymptotically k-strict pseudocontractive type mappings

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Abstract. Here, the approximations of common fixed points of a new iterative process for asymptotically k-strict pseudocontractive type mappings in Hilbert spaces are studied. Finally, some examples are presented to indicate the validity of our iterations.

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The iterative processes have many applications in mathematics (for example, the study of solution for an ordinary differential equations), Physics and engineering sciences. Because of this many researcher are interested to study this subject. Here we study a new modified iterative process. Due to do this, we need some facts (see [4, 6, 7]).

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Let E be a Banach space and C be a nonempty subset of E, $T : C \longrightarrow C$ be a mapping and

$$F(T) = \{x \in C, Tx = x\},$$

denotes the set of fixed points of T and $w_w(\{x_n\}) = \{x \in H : \exists x_{nj} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.
A mapping T, is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\longrightarrow\infty} k_n = 1$ such that for $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

In 2011, Ceng et al. [1], introduce the following concept of asymptotically k-strict pseudocontractive type mapping in the intermediate sense in a Hilbert space H.

Definition 0.1. [1] Let C be a nonempty subset of Hilbert space H. A mapping $T : C \longrightarrow C$ is called an asymptotically k-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\begin{array}{l} \limsup_{n\longrightarrow\infty} \sup_{x,y\in C}(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\ -k \max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2) \leq 0. \end{array} \tag{1}$$

Throughout this paper we assume that

$$\begin{aligned} \Theta_n := \ & \max\{0, \sup_{x,y\in C}(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\ & \quad -k \max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2)\}. \end{aligned}$$

Then $\Theta_n \geq 0 \ \ (\forall n \geq 1), \ \ \Theta_n \to 0 \ \ (n \to \infty)$, and (1) reduces to

$$\begin{aligned} \|T^n x - T^n y\|^2 \leq \ & (1 + \gamma_n)\|x - y\|^2 \\ & +k \max\{\|x - T^n x - (y - T^n y)\|, \|x - T^n x + (y - T^n y)\|\}^2 \\ & +\Theta_n, \end{aligned}$$

for all $x, y \in C$ and $n \geq 1$.

For an asymptotically k-strict pseudocontractive type mapping T with sequence $\{\gamma_n\}$, Ceng et al. [1] defined

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $0 < \delta \leq \alpha_n \leq 1-k-\delta < 1$ and $\sum_{n=1}^{\infty} \alpha_n \Theta_n < \infty$ for all $n \geq 1$. They proved the sequence $\{x_n\}$ is weakly convergent to a fixed point of T.

In this paper, a new modified iterative processes for the asymptotically k-strict pseudocontractive type mapping in the intermediate sense are presented. Finally, some numerical examples are also given.

Now, we collect some lemmas which will be used in the proofs of the main results.

Lemma 0.2. *[1] Suppose $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences of nonnegative numbers satisfying the recursive inequality,*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n, \qquad \forall n \geq 1,$$

if $\beta_n \geq 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 0.3. *[8, 12] Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in real number such that

(I) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.

(II) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 0.4. *[9] Let X be a uniformly convex Banach space, $\{t_n\}$ a sequence of real numbers in $(0,1)$ bounded away from 0 and 1, and $\{x_n\}$ and $\{y_n\}$ sequence of X such that $\limsup_{n\longrightarrow\infty} \|x_n\| \leq a$, $\limsup_{n\longrightarrow\infty} \|y_n\| \leq a$ and $\limsup_{n\longrightarrow\infty} \|t_n x_n + (1-t_n)y_n\| = a$, for some $a \geq 0$, then $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.*

Lemma 0.5. *[1] Let $\{x_n\}$ be a bounded sequence on a reflexive Banach space X. If $w_w(\{x_n\}) = \{x\}$, then $x \rightharpoonup x$.*

Lemma 0.6. *[1] Let H be a real Hilbert space. Then the following hold:*
(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2 \prec x-y, y \succ$ for all $x, y \in H$;
(ii) $\|(1-t)x+ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2$ for all $t \in [0,1]$ and for all $x, y \in H$;
(iii) If $\{x_n\}$ is a sequence in H, such that $x_n \rightharpoonup x$, it follows that,

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

Lemma 0.7. *[1] Let C be a nonempty subset of a Hilbert space H and $T : C \longrightarrow C$ be an asymptotically k-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$, then*

$$\|T^n x - T^n y\| \leq \frac{1}{1-k}(k\|x-y\| + \sqrt{(1+(1-k)\gamma_n)\|x-y\|^2 + (1-k)h_n(x,y)}),$$

for all $x, y \in C$ and $n \geq 1$, where $h_n(x,y) = 4k\|y - T^n y\|\|x - T^n x + y - T^n y\| + \Theta_n$. In particular, if $F(T) \neq \phi$, then the above inequality reduces to the following

$$\|T^n x - f\| \leq \frac{1}{1-k}(k\|x-f\| + \sqrt{(1+(1-k)\gamma_n)\|x-f\|^2 + (1-k)\Theta_n}),$$

for all $x \in C$, $f \in F(T)$ and $n \geq 1$.

Lemma 0.8. *[1] Let C be a nonempty subset of a Hilbert space H and $T : C \longrightarrow C$ be a uniformly continuous asymptotically k-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a bounded sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. If $F(T) \neq \phi$, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 0.9. *[1] Let H be a real Hilbert space. Suppose C is a closed and convex subset H and point $x, y, z \in H$. Assume $a \in \mathbb{R}$. The set $D := \{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \prec z, v \succ + a\}$ is convex and closed.*

Proposition 0.10. *[1] Let C be a nonempty, closed and convex subset of a Hilbert space H and $T : C \longrightarrow C$ be a continuous asymptotically*

k-strict pseudocontractive type mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \phi$. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.

1 Iterative approximation of asymptotically k-strict pseudocontractive type mappings

Let C be a nonempty subset of real Hilbert space H and $T_i : C \longrightarrow C$ be three uniformly continuous asymptotically k_i-strict pseudocontractive type mappings in the intermediate sense for $i = 1, 2, 3$. Consider the following iterative sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in C \\ u_n = \frac{1}{n+1} \sum_{j=0}^{n} T_1^j x_n \\ z_n = (1 - \lambda_n)x_n + \lambda_n u_n \quad n \geq 1, \\ y_n = (1 - \beta_n)x_n + \beta_n T_2^n z_n \quad n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_3^n y_n \quad n \geq 1, \end{cases} \tag{2}$$

where $0 < \alpha_n, \beta_n, \lambda_n < 1$.

In this section, we prove $\{x_n\}$ generated by (2) is weakly convergent in a Hilbert space H.

Theorem 1.1. *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Suppose $T_i : C \longrightarrow C$ are three uniformly continuous asymptotically k_i-strict pseudocontractive type mappings in the intermediate sense for $i = 1, 2, 3$, such that*

$$\begin{aligned} \|T_i^n x - T_i^n y\|^2 \leq \quad & (1 + \gamma_{ni})\|x - y\|^2 \\ & + k_i \max\{\|x - T_i^n x - (y - T_i^n y)\|, \|x - T_i^n x + (y - T_i^n y)\|\}^2 \\ & + \Theta_{ni}, \end{aligned}$$

for $i = 1, 2, 3$ and for all $x, y \in C$, $n \geq 1$ also for all $x \in C$, $\prec x - T_i x, T_i x \succ \geq 0$. Let $\sum_{n=1}^{\infty} \gamma_{ni} < \infty$ and $\{x_n\}$ be the sequence defined by (2). If $F = \bigcap_{i=1}^{3} F(T_i) \neq \phi$ and $0 < \liminf_{n\longrightarrow\infty} \alpha_n, \beta_n, \lambda_n \leq \limsup_{n\longrightarrow\infty} \alpha_n, \beta_n, \lambda_n < 1$, then

1) $\lim_{n\longrightarrow\infty}\|x_n - f\|$ *exists for all* $f \in F$.

2) $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$ *and* $\lim_{n\longrightarrow\infty}\|x_n - T_i x_n\| = 0$, $(i = 1, 2, 3)$.

3) The sequence $\{x_n\}$ *is weakly convergent to a common fixed point of* $T_i, (i = 1, 2, 3)$.

Proof. Let f be an element of $F = \bigcap_{i=1}^{3} F(T_i)$. By (ii) of Lemma 0.6,

$$\begin{aligned} \|z_n - f\|^2 = & \ \|(1-\lambda_n)x_n + \lambda_n u_n - f\|^2 \\ = & \ \|(1-\lambda_n)(x_n - f) + \lambda_n(u_n - f)\|^2 \\ = & \ (1-\lambda_n)\|x_n - f\|^2 + \lambda_n\|u_n - f\|^2 \\ & \ -\lambda_n(1-\lambda_n)\|x_n - u_n\|^2. \end{aligned}$$

By Holder inequality,

$$\begin{aligned} \|u_n - f\|^2 = & \ \|\tfrac{1}{n+1}\textstyle\sum_{j=0}^{n} T_1^j x_n - f\|^2 \\ \leq & \ \tfrac{n}{(n+1)^2}\textstyle\sum_{j=0}^{n}\|T_1^j x_n - f\|^2, \end{aligned}$$

for $1 \leq j \leq n$,

$$\|T_1^j x_n - f\|^2 \leq (1+\gamma_{j1})\|x_n - f\|^2 + k_1\|x_n - T_1^j x_n\|^2 + \Theta_{j1}.$$

By (i) of Lemma 0.6,

$$\begin{aligned} \|x_n - T_1^j x_n\|^2 = & \ \|(x_n - f) - (T_1^j x_n - f)\|^2 \\ = & \ \|x_n - f\|^2 - \|T_1^j x_n - f\|^2 - 2 \prec x_n - T_1^j x_n, T_1^j x_n \succ, \end{aligned}$$

since $\prec x_n - T_1^j x_n, T_1^j x_n \succ \geq 0$, then

$$\|T_1^j x_n - f\|^2 \leq (1+\gamma_{j1})\|x_n - f\|^2 + k_1\|x_n - f\|^2 - k_1\|T_1^j x_n - f\|^2 + \Theta_{j1},$$

therefore

$$\|T_1^j x_n - f\|^2 \leq (1 + \frac{\gamma_{j1}}{1+k_1})\|x_n - f\|^2 + \frac{\Theta_{j1}}{1+k_1}.$$

We have,

$$\begin{aligned} \|u_n - f\|^2 & \leq \tfrac{n}{(n+1)^2}(\|x_n - f\|^2 + \textstyle\sum_{j=1}^{n}\|T_1^j x_n - f\|^2) \\ & \leq \tfrac{n}{(n+1)^2}(\|x_n - f\|^2 + \textstyle\sum_{j=1}^{n}((1 + \tfrac{\gamma_{j1}}{1+k_1})\|x_n - f\|^2 + \tfrac{\Theta_{j1}}{1+k_1})) \\ & \leq \frac{n(n+1)(1+\frac{\varsigma_n}{1+k_1})}{(n+1)^2}\|x_n - f\|^2 + \tfrac{n^2}{(n+1)^2}\tfrac{\upsilon_n}{k_1+1} \\ & \leq (1 + \tfrac{\varsigma_n}{k_1+1})\|x_n - f\|^2 + \tfrac{\upsilon_n}{k_1+1}, \end{aligned}$$

where $\varsigma_n = \max\{\gamma_{j1}, 1 \leq j \leq n\}$ and $\upsilon_n = \max\{\Theta_{j1}, 1 \leq j \leq n\}$. Therefore

$$\begin{aligned}\|z_n - f\|^2 \leq\ & (1-\lambda_n)\|x_n - f\|^2 \\ & +\lambda_n[(1+\tfrac{\varsigma_n}{1+k_1})\|x_n - f\|^2 + \tfrac{\upsilon_n}{1+k_1}] \\ & -\lambda_n(1-\lambda_n)\|x_n - u_n\|^2,\end{aligned}$$

also

$$\|z_n - f\|^2 \leq (1+\lambda_n(\frac{\varsigma_n}{k_1+1}))\|x_n - f\|^2 + \lambda_n\frac{\upsilon_n}{1+k_1} - \lambda_n(1-\lambda_n)\|x_n - u_n\|^2. \tag{3}$$

And

$$\begin{aligned}\|y_n - f\|^2 =\ & \|(1-\beta_n)x_n + \beta_n T_2^n z_n - f\|^2 \\ =\ & \|(1-\beta_n)(x_n - f) + \beta_n(T_2^n z_n - f)\|^2 \\ =\ & (1-\beta_n)\|x_n - f\|^2 + \beta_n\|T_2^n z_n - f\|^2 \\ & -\beta_n(1-\beta_n)\|x_n - T_2^n z_n\|^2,\end{aligned}$$

by (i) of Lemma 0.6,

$$\|T_2^n z_n - f\|^2 \leq (1+\frac{\gamma_{n2}}{1+k_2})\|z_n - f\|^2 + \frac{\Theta_{n2}}{1+k_2}, \tag{4}$$

hence

$$\begin{aligned}\|y_n - f\|^2 \leq\ & (1-\beta_n)\|x_n - f\|^2 \\ & +\beta_n[(1+\tfrac{\gamma_{n2}}{1+k_2})\{(1+\lambda_n(\tfrac{\varsigma_n}{1+k_1}))\|x_n - f\|^2 \\ & +\lambda_n\tfrac{\upsilon_n}{1+k_1} - \lambda_n(1-\lambda_n)\|x_n - u_n\|^2\} \\ & +\tfrac{\Theta_{n2}}{1+k_2}] - \beta_n(1-\beta_n)\|x_n - T_2^n z_n\|^2,\end{aligned}$$

therefore

$$\begin{aligned}\|y_n - f\|^2 \leq\ & [(1-\beta_n) + \beta_n(1+\lambda_n(\tfrac{\varsigma_n}{1+k_1}))(1+\tfrac{\gamma_{n2}}{1+k_2})]\|x_n - f\|^2 \\ & +\lambda_n\beta_n(1+\tfrac{\gamma_{n2}}{1+k_2})\tfrac{\upsilon_n}{1+k_1} + \beta_n\tfrac{\Theta_{n2}}{1+k_2} \\ & -\lambda_n\beta_n(1-\lambda_n)(1+\tfrac{\gamma_{n2}}{1+k_2})\|x_n - u_n\|^2 \\ & -\beta_n(1-\beta_n)\|x_n - T_2^n z_n\|^2.\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}\|x_{n+1} - f\|^2 =\ & \|(1-\alpha_n)(x_n - f) + \alpha_n(T_3^n y_n - f)\|^2 \\ =\ & (1-\alpha_n)\|x_n - f\|^2 + \alpha_n\|T_3^n y_n - f\|^2 \\ & -\alpha_n(1-\alpha_n)\|x_n - T_3^n y_n\|^2,\end{aligned}$$

since

$$\|T_3^n y_n - f\|^2 \leq (1 + \frac{\gamma_{n3}}{1+k_3})\|y_n - f\|^2 + \frac{\Theta_{n3}}{1+k_3}, \tag{6}$$

then

$$\begin{aligned}\|x_{n+1} - f\|^2 \leq \quad & [(1-\alpha_n) + \alpha_n\{(1-\beta_n) \\ & +\beta_n(1+\lambda_n\tfrac{\varsigma_n}{1+k_1})(1+\tfrac{\gamma_{n2}}{1+k_2})\}(1+\tfrac{\gamma_{n3}}{1+k_3})]\|x_n - f\|^2 \\ & +\lambda_n\beta_n\alpha_n(1+\tfrac{\gamma_{n2}}{1+k_2})(1+\tfrac{\gamma_{n3}}{1+k_3})\tfrac{\upsilon_n}{1+k_1} \\ & +\beta_n\alpha_n(1+\tfrac{\gamma_{n3}}{1+k_3})\tfrac{\Theta_{n2}}{1+k_2} \\ & +\alpha_n\tfrac{\Theta_{n3}}{1+k_3} \\ & -\lambda_n\beta_n\alpha_n(1+\tfrac{\gamma_{n2}}{1+k_2})(1+\tfrac{\gamma_{n3}}{1+k_3})(1-\lambda_n)\|x_n - u_n\|^2 \\ & -\beta_n\alpha_n(1-\beta_n)(1+\tfrac{\gamma_{n3}}{1+k_3})\|x_n - T_2^n z_n\|^2 \\ & -\alpha_n(1-\alpha_n)\|x_n - T_3^n y_n\|^2,\end{aligned}$$

hence,

$$\begin{aligned}\|x_{n+1} - f\|^2 \leq \quad & [(1-\alpha_n) + \alpha_n\{(1-\beta_n) + \beta_n\rho_1\rho_2\}\rho_3]\|x_n - f\|^2 \\ & +\lambda_n\beta_n\alpha_n\rho_2\rho_3\tfrac{\upsilon_n}{1+k_1} \\ & +\beta_n\alpha_n\rho_3\tfrac{\Theta_{n2}}{1+k_2} \\ & +\alpha_n\tfrac{\Theta_{n3}}{1+k_3} \\ & -\lambda_n\beta_n\alpha_n\rho_2\rho_3(1-\lambda_n)\|u_n - x_n\|^2 \\ & -\beta_n\alpha_n(1-\beta_n)\rho_3\|x_n - T_2^n z_n\|^2 \\ & -\alpha_n(1-\alpha_n)\|x_n - T_3^n y_n\|^2,\end{aligned} \tag{7}$$

where $\rho_1 = 1 + \frac{\varsigma_n}{1+k_1}$ and $\rho_i = 1 + \frac{\gamma_{ni}}{1+k_i}$ for $i = 2, 3$. Therefore

$$\|x_{n+1} - f\|^2 \leq \mu_n\|x_n - f\|^2 + \alpha_n\eta_n, \tag{8}$$

where, $\mu_n = (1-\alpha_n) + \alpha_n(1-\beta_n)\rho_3 + \beta_n\alpha_n\rho_1\rho_2\rho_3$ and $\eta_n = \rho_2\rho_3\frac{\upsilon_n}{1+k_1} + \rho_3\frac{\Theta_{n2}}{1+k_2} + \frac{\Theta_{n3}}{1+k_3}$. Since $\lim_{n\longrightarrow\infty}\mu_n = 1$ and $\lim_{n\longrightarrow\infty}\eta_n = 0$, therefore by Lemma 0.2 and inequality (8), we deduce that $\lim_{n\to\infty}\|x_n - f\| = r$ exists for some $r > 0$. By inequality (7),

$$\lambda_n\beta_n\alpha_n\rho_2\rho_3(1-\lambda_n)\|x_n - u_n\|^2 \leq \mu_n\|x_n - f\|^2 - \|x_{n+1} - f\|^2 + \alpha_n\eta_n,$$

which implies that $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$. Since $\|x_n - u_n\| = \|\frac{1}{n+1}\sum_{j=0}^n(T_1^j x_n - x_n)\|$ then $\lim_{n\longrightarrow\infty}\|x_n - T_1^n x_n\| = 0$. By inequality (7),

$$\beta_n\alpha_n(1-\beta_n)\rho_3\|x_n - T_2^n z_n\|^2 \leq \mu_n\|x_n - f\|^2 - \|x_{n+1} - f\|^2 + \alpha_n\eta_n,$$

which implies that $\lim_{n\longrightarrow\infty}\|x_n - T_2^n z_n\| = 0$. By inequality (7),

$$\alpha_n(1-\alpha_n)\|x_n - T_3^n y_n\|^2 \leq \mu_n\|x_n - f\|^2 - \|x_{n+1} - f\|^2 + \alpha_n\eta_n,$$

which implies that $\lim_{n\longrightarrow\infty}\|x_n - T_3^n y_n\| = 0$. we have

$$\|z_n - x_n\| = \lambda_n\|x_n - u_n\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

and

$$\|y_n - x_n\| = \beta_n\|x_n - T_2^n z_n\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

and

$$\|x_{n+1} - x_n\| = \alpha_n\|x_n - T_3^n y_n\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

also

$$\|x_n - T_2^n x_n\| \leq \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\|,$$

and by Lemma 0.7,

$$\begin{aligned}&\|T_2^n x_n - T_2^n z_n\|\\ &\leq \tfrac{1}{1-k_2}(k_2\|x_n - z_n\| + \sqrt{(1+(1-k_2)\gamma_{n2})\|x_n - z_n\|^2 + (1-k_2)h_n(x_n, z_n)}),\end{aligned}$$

where $h_n(x_n, z_n) = 4k_2\|z_n - T_2^n z_n\|\|x_n - T_2^n x_n + z_n - T_2^n z_n\| + \Theta_{n2}$. We have,

$$\begin{aligned}\|z_n - T_2^n z_n\|^2 =\ & \|(1-\lambda_n)(x_n - T_2^n z_n) + \lambda_n(u_n - T_2^n z_n)\|^2\\
=\ & (1-\lambda_n)\|x_n - T_2^n z_n\|^2 + \lambda_n\|u_n - T_2^n z_n\|^2\\
& \qquad -\lambda_n(1-\lambda_n)\|x_n - u_n\|^2\\
\leq\ & (1-\lambda_n)[(1+\gamma_{n2})\|x_n - z_n\|^2 + k_2\|z_n - T_2^n z_n\|^2 + \Theta_{n2}]\\
& \quad +\lambda_n[(1+\gamma_{n2})\|u_n - z_n\|^2 + k_2\|z_n - T_2^n z_n\|^2 + \Theta_{n2}]\\
& \qquad -\lambda_n(1-\lambda_n)\|x_n - u_n\|^2\\
=\ & (1-\lambda_n)[(1+\gamma_{n2})\lambda_n^2\|x_n - u_n\|^2 + k_2\|z_n - T_2^n z_n\|^2 + \Theta_{n2}]\\
& +\lambda_n[(1+\gamma_{n2})(1-\lambda_n)^2\|x_n - u_n\|^2 + k_2\|z_n - T_2^n z_n\|^2 + \Theta_{n2}]\\
& \qquad -\lambda_n(1-\lambda_n)\|x_n - u_n\|^2\\
=\ & k_2\|z_n - T_2^n z_n\|^2 + \lambda_n(1-\lambda_n)\gamma_{n2}\|x_n - u_n\|^2 + \Theta_{n2},\end{aligned}$$

therefore

$$(1-k_2)\|z_n - T_2^n z_n\|^2 \leq \lambda_n(1-\lambda_n)\gamma_{n2}\|x_n - u_n\|^2 + \Theta_{n2}.$$

This means that $\|z_n - T_2^n z_n\| \longrightarrow 0$, $h_n(x_n, z_n) \longrightarrow 0$ and $\|T_2^n z_n - T_2^n x_n\| \longrightarrow 0$, then we have $\|x_n - T_2^n x_n\| \longrightarrow 0$. Also,

$$\|x_n - T_3^n x_n\| \leq \|x_n - T_3^n y_n\| + \|T_3^n y_n - T_3^n x_n\|,$$

and by Lemma 0.7,

$$\begin{aligned}&\|T_3^n x_n - T_3^n y_n\| \\ &\leq \tfrac{1}{1-k_3}(k_3\|x_n - y_n\| + \sqrt{(1+(1-k_3)\gamma_{n3})\|x_n - y_n\|^2 + (1-k_3)h_n(x_n, y_n)}),\end{aligned}$$

where $h_n(x_n, y_n) = 4k_3\|y_n - T_3^n y_n\|\|x_n - T_3^n x_n + y_n - T_3^n y_n\| + \Theta_{n3}$. Now

$$(1-k_3)\|y_n - T_3^n y_n\|^2 \leq \beta_n(1-\beta_n)\gamma_{n3}\|x_n - T_2^n z_n\|^2 + \Theta_{n3}.$$

This means that $\|y_n - T_3^n y_n\| \longrightarrow 0$, then $\|x_n - T_3^n x_n\| \longrightarrow 0$. Therefore, by Lemma 0.8 we have, $\|x_n - T_i x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ for $i = 1, 2, 3$. Assume that $x_{ni} \longrightarrow u$ weakly and $x_{nj} \longrightarrow v$ weakly as $n \longrightarrow \infty$. Then $u, v \in F$. We prove $u = v$. If not, by Opial's condition,

$$\begin{aligned} lim_{n\longrightarrow\infty}\|x_n - u\| &= lim_{i\longrightarrow\infty}\|x_{ni} - u\| \\ &< lim_{i\longrightarrow\infty}\|x_{ni} - v\| \\ &= lim_{n\longrightarrow\infty}\|x_n - v\| \\ &< lim_{j\longrightarrow\infty}\|x_{nj} - u\| \\ &= lim_{n\longrightarrow\infty}\|x_n - u\|, \end{aligned}$$

which is a contradiction. □

Corollary 1.2. *In Theorem 1.1, suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$, such that satisfying the following conditions,*

I) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

II) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

III) $0 < \liminf_{n\longrightarrow\infty} \lambda_n \leq \limsup_{n\longrightarrow\infty} \lambda_n < 1$.

Then

1) $\lim_{n\longrightarrow\infty} \|x_n - f\|$ *exists for all* $f \in F$.

2) $\lim_{n\longrightarrow\infty} \|x_n - T_i x_n\| = 0$, $(i = 1, 2, 3)$.

3) *The sequence* $\{x_n\}$ *defined by* (2)*, is strongly convergent to a common fixed point of* $T_i, (i = 1, 2, 3)$.

Proof. Using the same argument in the proof of Theorem 1.1, we have

$$\|x_{n+1} - f\|^2 \leq \mu_n \|x_n - f\|^2 + \alpha_n \eta_n,$$

then by Lemma 0.3, $\lim_{n \longrightarrow \infty} \|x_n - f\| = 0$. Furthermore,

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - f\| = \lim_{n \longrightarrow \infty} \|(1 - \alpha_n)(x_n - f) + \alpha_n (T_3^n y_n - f)\|,$$

by inequality (6), (5) and by Lemma 0.4, $\lim_{n \longrightarrow \infty} \|T_3^n y_n - x_n\| = 0$. Furthermore, by inequality (5),

$$\lim_{n \longrightarrow \infty} \|x_n - f\| = \lim_{n \longrightarrow \infty} \|y_n - f\|.$$

Also

$$\limsup_{n \longrightarrow \infty} \|y_n - f\| = \limsup_{n \longrightarrow \infty} \|(1 - \beta_n)(x_n - f) + \beta_n (T_2^n z_n - f)\|,$$

and by inequalities (4) and (3), and by Lemma 0.4, we have $\lim_{n \longrightarrow \infty} \|T_2^n z_n - x_n\| = 0$.
Similarly, $\lim_{n \longrightarrow \infty} \|z_n - f\| = 0$ and $\lim_{n \longrightarrow \infty} \|u_n - x_n\| = 0$. The same argument in the proof of Theorem 1.1 shows that the sequence $\{x_n\}$ defined by (2), is strongly convergent to a common fixed point of T_i for $i = 1, 2, 3$. □

Corollary 1.3. *Let* H *be a real Hilbert space and* C *be a nonempty, closed and convex subset of* H*. Suppose* $T_i : C \longrightarrow C$ *are three uniformly continuous asymptotically* k_i*-strict pseudocontractive type mappings in the intermediate sense for* $i = 1, 2, 3$ *and for all* $x, y \in C$, $n \geq 1$ *and for all* $x \in C$, $\prec x - T_i x, T_i x \succ \geq 0$, $i = 1, 2, 3$*. Let* $\sum_{n=1}^{\infty} \gamma_{ni} < \infty$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$*. Let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} x_1 \in C \\ z_n = (1 - \lambda_n) x_n + \lambda_n T_1^n x_n \quad n \geq 1, \\ y_n = (1 - \beta_n) x_n + \beta_n T_2^n z_n \quad n \geq 1, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_3^n y_n \quad n \geq 1, \end{cases} \tag{9}$$

if $F = \bigcap_{i=1}^{3} F(T_i) \neq \phi$, $0 < \liminf_{n \longrightarrow \infty} \alpha_n, \beta_n, \lambda_n$ *and* $\limsup_{n \longrightarrow \infty} \alpha_n, \beta_n, \lambda_n < 1$ *then*

1) $\lim_{n\longrightarrow\infty}\|x_n - f\|$ *exists for all* $f \in F$.

2) $\lim_{n\longrightarrow\infty}\|x_n - T_i x_n\| = 0$, $(i = 1, 2, 3)$.

3) The sequence $\{x_n\}$ *is weakly convergent to a common fixed point of* $T_i, (i = 1, 2, 3)$.

The next theorem is a new generalization of [1, Theorem 4.1] and [5, Theorem 3.1].

Theorem 1.4. *Let* C *be a nonempty, bounded, closed and convex subset of a real Hilbert space* H *and let* $T_i : C \longrightarrow C$ *for* $i = 1, 2$, *be uniformly continuous asymptotically* k_i-*strict pseudocontractive type mappings in the intermediate sense with sequence* $\{\gamma_{ni}\}$ *such that* $\sum_{n=1}^{\infty}\gamma_{ni} < \infty$, $F = F(T_1)\bigcap F(T_2)$ *is nonempty and bounded and* $\prec x - T_i x, T_i x \succ \geq 0$ *for all* $x \in C$. *Let* $\{t_n\}$ *and* $\{s_n\}$ *are real sequence in* $(0, 1)$, *such that* $\sum t_n < \infty$ *and* $\{s_n\}$ *is bounded away from* 0 *and* 1. *Suppose the sequence* $\{x_n\}$ *is generated by,*

$$\begin{cases} v = x_1 \in C \quad chosen \quad arbitrary, \\ y_n = (1 - t_n)x_n + t_n T_2^n z_n, \\ z_n = (1 - s_n)x_n + s_n u_n, \\ u_n = \frac{1}{n+1}\sum_{j=0}^{n} T_1^j x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \eta_n\}, \\ Q_n = \{z \in C : \prec x_n - z, v - x_n \succ \geq 0\}, \\ x_{n+1} = P_{C_n\bigcap Q_n}(v), \quad \forall n \geq 1, \end{cases} \tag{10}$$

where P_K *denotes the metric projection from* H *onto a closed convex subset* K *of* H, $\eta_n = \dot{\Theta}_n + \Delta_n$, $\dot{\Theta}_n = s_n t_n(1 + \frac{\gamma_{n2}}{1+k_2})\frac{\upsilon_n}{1+k_1} + t_n\frac{\Theta_{n2}}{1+k_2}$ *where* $\upsilon_n = \max\{\Theta_{j1}, 1 \leq j \leq n\}$ *and* $\Delta_n = [t_n(1 + \frac{\gamma_{n2}}{1+k_2})(1 + \frac{\varsigma_n}{1+k_1}) - t_n](dimC)^2 \longrightarrow 0$ *as* $n \longrightarrow \infty$ *where* $\varsigma_n = \max\{\gamma_{j1}, 1 \leq j \leq n\}$.
Then $\{x_n\}$ *is strongly convergent to* $P_F(v)$.

Proof. Lemma 0.9 implies C_n is convex. Notice that $F \subset C_n$ for all n. Indeed, for all $f \in F$,

$$\begin{aligned} \|y_n - f\|^2 &= \|(1 - t_n)x_n + t_n T_2^n z_n - f\|^2 \\ &= \|(1 - t_n)(x_n - f) + t_n(T_2^n z_n - f)\|^2 \\ &= (1 - t_n)\|x_n - f\|^2 + t_n\|T_2^n z_n - f\|^2 \\ &\quad - t_n(1 - t_n)\|x_n - T_2^n z_n\|^2, \end{aligned}$$

using (i) of Lemma 0.6,

$$\|T_2^n z_n - f\|^2 \leq (1 + \frac{\gamma_{n2}}{1+k_2})\|z_n - f\|^2 + \frac{\Theta_{n2}}{1+k_2},$$

also

$$\begin{aligned} \|z_n - f\|^2 = & \ \|(1-s_n)x_n + s_n u_n - f\|^2 \\ = & \ \|(1-s_n)(x_n - f) + s_n(u_n - f)\|^2 \\ = & \ (1-s_n)\|x_n - f\|^2 + s_n\|u_n - f\|^2 \\ & \ -s_n(1-s_n)\|x_n - u_n\|^2. \end{aligned}$$

The same argument in the proof of Theorem 1.1 shows

$$\|z_n - f\|^2 \leq (1 + s_n\frac{\varsigma_n}{1+k_1})\|x_n - f\|^2 + s_n\frac{\upsilon_n}{1+k_1} - s_n(1-s_n)\|x_n - u_n\|^2. \tag{11}$$

Therefore,

$$\begin{aligned} \|y_n - f\|^2 \leq & \ (1-t_n)\|x_n - f\|^2 + t_n[(1+\tfrac{\gamma_{n2}}{1+k_2})\|z_n - f\|^2 + \tfrac{\Theta_{n2}}{1+k_2}] \\ & \ -t_n(1-t_n)\|x_n - T_2^n z_n\|^2 \\ \leq & \ [(1-t_n) + t_n(1+\tfrac{\gamma_{n2}}{1+k_2})(1+s_n\tfrac{\varsigma_n}{1+k_1})]\|x_n - f\|^2 \\ & \ -t_n s_n(1+\tfrac{\gamma_{n2}}{1+k_2})(1-s_n)\|x_n - u_n\|^2 \\ & \ +s_n t_n(1+\tfrac{\gamma_{n2}}{1+k_2})\tfrac{\upsilon_n}{1+k_1} + t_n\tfrac{\Theta_{n2}}{1+k_2} \\ & \ -t_n(1-t_n)\|x_n - T_2^n z_n\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \|y_n - f\|^2 \leq & \ \|x_n - f\|^2 + [t_n(1+\tfrac{\gamma_{n2}}{1+k_2})(1+\tfrac{\varsigma_n}{1+k_1}) - t_n]\|x_n - f\|^2 \\ + & \ s_n t_n(1+\tfrac{\gamma_{n2}}{1+k_2})\tfrac{\upsilon_n}{1+k_1} + t_n\tfrac{\Theta_{n2}}{1+k_2}. \end{aligned}$$

Then

$$\|y_n - f\|^2 \leq \|x_n - f\|^2 + \eta_n.$$

Hence $f \in C_n$. The same argument as in the proof of [1, Theorem 4.1] shoes $F \subset C_n \cap Q_n$ for all $n \geq 1$ and $\|x_n - x_{n+1}\| \longrightarrow 0$ as $n \longrightarrow \infty$. Now, we claim that $\|T_i x_n - x_n\| \longrightarrow 0$, for $i = 1, 2$. By definition of y_n,

$$\begin{aligned} \|T_2^n z_n - x_n\| = & \ \tfrac{1}{t_n}\|y_n - x_n\| \\ \leq & \ \tfrac{1}{t_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|), \end{aligned}$$

since $x_{n+1} \in C_n$, $\|y_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_n\|^2 + \eta_n \longrightarrow 0$ as $n \longrightarrow \infty$, this implies that $\|T_2^n z_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{nj}\}$ of $\{x_n\}$ such that $\lim_{j\longrightarrow\infty} \|x_{nj} - f\| = \lim_{j\longrightarrow\infty} \|y_{nj} - f\| = \limsup_{n\longrightarrow\infty} \|x_n - f\| =: r$. Since

$$t_{nj}s_{nj}(1+\frac{\gamma_{nj2}}{1+k_2})(1-s_{nj})\|x_{nj} - u_{nj}\|^2 \leq \|x_{nj} - f\|^2 - \|y_{nj} - f\|^2 + \eta_{nj},$$

and $\lim_{j\longrightarrow\infty} \eta_{nj} = 0$ then $\lim_{j\longrightarrow\infty} \|x_{nj} - u_{nj}\| = 0$ and also $\lim_{j\longrightarrow\infty} \|x_{nj} - T_1^{nj} x_{nj}\| = 0$. Also since $\|z_n - x_n\| = s_n\|x_n - u_n\|$ then $\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0$. We have

$$\|x_n - T_2^n x_n\| \leq \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\|.$$

Since $\|z_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ and the same argument in the proof of Theorem 1.1, we have $\|T_2^n z_n - T_2^n x_n\| \longrightarrow 0$, then $\|x_n - T_2^n x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Therefore, by Lemma 0.8, $\|x_n - T_i x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ for $i = 1, 2$.
Since H is reflexive and $\{x_n\}$ is bounded we obtain that $w_w(\{x_n\})$ is nonempty. By the fact that $\|x_n - v\| \leq \|f - v\|$ for all $n \geq 0$ where $f := P_F(v)$ and the weak lower semi-continuity of the norm, we have $\|w - v\| \leq \|f - v\|$ for all $w \in w_w(\{x_n\})$. However, since $w_w(\{x_n\}) \subset F$, we have $w = f$ for all $w \in w_w(\{x_n\})$. Thus $w_w(\{x_n\}) = \{f\}$ and then $x_n \rightharpoonup f = P_F(v)$ by

$$\begin{aligned} \|x_n - f\|^2 = &\ \|x_n - v\|^2 + 2 \prec x_n - v, v - f \succ + \|v - f\|^2 \\ \leq &\ 2(\|f - v\|^2 + \prec x_n - v, v - f \succ) \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \end{aligned}$$

□

2 Example

In this section (based on the similar arguments in [1] and [8]) some examples are presented to grantee the statement of Theorem 1.1.

Example 2.1. Let $X = \mathbb{R}$ the set of real numbers, and $C = [0, \infty)$. $T : C \longrightarrow C$ be defined by [1],

$$T(x) = \begin{cases} kx & if \quad x \in [0, 1], \\ 0, & if \quad x \in (1, \infty), \end{cases} \tag{12}$$

where $0 < k < \frac{1}{4}$. Then $T : C \longrightarrow C$ is an asymptotically $\frac{1}{4}$-strict pseudocontractive type mappings in the intermediate sense. For all $x, y \in C$ and $n \geq 1$; we have,

$$\begin{array}{rl} |T^n x - T^n y|^2 \leq & |x-y|^2 + \frac{1}{4}\max\{|x - T^n x - (y - T^n y)| \\ & , |x - T^n x + (y - T^n y)|\}^2 + \frac{1}{4}k^{2(n-1)}. \end{array}$$

Example 2.2. Let $X = \mathbb{R}$ the set of real numbers, and $C = [0, \infty)$. For each $x \in C$, we defined,

$$T(x) = \begin{cases} \frac{kx}{1+x} \quad if \quad x \in [0, \frac{1}{2}], \\ 0, \quad if \quad x \in (\frac{1}{2}, \infty), \end{cases} \tag{13}$$

where $0 < k < \frac{1}{2}$. Set $C_1 = [0, \frac{1}{2}]$ and $C_2 = (\frac{1}{2}, \infty)$. Then for all $x, y \in C_1$ and $n \geq 1$ we have;

$$|Tx - Ty| = |\frac{kx}{1+x} - \frac{ky}{1+y}| \leq k|x - y|,$$

and

$$|T^2 x - T^2 y| = |\frac{kTx}{1+Tx} - \frac{kTy}{1+Ty}| \leq k|Tx - Ty| \leq k^2|x - y|,$$

then for all $n \geq 1$,

$$|T^n x - T^n y| \leq k^n |x - y|.$$

For all $x, y \in C_2$ and $n \geq 1$ we have,

$$|T^n x - T^n y| = 0 \leq |x - y|.$$

For $x \in C_1$ and $y \in C_2$, we have,

$$|Tx - Ty| = |\frac{kx}{1+x} - 0| \leq |kx - 0|,$$

therefore

$$\begin{array}{rl} |T^n x - T^n y|^2 \leq & |k^n x - 0|^2 = |k^n(x-y) + k^n y|^2 \\ \leq & (\frac{k^{n-1}|x-y| + k^{n-1}|y|}{2})^2 \\ \leq & \frac{1}{4}k^{2(n-1)}|x-y| + \frac{1}{4}k^{2(n-1)}|(y + x - T^n x) - (x - T^n y)|^2 \\ \leq & |x-y|^2 + \frac{1}{2}max\{|x - T^n x - (y - T^n y)|, \\ & |x - T^n x + y - T^n y|\}^2 + \frac{1}{2}k^{2(n-1)}. \end{array}$$

Therefore $T : C \longrightarrow C$ is an asymptotically $\frac{1}{2}$-strict pseudocontractive type mappings in the intermediate sense.

Example 2.3. Let $H = \mathbb{R}$ the set of real numbers and $C = [0, \infty)$. Let $T : C \longrightarrow C$ be defined by Example 2.1, we have $F(T) = 0$. Let $x_1 = 1$, $\alpha_n = \frac{3n+1}{4n+1}$ and $\beta_n = \frac{2n+1}{3n+1}$, for all $n \in \mathbb{N}$. Sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = \frac{3n+1}{4n+1} T^n y_n + \frac{n}{4n+1} x_n, \\ y_n = \frac{2n+1}{3n+1} T^n x_n + \frac{n}{3n+1} x_n. \end{cases} \tag{14}$$

Since $x_1 = 1$ then $Tx_1 = k$, $y_1 = \frac{2+1}{3+1}k + \frac{1}{3+1}$ and $x_2 = \frac{3+1}{4+1}k(\frac{2+1}{3+1}k + \frac{1}{3+1}) + \frac{1}{4+1}$. Since $k \in (0, \frac{1}{4})$, then $x_2 < x_1$. By induction

$$x_{n+1} = (\frac{3n+1}{4n+1}k^n(\frac{2n+1}{3n+1}k^n + \frac{n}{3n+1}) + \frac{n}{4n+1})x_n,$$

for all $n \geq 1$. Therefore $\{x_n\}$ is a strictly decreasing sequence and

$$\lim_{n\to\infty} (\frac{3n+1}{4n+1}k^n(\frac{2n+1}{3n+1}k^n + \frac{n}{3n+1}) + \frac{n}{4n+1}) = 0.$$

Let $r = \lim_{n\to\infty} |\frac{x_{n+1}-0}{x_n-0}|$, then

$$lim_{n\to\infty} \frac{(\frac{3n+1}{4n+1}k^n(\frac{2n+1}{3n+1}k^n + \frac{n}{3n+1}) + \frac{n}{4n+1})x_n}{x_n} = \frac{1}{4},$$

and $r = \frac{1}{4}$. Hence, the sequence $\{x_n\}$ is convergent to zero and the rate of the convergence is $\frac{1}{4}$.
Let $k = \frac{1}{8}$, then

$$\begin{cases} x_{n+1} = \frac{3n+1}{4n+1} \frac{y_n}{8^n} + \frac{n}{4n+1} x_n, \\ y_n = \frac{2n+1}{3n+1} \frac{x_n}{8^n} + \frac{n}{3n+1} x_n. \end{cases} \tag{15}$$

n	x_n	n	x_n
1	1	16	5.067×10^{-10}
2	0.2343	17	1.247×10^{-10}
3	0.0529	18	3.073×10^{-11}
4	0.0122	19	7.577×10^{-12}
5	0.0028	20	1.869×10^{-12}
6	0.0006	21	4.617×10^{-13}
7	0.0001	22	1.14×10^{-13}
8	3.972×10^{-5}	23	2.819×10^{-14}
9	9.631×10^{-6}	24	6.973×10^{-15}
10	2.342×10^{-6}	25	1.725×10^{-16}
11	5.714×10^{-7}	26	4.27×10^{-16}
12	1.396×10^{-7}	27	1.057×10^{-16}
13	3.42×10^{-8}	28	2.619×10^{-17}
14	8.39×10^{-9}	29	6.49×10^{-18}
15	2.06×10^{-9}	30	1.608×10^{-18}

Example 2.4. Let $H = \mathbb{R}$ and $C = [0, \infty)$. Assume $\{x_n\}$ is a sequence defined by (2), where $T_1 x = \frac{x}{20}$, $T_2 x = \frac{x}{100}$ and $T_3 x = \frac{x}{100(1+x)}$. Also $\alpha_n = \frac{6n+1}{7n+1}$, $\beta_n = \frac{4n+1}{5n+1}$ and $\lambda_n = \frac{2n+1}{3n+1}$. We have

$$\begin{cases} u_n = \frac{1}{n+1} \sum_{j=0}^{n} \frac{x_n}{20^j}, \\ z_n = \frac{n}{3n+1} x_n + (\frac{2n+1}{3n+1}) u_n, \\ y_n = \frac{n}{5n+1} x_n + (\frac{4n+1}{5n+1})(\frac{1}{100^n}) z_n, \\ x_{n+1} = \frac{n}{7n+1} x_n + (\frac{6n+1}{7n+1}) T_3^n y_n, \end{cases}$$

and $\bigcap_{i=1}^{3} F(T_i) = \{0\}$. If $x_1 = \frac{1}{2}$, so

n	x_n	n	x_n
1	0.5	16	6.729×10^{-14}
2	6.319×10^{-2}	17	9.529×10^{-15}
3	8.426×10^{-3}	18	1.349×10^{-15}
4	1.149×10^{-3}	19	1.913×10^{-16}
5	1.584×10^{-4}	20	2.712×10^{-17}
6	2.201×10^{-5}	21	3.848×10^{-18}
7	3.071×10^{-6}	22	5.46×10^{-19}
8	4.3×10^{-7}	23	7.749×10^{-20}
9	6.035×10^{-8}	24	1.1×10^{-20}
10	8.487×10^{-9}	25	1.562×10^{-21}
11	1.195×10^{-9}	26	2.219×10^{-22}
12	1.685×10^{-10}	27	3.153×10^{-23}
13	2.38×10^{-11}	28	4.481×10^{-24}
14	3.363×10^{-12}	29	6.369×10^{-25}
15	4.755×10^{-13}	30	9.054×10^{-26}

Example 2.5. Assume in Example 2.4, $\alpha_n = \frac{7}{10n}$, then

$$\begin{cases} u_n = \frac{1}{n+1}\sum_{j=0}^{n} \frac{x_n}{20^j}, \\ z_n = \frac{n}{3n+1}x_n + (\frac{2n+1}{3n+1})u_n, \\ y_n = \frac{n}{5n+1}x_n + (\frac{4n+1}{5n+1})(\frac{1}{100^n})z_n, \\ x_{n+1} = \frac{10n-7}{10n}x_n + (\frac{7}{10n})T_3^n y_n, \end{cases}$$

and $\bigcap_{i=1}^{3} F(T_i) = \{0\}$. If $x_1 = \frac{1}{2}$, so

n	x_n	n	x_n
1	0.5	16	0.02502
2	0.15055	17	0.02392
3	0.09786	18	0.02294
4	0.07502	19	0.02205
5	0.06189	20	0.02123
6	0.05323	21	0.02049
7	0.04702	22	0.01981
8	0.04231	23	0.01918
9	0.03861	24	0.01859
10	0.03561	25	0.01805
11	0.03311	26	0.01755
12	0.03101	27	0.01707
13	0.0292	28	0.01663
14	0.02763	29	0.01621
15	0.02624	30	0.01582

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