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On z-filters and coz-ultrafilters

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Abstract. In this article we introduce the concepts of minimal prime z-filter, essential z-filter and r-filter. We investigate and study the behavior of minimal prime z-filters and compare them with minimal prime ideals and coz-ultrafilters. We show that X is a P-space if and only if every fixed prime z-filter is minimal prime. It is observed that if X is a ∂-space then X is a P-space if and only if $Z[M_f]$ is an r-filter, for every $f \in C(X)$. The collection of all minimal prime z-filters will be topologized and it is proved that the space of minimal prime z-filters is homeomorphic with the space of coz-ultrafilters. Finally, it is obtained several properties and relations between the space of minimal prime z-filters and the space of minimal prime ideals in $C(X)$.

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1 Introduction

We consider X to be a completely regular Hausdorff space and also all rings are commutative with identity. We denote by $C(X)$ the ring of all real-valued continuous functions on the space X. For each $f \in C(X)$,

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the set $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f and the set $coz(f) = \{x \in X : f(x) \neq 0\}$ is the cozero-set of f. Also $f \in C(X)$ is a zero divisor if and only if $Z^\circ(f) = \text{int}_X Z(f) \neq \emptyset$. Denote the collection of all zero-sets of X as $Z(X)$ and the collection of all cozero-sets of X by $Coz(X)$. For each $f \in C(X)$ let M_f be the intersection of all maximal ideals containing f. It is easy to see that $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. If I is an ideal in $C(X)$ and $\mathcal{F}$ is a z-filter on X, then we note $Z[I] = \{Z(f) : f \in I\}$ and $Z^{-1}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. An ideal I of $C(X)$ is a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$, that is to say, if $I = Z^{-1}[Z[I]]$. Every minimal prime ideal, every maximal ideal in $C(X)$ and every M_f, for each $f \in C(X)$, are z-ideals. Note $\mathcal{F} = Z[Z^{-1}[\mathcal{F}]]$ is always true for a z-filter $\mathcal{F}$. For $x \in X$, we have a z-ideal $O_x = \{f \in C(X) : x \in Z^\circ(f)\}$ contained in the maximal ideal $M_x = \{f \in C(X) : x \in Z(f)\}$. A space X is said to be P-space if $O_x = M_x$, for any $x \in X$ or equivalently, $Z(f) = Z^\circ(f)$, for every $f \in C(X)$ or equivalently, every prime ideal in $C(X)$ is a minimal prime ideal. For more details and undefined terms and notations, see [6], [7].

In Section 2, we introduce the concept of minimal prime z-filter and compare them with minimal prime ideal. In Section 3, we introduce the concept of r-filter and compare them with r-ideal. Section 4, devoted to coz-ultrafilters. Finally, in Section 5 we investigate the relationship between the space of minimal prime z-filters and the space of coz-ultrafilters.

2 Minimal Prime z-filters

In this section we introduce the concept of minimal prime z-filter. We begin with the following definitions.

Definition 2.1. *a) A prime z-filter $\mathcal{F}$ is called minimal prime if it does not properly contain any other prime z-filter.*
b) If $\mathcal{F}$ is a z-filter on X, then $Ann(\mathcal{F}) = \{Z \in Z(X) : Z \cup Z' = X \text{ for any } Z' \in \mathcal{F}\}$ is called the annihilator of $\mathcal{F}$.

One can easily show that if $\mathcal{F}$ is a nontrivial z-filter on X, then $Ann(\mathcal{F})$ is a z-filter on X, too. The following result shows that the minimal prime z-filters behave like the prime z-filters and z-ultrafilters.

Proposition 2.2. *The following statements hold.*
a) If $\mathcal{F}$ is a minimal prime z-filter on X, then $Z^{-1}[\mathcal{F}]$ is a minimal prime ideal in $C(X)$.
b) If P is a minimal prime ideal of $C(X)$, then $Z[P]$ is a minimal prime z-filter on X.

Proof. a) It is known that $Z^{-1}[\mathcal{F}]$ is a prime ideal. Now suppose that P_0 is a minimal prime ideal such that $P_0 \subseteq Z^{-1}[\mathcal{F}]$. Hence $Z[P_0] \subseteq ZZ^{-1}[\mathcal{F}] = \mathcal{F}$. By hypothesis we have $Z[P_0] = \mathcal{F}$. Since P_0 is a z-ideal, it follows that $P_0 = Z^{-1}Z[P_0] = Z^{-1}[\mathcal{F}]$, that is $Z^{-1}[\mathcal{F}]$ is a minimal prime ideal.
b) Clearly $Z[P]$ is a prime z-filter. Assume that $\mathcal{F}$ is a minimal prime z-filter such that $\mathcal{F} \subseteq Z[P]$. Thus $Z^{-1}[\mathcal{F}] \subseteq Z^{-1}Z[P] = P$, for P is a z-ideal. Since P is minimal prime, it conclude that $Z^{-1}[\mathcal{F}] = P$, and hence $\mathcal{F} = Z[P]$, i.e., $Z[P]$ is a minimal prime z-filter. □

Recall that in reduced ring R a prime ideal P is minimal if and only if for every $x \in P$ there exists an $r \notin P$ such that $xr = 0$, see [9]. The following proposition is a counterpart of the previous result for minimal prime z-filters.

Proposition 2.3. *A prime z-filter $\mathcal{F}$ on X is minimal prime if and only if for every $Z \in \mathcal{F}$ there exists $Z' \notin \mathcal{F}$ such that $Z \cup Z' = X$.*

Proof. Suppose that $\mathcal{F}$ is minimal prime z-filter and $Z = Z(f) \in \mathcal{F}$. Hence $f \in Z^{-1}[\mathcal{F}]$ and so by part (a) of the previous proposition there is a $g \notin Z^{-1}[\mathcal{F}]$ such that $fg = 0$. Therefore $Z' = Z(g) \notin \mathcal{F}$ and it is obvious that $Z \cup Z' = X$. Conversely, let $\mathcal{E}$ be a minimal prime z-filter, $\mathcal{E} \subseteq \mathcal{F}$ and $Z \in \mathcal{F}$ be an arbitrary zero-set. By hypothesis there is a $Z' \notin \mathcal{F}$ such that $Z \cup Z' = X$. Now $X \in \mathcal{E}$ and $Z' \notin \mathcal{E}$ implies that $Z \in \mathcal{E}$. Consequently, $\mathcal{E} = \mathcal{F}$, that is $\mathcal{F}$ is a minimal prime z-filter. □

It is well-known that if I be a finitely generated ideal in reduced ring R, then I is contained in a minimal prime ideal if and only if $Ann(I) \neq (0)$, see [9]. The following proposition is a counterpart of the previous statement for minimal prime z-filters.

Proposition 2.4. *Suppose that I be a finitely generated ideal in $C(X)$. Then z-filter $Z[I]$ is contained in a minimal prime z-filter if and only if $Ann(Z[I]) \neq \{X\}$.*

Proof. Let $Z[I] \subseteq \mathcal{F}$, which $\mathcal{F}$ is a minimal prime z-filter on X. Hence $I \subseteq Z^{-1}Z[I] \subseteq Z^{-1}[\mathcal{F}]$. Since $Z^{-1}[\mathcal{F}]$ is a minimal prime ideal, we have $Ann(I) \neq (0)$ and so $Ann(Z[I]) \neq \{X\}$, for $Ann(Z[I]) = Z[Ann(I)]$. For the converse, assume that $Ann(Z[I]) \neq \{X\}$, hence $Ann(I) \neq (0)$. Thus there exists minimal prime ideal P such that $I \subseteq P$. It implies that $Z[I] \subseteq Z[P]$ and $Z[P]$ by part (b) of Proposition 2.2 is a minimal prime z-filter on X. □

A subset S of a ring R is called a multiplicatively closed set if $1 \in S$ and for all $a, b \in S$ the product $ab \in S$. Let I be an ideal and S be multiplicatively closed set in R whit $I \cap S = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap S = \emptyset$, see [12]. The following proposition is a counterpart of the previous result for prime z-filters. First we need the next definition.

Definition 2.5. *A nonempty subfamily $\mathcal{S} \subseteq Z(X)$ is called a z- multiplicatively closed set if $Z(1) \in \mathcal{S}$ and for all $Z(f), Z(g) \in \mathcal{S}$ we have $Z(fg) \in \mathcal{S}$.*

Proposition 2.6. *Let $\mathcal{F}$ be a z-filter and $\mathcal{S}$ be a z-multiplicatively closed set with $\mathcal{F} \cap \mathcal{S} = \emptyset$. Then there exists a prime z-filter $\mathcal{E}$ such that $\mathcal{F} \subseteq \mathcal{E}$ and $\mathcal{E} \cap \mathcal{S} = \emptyset$.*

Proof. It is clear that $Z^{-1}[\mathcal{S}]$ is a multiplicatively closed set in $C(X)$ and $Z^{-1}[\mathcal{S} \cap \mathcal{F}] = Z^{-1}[\mathcal{S}] \cap Z^{-1}[\mathcal{F}] = \emptyset$. Now there exists a prime ideal P such that $Z^{-1}[\mathcal{F}] \subseteq P$ and $Z^{-1}[\mathcal{S}] \cap P = \emptyset$. Hence $\mathcal{F} \subseteq Z[P]$ and $\mathcal{S} \cap Z[P] = \emptyset$. It suffices that $\mathcal{E} = Z[P]$, and we are done. □

For each $f \in C(X)$ let $pos(f) = \{x \in X : f(x) > 0\}$ and $neg(f) = \{x \in X : f(x) < 0\}$. The next proposition is similar to Theorem 2.9 in [7], which is about prime z-ideals.

Proposition 2.7. *For a z-filter $\mathcal{F}$, the following statements are equivalent:*
a) $\mathcal{F}$ is prime.

b) $\mathcal{F}$ *contains a prime* z*-filter.*
c) If $Z_1 \cup Z_2 = X$*, then* $Z_1 \in \mathcal{F}$ *or* $Z_2 \in \mathcal{F}$*, for any* $Z_1, Z_2 \in Z(X)$*.*
d) For every $Z(f) \in Z(X)$*, there is a* $Z \in \mathcal{F}$*such that* $Z \subseteq Z(f) \cup pos(f)$ *or* $Z \subseteq Z(f) \cup neg(f)$*.*

Proof. The implications ($a \Rightarrow b \Rightarrow c \Rightarrow d$) are clear. We prove that ($d \Rightarrow a$). Consider $Z(gh) \in \mathcal{F}$. Suppose, without loss of generality, that $Z \in \mathcal{F}$ exists with $Z \subseteq Z(|g| - |h|) \cup pos(|g| - |h|)$. Therefore $Z \cap Z(g) \subseteq Z \cap Z(h)$ and so $Z \cap Z(gh) = Z \cap Z(h) \subseteq Z(h)$. Since $Z \cap Z(gh) \in \mathcal{F}$, it implies that $Z(h) \in \mathcal{F}$. □

We should remind the reader that a z-filter $\mathcal{F}$ on X is fixed if $\cap Z_{Z \in \mathcal{F}} \neq \emptyset$. Clearly $Z[M_x]$ is a fixed z-filter on X, for any $x \in X$. Now by using minimal prime z-filters we obtain an equivalent condition for P-spaces, which is one of the main results of this paper. For more details about P-spaces, see [7].

Theorem 2.8. *X is a P-space if and only if every fixed prime z-filter is minimal prime.*

Proof. First, assume that X is a P-space and let $\mathcal{F}$ be a fixed prime z-filter. Hence $Z^{-1}[\mathcal{F}]$ is a prime ideal and so is a minimal prime ideal. Therefore by part (b) of Proposition 2.2, $Z[Z^{-1}[\mathcal{F}]] = \mathcal{F}$ is a minimal prime z-filter. Conversely, suppose that $x \in X$ be an arbitrary element and on the contrary let $O_x \neq M_x$. Hence by Exercise 4I.5 of [7] there is a prime ideal P which is not a z-ideal and $O_x \subsetneq P \subsetneq M_x$. Now $Z[P]$ is a fixed prime z-filter which is not minimal prime, for P is not a minimal prime ideal. This is a contradiction. □

A nonzero ideal I in a ring R is called essential if it intersects every nonzero ideal nontrivially. It is shown that an ideal I in a reduced ring is essential if and only if $Ann(I) = (0)$, where $Ann(I) = \{r \in R : rI = (0)\}$. Recall that from [2] a z-filter $\mathcal{F}$ is called essential if $\mathcal{F} \cap \mathcal{E} \neq \{X\}$ for every z-filter $\mathcal{E} \neq \{X\}$.

Proposition 2.9. *The following statements hold.*
a) $\mathcal{F}$ *is an essential* z*-filter on* X *if and only if* $Z^{-1}[\mathcal{F}]$ *is an essential ideal in* $C(X)$*.*

b) I is an essential ideal in $C(X)$ if and only if $Z[I]$ is an essential z-filter on X.

Proof. a) Let $J \neq (0)$ be an ideal and assume that $0 \neq f \in J$. Hence $X \neq Z(f) \in Z[J]$. Therefore by hypothesis there exists $X \neq Z(g) \in \mathcal{F} \cap Z[J]$. Thus $g \in Z^{-1}[\mathcal{F}]$ and $Z(g) = Z(h)$, for some $0 \neq h \in J$. Now it is clear that $0 \neq gh \in Z^{-1}[\mathcal{F}] \cap J$. For the converse, suppose that $\mathcal{E} \neq \{X\}$ be a z-filter. Hence there is a $X \neq Z(f) \in \mathcal{E}$ and so $0 \neq f \in Z^{-1}[\mathcal{E}]$. Now since $Z^{-1}[\mathcal{F}] \cap Z^{-1}[\mathcal{E}] \neq \emptyset$ there exists $0 \neq g \in Z^{-1}[\mathcal{F}] \cap Z^{-1}[\mathcal{E}]$. It implies that $X \neq Z(g) \in \mathcal{F} \cap \mathcal{E}$.
b) It is similar to part (a). □

Proposition 2.10. *If I is an ideal in $C(X)$ and $\mathcal{F}$ is a z-filter on X, the following statements hold.*
a) $Ann(\mathcal{F}) = Z[Ann(Z^{-1}[\mathcal{F}])]$, for any z-filter $\mathcal{F}$.
b) A z-filter $\mathcal{F} \neq \{X\}$ is essential if and only if $Ann(\mathcal{F}) = \{X\}$.
c) An ideal I is essential if and only if $Ann(Z[I]) = \{X\}$.

Proof. a) It is clear that, $Ann(\mathcal{F}) \subseteq Z[Ann(Z^{-1}[\mathcal{F}])]$. Let $Z(f) \in Z[Ann(Z^{-1}[\mathcal{F}])]$ be an arbitrary zero-set. Since $Ann(Z^{-1}[\mathcal{F}])$ is a z-ideal it follows that $f \in Ann(Z^{-1}[\mathcal{F}])$. Now if $Z(g) \in \mathcal{F}$, then $g \in Z^{-1}[\mathcal{F}]$. Clearly, $fg = 0$ and so $Z(fg) = X$. It consequence that $Z(f) \in Ann(\mathcal{F})$.
b) If $\mathcal{F} \neq \{X\}$ is an essential z-filter then by part (a) of the Proposition 2.9 $Z^{-1}[\mathcal{F}]$ is an essential ideal. Hence by Theorem 3.1 of [2] we have $Ann(Z^{-1}[\mathcal{F}]) = (0)$ and so by part (a) $Ann(\mathcal{F}) = Z[(0)] = \{X\}$. For the converse, if $Ann(\mathcal{F}) = \{X\}$ then by part (a) we have $Z[Ann(Z^{-1}[\mathcal{F}])] = \{X\}$ and hence $Ann(Z^{-1}[\mathcal{F}]) = (0)$. Therefore $Z^{-1}[\mathcal{F}]$ is an essential ideal and so by part (a) of Proposition 2.9, $\mathcal{F}$ is an essential z-filter.
c) If I is an essential ideal, then $Ann(I) = (0)$. Hence $Ann(Z[I]) = Z[Ann(I)] = Z[(0)] = \{X\}$. Conversely, if $Ann(Z[I]) = \{X\}$ then $Z[Ann(I)] = \{X\}$ and hence $Ann(I) = (0)$. Thus I is an essential ideal. □

Example 2.11. a) z-filter $\mathcal{F} = \{Z(f) : (0,1) \subseteq Z(f)\}$ on $\mathbb{R}$ is not essential. Suppose that $g \in C(\mathbb{R})$ such that $Z(g) = (-\infty, 0] \cup [1, \infty)$. Clearly, $fg = 0$, for every $Z(f) \in \mathcal{F}$. Hence $X \neq Z(g) \in Ann(\mathcal{F})$.
b) z-filter $\mathcal{F} = \{Z(f) : [0, \epsilon) \subseteq Z(f)$ *for some* $\epsilon > 0\}$ on $X = [0, \infty)$ is

essential. On the contrary, assume that $X \neq Z(g) \in Ann(\mathcal{F})$. Hence there is $x_0 \in X$ such that $g(x_0) \neq 0$. Since $g \in C(X)$, without loss of generality we assume that $x_0 > 0$. Consider $Z(f) = [0, \frac{x_0}{2}]$, then $f \in \mathcal{F}$, for it is sufficient put $\epsilon = \frac{x_0}{3}$. Furthermore, it is clear that $fg \neq 0$ and this is a desired contradiction.

3 r-filters

An ideal I of a ring R is called an r-ideal if for each non zerodivisor $a \in R$ and each $b \in R$, $ab \in I$ implies that $b \in I$. Every minimal prime ideal in R is an r-ideal. Also an ideal I in R is called nonregular if contains entirely of zerodivisors. Every r-ideal is a nonregular ideal. For more information about r-ideals and nonregular ideals see [10] and [4], respectively. In this section we introduce the concepts of r-filter and nonregular z-filter. We begin with the following definition.

Definition 3.1. *a) A z-filter $\mathcal{F}$ on X is called r-filter, whenever $Z_1 \cup Z_2 \in \mathcal{F}$ and $Z_1^\circ = \emptyset$ then $Z_2 \in \mathcal{F}$, for any $Z_1, Z_2 \in Z(X)$.*
b) A z-filter $\mathcal{F}$ on X is called nonregular if $Z^\circ \neq \emptyset$, for any $Z \in \mathcal{F}$.

It is clear that an ideal I in $C(X)$ is nonregular if and only if $Z[I]$ is a nonregular z-filter on X. Also, if I is a z-ideal, then I is an r-ideal if and only if $Z[I]$ is an r-filter on X. For instance, if P is a minimal prime ideal of $C(X)$, then $Z[P]$ is an r-filter. Furthermore, if $x \in X$ is an isolated point of X then it is easy to see that $\mathcal{A}_x = \{Z(f) : x \in Z(f)\}$ is an r-filter. The converse of this statement is true if X is a perfectly normal space. Recall that a space X is called perfectly normal space if every closed set is a zero-set. For example, every metric space is perfectly normal. For more details about these spaces, see [6]. Clearly, every r-filter is nonregular but the converse is not true. For example, we consider $\mathcal{F} = \{Z(f) \in Z(\mathbb{R}) : [0, 1] \cup \{2\} \subseteq Z(f)\}$. Clearly, $\mathcal{F}$ is a nonregular z-filter on $\mathbb{R}$. Now suppose that $Z(g) = [0, 1]$ and $Z(h) = \{2\}$, where $g, h \in C(\mathbb{R})$. It is obvious that $Z(gh) \in \mathcal{F}$, $Z^\circ(h) = \emptyset$ and $Z(g) \notin \mathcal{F}$. It implies that $\mathcal{F}$ is not an r-filter.

Recall that X is a ∂-space if for every zero-set $Z \in Z(X)$ there exists a zero-set $F \in Z(X)$ such that $\partial(Z) \subseteq F$ and $F^\circ = \emptyset$, where $\partial(Z) = Z - Z^\circ$ is the boundary of Z. Also X is an almost P-space if

$Z(f)$ is regular closed set in X for any $f \in C(X)$, that is $\overline{Z^\circ(f)} = Z(f)$ or equivalently if $Z(f) \neq \emptyset$, then $Z^\circ(f) \neq \emptyset$ for any $f \in C(X)$. It is easy to see that the space X is an almost P-space and a ∂-space if and only if X is a P-space. Clearly, X is an almost P-space if and only if every z-filter is an r-filter. For more information about these spaces see [3] and [4].

Proposition 3.2. *Let X is a ∂-space and $f \in C(X)$. Then $Z(f)$ is a regular closed set if and only if $Z[M_f]$ is an r-filter.*

Proof. First, assume that $Z_1 \cup Z_2 \in Z[M_f]$ and $Z_1^\circ = \emptyset$. Hence $Z(f) \subseteq Z_1 \cup Z_2$ and so $Z^\circ(f) \subseteq Z_2$ therefore $\overline{Z^\circ(f)} = Z(f) \subseteq Z_2$. Thus $Z_2 \in Z[M_f]$, that is $Z[M_f]$ is an r-filter. Conversely, and on the contrary suppose that there exists $x \in Z(f)$ and $x \notin \overline{Z^\circ(f)}$. Hence there exist $g, h \in C(X)$ such that $x \in Z(g)$, $\overline{Z^\circ(f)} \subseteq Z(h)$ and $Z(g) \cap Z(h) = \emptyset$. On the other hand there is $k \in C(X)$ such that $\partial(Z(f)) \subseteq Z(k)$ and $Z^\circ(k) = \emptyset$. Clearly, $Z(f) = Z^\circ(f) \cup \partial(Z(f)) \subseteq Z(h) \cup Z(k)$ and hence $Z(h) \cup Z(k) \in Z[M_f]$. Now by hypothesis we infer that $Z(h) \in Z[M_f]$ that is $Z(f) \subseteq Z(h)$ which is a contradiction and we are done. □

Corollary 3.3. *Let X is a ∂-space. The following statements are equivalent:*
a) X is a P-space.
b) M_f is an r-ideal, for any $f \in C(X)$.
c) $Z[M_f]$ is an r-filter, for any $f \in C(X)$.

Proof. The implications ($a \Rightarrow b \Rightarrow c$) are clear. In fact, they are valid for any space.
($c \Rightarrow a$) By Proposition 3.2 we conclude that X is an almost P-space. Therefore X both an almost P-space and a ∂-space. This implies that X is a P-space. □

4 *coz*-ultrafilters

Recall that from [5], a nonempty subfamily $\mathcal{E}$ of $Coz(X)$ is called a *coz*-filter on X, if $\emptyset \notin \mathcal{E}$, the intersection of any two members of $\mathcal{E}$ is again a member of $\mathcal{E}$, and any member of $Coz(X)$ containing a member of $\mathcal{E}$ also belongs to $\mathcal{E}$. A *coz*-filter $\mathcal{E}$ is called prime if $coz(f) \cup coz(g) \in \mathcal{E}$, implies that $coz(f) \in \mathcal{E}$ or $coz(g) \in \mathcal{E}$. A coz-filter $\mathcal{E} \subseteq Coz(X)$ is a coz-ultrafiler if whenever $\mathcal{E} \subseteq \mathcal{U}$ and $\mathcal{E} \neq \mathcal{U}$, where $\mathcal{U}$ is a coz-filter, then $\mathcal{U} = Coz(X)$. For more details see [5], [11].

Let $\mathcal{F}$ be a z-filter and $\mathcal{E}$ be a coz-filter. Then we note $c(\mathcal{F}^c) = \{coz(f) : Z(f) \in \mathcal{F}^c\}$, where $\mathcal{F}^c = Z(X) \setminus \mathcal{F}$ and $z(\mathcal{E}^c) = \{Z(f) : coz(f) \in \mathcal{E}^c\}$, where $\mathcal{E}^c = Coz(X) \setminus \mathcal{E}$.

Proposition 4.1. *The following statements hold.*
a) If $\mathcal{F}$ is a prime z-filter, then $c(\mathcal{F}^c)$ is a prime coz-filter.
b) If $\mathcal{E}$ is a prime coz-filter, then $z(\mathcal{E}^c)$ is a prime z-filter.

Proof. It is evident. □

Proposition 4.2. *The following statements hold.*
a) If $\mathcal{F}$ is a minimal prime z-filter, then $c(\mathcal{F}^c)$ is a coz-ultrafilter.
b) If $\mathcal{U}$ is a coz-ultrafilter, then $z(\mathcal{U}^c)$ is a minimal prime z-filter.

Proof. a) Assume that $coz(f) \notin c(\mathcal{F}^c)$. Hence $Z(f) \in \mathcal{F}$ and therefore there exists $Z(g) \notin \mathcal{F}$ such that $fg = 0$. Thus $coz(g) \in c(\mathcal{F}^c)$. Now Proposition 1.2 of [5] implies that $c(\mathcal{F}^c)$ is a coz-ultrafilter.
b) It is similar to part (a). □

Remark 4.3. Three notations $coz(P^c)$, B_p and C_p have used in [5]. By the above notation we have:
a) $coz(P^c) = c(Z[P]^c)$ and $z(coz(P^c)^c) = Z[P]$.
b) $B_p = c(\mathcal{A}_p^c)$ and $\mathcal{A}_p = z(B_p^c)$.
c) $C_p = c(Z[O_p]^c)$ and $Z[O_p] = z(C_p^c)$.

Recall that when O_x is a prime ideal, it is a minimal prime ideal. Similarly, if $Z[O_x]$ is a prime z-filter then it is a minimal prime z-filter. Now by using the above two propositions and remark, Theorem 2.12, Proposition 2.13 and Theorem 2.14 in [5] is clear.
A subset S of a ring R is called a saturated multiplicatively closed set if it is a multiplicatively closed set and if $ab \in S$ then $a, b \in S$, for every $a, b \in R$. Let $S \subseteq C(X)$ and $\mathcal{E}$ is a coz-filter. Then we note $coz[S] = \{coz(f) : f \in S\}$ and $coz^{-1}[\mathcal{E}] = \{f : coz(f) \in \mathcal{E}\}$. The next two results is generalization of Theorems 2.3, 2.5, 2.7 and 2.11 of [5].

Proposition 4.4. *The following statements hold.*
a) If S is a saturated multiplicatively closed set in $C(X)$, then $coz[S]$ is

a coz-filter on X.
b) If $\mathcal{E}$ is a coz-filter on X, then $S = coz^{-1}[\mathcal{E}]$ is a saturated multiplicatively closed set in $C(X)$.

Proof. a) It is known that $S = R \setminus \cup_{\alpha\in A} P_\alpha$, which P_α is a prime ideal of $C(X)$ for every $\alpha \in A$. The first two axioms of being a *coz*-filter can be easily verified. To show that the third axiom suppose that $coz(f) \subseteq coz(g)$ and $f \in S$. Hence $coz(f^2 + g^2) = coz(g)$ and so it suffices to show that $f^2 + g^2 \in S$. On the contrary, there exists $\alpha_0 \in A$ such that $f^2 + g^2 \in P_{\alpha_0}$. Therefore $f \in P_{\alpha_0}$, which is a contradiction, for $f \in R \setminus P_{\alpha_0}$.
b) Suppose that $f, g \in C(X)$. Then $f, g \in S$ if and only if $coz(fg) = coz(f) \cap coz(g) \in \mathcal{E}$ if and only if $fg \in S$. □

Proposition 4.5. *$\mathcal{E}$ is a prime coz-filter on X if and only if $S = coz^{-1}[\mathcal{E}]$ is a saturated multiplicatively closed set and $I = C(X) \setminus S$ is an ideal in $C(X)$.*

Proof. Let $\mathcal{E}$ is a prime *coz*-filter. By the above proposition it is sufficient to show that I is an ideal. Suppose that $f, g \in I$. Then $f, g \notin S$, that is $coz(f), coz(g) \notin \mathcal{E}$. Note that $coz(f-g) \subseteq coz(f) \cup coz(g) \notin \mathcal{E}$. This implies that $coz(f - g) \notin \mathcal{E}$ and hence $f - g \notin S$. Consequently, $f - g \in I$. Now let $f \in I$ and $g \in C(X)$. Since $coz(f) \notin \mathcal{E}$ and $coz(fg) \subseteq coz(f)$, therefore $coz(fg) \notin \mathcal{E}$. It consequence that $fg \notin S$ and so $fg \in I$. On the other hand, it suffices to show that $\mathcal{E}$ is prime. Let $coz(f^2 + g^2) = coz(f) \cup coz(g) \in \mathcal{E}$. Hence $f^2 + g^2 \in S$ and so $f^2 + g^2 \notin I$. Since I is an ideal, therefore $f \notin I$ or $g \notin I$. Thus $f \in S$ or $g \in S$. This implies that $coz(f) \in \mathcal{E}$ or $coz(g) \in \mathcal{E}$. □

5 The Space of Minimal Prime z-filters

In this section we investigate the relationship between the space of minimal prime z-filters and the space of *coz*-ultrafilters. Recall that from [5], $\mathcal{U}(X)$ denote the collection of all *coz*-ultrafilters and the collection $\{U_{\mathcal{U}}(f) : f \in C(X)\}$ forms a basis for open sets a topology on $\mathcal{U}(X)$, which $U_{\mathcal{U}}(f) = \{\mathcal{E} \in \mathcal{U}(X) : coz(f) \notin \mathcal{E}\}$.

We denote by $MF(X)$ the collection of all minimal prime z-filters. For each $f \in C(X)$, let us use the notation $U_{MF}(f) = \{\mathcal{F} : Z(f) \in \mathcal{F}\}$.

Lemma 5.1. *For $f, g \in C(X)$ we have:*
a) $U_{MF}(f) \cup U_{MF}(g) = U_{MF}(fg)$.
b) $U_{MF}(f) \cap U_{MF}(g) = U_{MF}(f^2 + g^2)$.
c) $U_{MF}(f) = MF(X)$ if and only if $f = 0$.
d) $U_{MF}(f) = \emptyset$ if and only if $Z^\circ(f) = \emptyset$.

Proof. It is similar to Proposition 3.1 in [5]. □

The collection $\{U_{MF}(f) : f \in C(X)\}$ forms a basis for open sets a topology on $MF(X)$.

We conclude this section by the main result which is as follows.

Theorem 5.2. *The space $MF(X)$ is homeomorphic with the space $\mathcal{U}(X)$.*

Proof. Let $\phi : MF(X) \to \mathcal{U}(X)$ be defined by $\phi(\mathcal{F}) = c(\mathcal{F}^c)$, it is clear that ϕ is a well-defined. By Proposition 4.2, ϕ is a bijection. We claim that ϕ is a continuous function. To see this we show that $\phi^{-1}(U_{\mathcal{U}}(f)) = U_{MF}(f)$. Consider $\mathcal{F} \in MF(X)$. It is obvious that $\mathcal{F} \in \phi^{-1}(U_{\mathcal{U}}(f))$ if and only if $\phi(\mathcal{F}) \in U_{\mathcal{U}}(f)$ if and only if $c(\mathcal{F}^c) \in U_{\mathcal{U}}(f)$ if and only if $coz(f) \notin c(\mathcal{F}^c)$ if and only if $Z(f) \in \mathcal{F}$ if and only if $\mathcal{F} \in U_{MF}(f)$. Finally, we show that ϕ is an open map. To see this we claim that $\phi(U_{MF}(f)) = U_{\mathcal{U}}(f)$. Consider $\mathcal{E} \in U_{\mathcal{U}}(f)$, then $coz(f) \in \mathcal{E}^c$ and hence $Z(f) \in z(\mathcal{E}^c) = \mathcal{F}$. It follows that $\mathcal{F} \in U_{MF}(f)$. Furthermore, we have $\phi(\mathcal{F}) = c(\mathcal{F}^c) = c(z(\mathcal{E}^c)^c) = \mathcal{E}$. It implies that $\mathcal{E} \in \phi(U_{MF}(f))$. On the other hand, suppose that $\mathcal{E} \in \phi(U_{MF}(f))$, so there exists $\mathcal{F} \in U_{MF}(f)$ such that $\mathcal{E} = \phi(\mathcal{F})$. Since $Z(f) \in \mathcal{F}$, therefore $coz(f) \notin c(\mathcal{F}^c) = \mathcal{E}$ and hence $\mathcal{E} \in U_{\mathcal{U}}(f)$. Therefore ϕ is a homeomorphism and we are done. □

Using the above theorem we can replaced the space $\mathcal{U}(X)$ by the space $MF(X)$ throughout Section 3 of [5]. Indeed, we obtain several properties of them and relations between the space of minimal prime z-filters and the space of minimal prime ideals in $C(X)$, i.e., $Min(C(X))$. But to avoid the repetition, we leave it to the reader. The interested reader can refer to that section. For more information about the space of minimal prime ideals see also [8].

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